Novi Sad J. Math.
Vol. 33, No. 2, 2003, 89–94

ON STABILITY OF SPLINE DIFFERENCE SCHEME FOR REACTION–DIFFUSION TIME–DEPENDENT SINGULARLY PERTURBED PROBLEM[1]

Katarina Surla[2], **Zorica Uzelac**[3]

Abstract. The singularly perturbed parabolic boundary value problem is considered. Difference scheme is obtained by using cubic spline difference scheme on Shishkin's mesh in space and classical discretization on uniform mesh in time. To obtain better stability and simpler matrix the fitting factor in polynomial form is used. The uniform convergence is achieved. Numerical results are presented.

AMS Mathematics Subject Classification (2000): 65L10, 65L12

Key words and phrases: spline difference scheme, reaction–diffusion

1. Introduction

For the open interval $D = (0,1)$, on the domain

$$Q = D \times (0,T], \qquad S = \bar{Q}/Q,$$

we consider a boundary value problem for the parabolic equation

$$(1) \qquad \begin{cases} Ly(x,t) \equiv -\varepsilon^2 y_{xx}(x,t) + c(x,t)y(x,t) + d(x,t)y_t(x,t) = f(x,t), \\ (x,t) \in Q, \\ y(x,t) = \varphi(x,t), \qquad (x,t) \in S. \end{cases}$$

Here the functions $c(x,t)$, $d(x,t)$, $f(x,t)$, and also the function $\varphi(x,t)$, are sufficiently smooth functions on the sets $\bar{Q}$ and S respectively. Moreover,

$$c(x,t) \geq d_0 > 0, \qquad d(x,t) \geq r_0 > 0, \qquad (x,t) \in \bar{Q},$$

$\varepsilon \in (0,1]$. The solution of the boundary value problem is a smooth function y, which satisfies the equation on Q and the boundary condition on S.

[1]This paper was supported by the Ministry of Science, Technology and Development of the Republic of Serbia, Grant No. 1840

[2]Department of Mathematics and Informatics, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, SCG, ksurla@im.ns.ac.yu

[3]Faculty of Technical Sciences, University of Novi Sad, Trg Dositeja Obradovića 6, 21000 Novi Sad, SCG, zora@uns.ns.ac.yu

The estimates of the derivatives are given in [4]. Namely, according to [4] we can assume that

$$y(x,t) = g(x,t) + v(x,t),$$

where $|\frac{\partial^k g(x,t)}{\partial x^k}| \leq M,\ |\frac{\partial^k v(x,t)}{\partial x^k}| \leq M\varepsilon^{-k}(\exp(-p\varepsilon^{-1}x) + \exp(p\varepsilon^{-1}(1-x)),\ k = 0,1,...,4,\ |\frac{\partial^k y(x,t)}{\partial t^k}|, k = 1,2$; p is a positive constant.

When the parameter ε tends to zero, a parabolic boundary layer appears in the neighborhood of the lateral boundary of the set Q. On the set $\bar{Q}$ we introduce the grid

$$\bar{Q}_h = \bar{\omega}_1 \times \bar{\omega}_0, \tag{2}$$

where $\bar{\omega}_1$ is a grid, generally nonuniform, on the interval $[0,1]$ and $\bar{\omega}_0$ is a uniform grid on the interval $[0,T]$. Let $h_{i+1} = x_{i+1} - x_i,\ \ x_i, x_{i+1} \in \bar{\omega}_1$ and $\tau = t_{k+1} - t_k\ ,\ t_k, t_{k+1} \in \bar{\omega}_0$. By $N+1$ and N_0+1 we denote the number of nodes in the grids $\bar{\omega}_1$ and $\bar{\omega}_0$ respectively. By $g_{j,k} = g(x_j, t_k)$ we denote the value of the function $g(x,t)$ at the grid point (x_j, t_k).

Different schemes for the problem (1) when $c(x,t) = 0$, is considered in [5]. Here we use one of them and introduce the fitting factor in order to simplify matrix and to magnify stability.

2. Construction of the scheme

In the grid $\bar{Q}_h$ we define the following collocation equation for problem (1)

$$\begin{cases} -\sigma_{j,k}u(x_j,t_k) + c_{j,k}u(x_j,t_k) = F_{j,k},\ (x_j,t_k) \in Q_h = Q \bigcap \bar{Q}_h \\ u(x_j,t_k) = \varphi(x_j,t_k), \qquad (x_j,t_k) \in S_h = S \bigcap \bar{Q}_h, \end{cases} \tag{3}$$

where $F_{j.k} = -d_{j,k}\ \ u_t(x_j,t_k) + f_{j.k}$. Using cubic spline discretization for the fixed t_k as in [6], we obtain

$$\begin{cases} r^-_{j,k}\ u_{j-1,k} + r^c_{j,k}\ u_{j,k} + r^+_{j,k}\ u_{j+1,k} = q^-_{j,k}\ F_{j-1,k} + q^c_{j,k}\ F_{j,k} + f^+_{j,k}\ F_{j+1,k}, \\ r^-_{j,k} = -(1 - h_j^2 c_{j-1,k}/(6\sigma_{j-1,k}))/h_j, \\ r^+_{j,k} = -(1 - h_{j+1}^2 c_{j+1,k}/(6\sigma_{j+1,k}))/h_{j+1}, \\ r^c_{j,k} = (1 + h_j^2 c_{j,k}/(6\sigma_{j,k}))/h_j + (1 + h_{j+1}^2 c_{j,k}/(6\sigma_{j,k}))/h_{j+1}, \\ q^-_{j,k}\ = h_j/(6\sigma_{j-1,k}),\ \ q^+_{j,k}\ = h_{\bar{j}1}/(6\sigma_{j+1,k}),\ \ q^c_{j,k}\ = (h_j + h_{j+1})/(6\sigma_{j,k}), \\ u(x_j,t_k) = \varphi(x_j,t_k), \qquad (x_j,t_k) \in S_h = S \bigcap \bar{Q}_h. \end{cases} \tag{4}$$

If we replace $F_{j,k}$ in (4) by $\Phi_{j,k} = -d_{j,k}\ \ (u_{j,k} - u_{j,k-1})/\tau + f_{j.k}$, we obtain the scheme

$$
(5)\qquad \begin{cases}
R\,u_{j,k} = Q\ u_{j,k-1} + qf_{j,k}, \\
R\,u_{j,k} = R^-_{j,k}\ u_{j-1,k} + R^c_{j,k}\ u_{j,k} + R^+_{j,k}\ u_{j+1,k} \\
Q\,u_{j,k-1} = Q^-_{j,k}\ u_{j-1,k-1} + Q^c_{j,k}\ u_{j,k-1} + Q^+_{j,k}\ u_{j+1,k-1} \\
q\ f^-_{j,k}\ = q^-_{j,k}\ \ f_{j-1,k} + q^c_{j,k}\ \ f_{j,k} + f^+_{j,k}\ \ f_{j+1,k}, \\
R^-_{j,k}\ = r^-_{j,k} + d_{j-1,k}\ q^-_{j,k}/\tau,\ \ R^+_{j,k}\ = r^+_{j,k} + d_{j+1,k}\ q^+_{j,k}/\tau, \\
R^c_{j,k}\ = r^c_{j,k} + d_{j,k}\ q^c_{j,k}/\tau, \\
Q^-_{j,k} = d_{j-1,k}\ q^-_{j,k}/\tau \\
Q^+_{j,k} = d_{j+1,k}\ q^+_{j,k}/\tau \\
Q^c_{j,k} = d_{j,k}\ q^c_{j,k}/\tau.
\end{cases}
$$

On the set $\bar{Q}$ we introduce a special grid, condensed in the boundary layer, similar to the grid constructed in [1],

$$
(6)\qquad \bar{Q}^*_h = \bar{\omega}^*_1 \times \bar{\omega}_0,
$$

where $\bar{\omega}^*_1$ is a piecewise grid on $[0,1]$. The step size of the grid $\bar{\omega}^*_1$ on the intervals $[0,\delta], [1-\delta,1]$ and on the interval $[\delta, 1-\delta]$ are constant and equal

$$
h_1 = 4\delta N^{-1} \qquad \text{and} \qquad h_2 = 2(1-2\delta)N^{-1},
$$

respectively. The value δ is chosen to satisfy the condition

$$
\delta = \min\,(1/4, 4\,\varepsilon\ \ \ln N).
$$

If $\varepsilon = \sigma_{j-1,k}$ and $h_j = h_2$, $r^-_{j,k} \to \infty$ when $\varepsilon \to 0$. The stability of the system becomes weak. Because of that we put

$$
\begin{aligned}
\sigma_{j,k} &= h^2_{j+1}c_{j,k}/6\ , \text{ when } x_j \le 1/2, \\
\sigma_{j,k} &= h^2_{j}c_{j,k}/6\ , \text{ when } x_j \ge 1/2.
\end{aligned}
$$

Then $r^-_{j,k} = r^+_{j,k} = 0$, for $j \ne i_0 - 1$ and $j \ne n - i_0 + 1,\ i_0 = \delta$.

At the end of the paper we present numerical results when fitting factor is used only outside the boundary layers. The results are very similar, but in the previous case the matrix is simpler.

3. Convergence of the method

Let $z_{j,k} = y_{j,k} - u_{j,k}$ and $\tau_{j,k}(y)$ be the truncation error at the point (x_j, t_k). Then

$$(7)\qquad \begin{aligned} Rz_{j,k} - Qz_{j,k-1} &= \tau_{j,k}(y), \\ \tau_{j,k}(y) &= \tilde{\tau}_{j,k}(y) + M\tau/h_j, \end{aligned}$$

where $\tilde{\tau}_{j,k}(y)$ is the truncation error for the cubic spline difference scheme for the fixed t_k,

$$(8)\qquad \tilde{\tau}_{j,k}(y) = \Phi_{2,j+1,k}/h_{j+1} - \Phi_{2,j,k}/h_j + \Phi_{1,j,k},$$

$$\begin{cases} \Phi_{2,j,k} = \Psi_{0,j,k} + h_j^2(\eta_{j-1,k}/3\sigma_{j-1,k} + \eta_{j,k}/6\sigma_{j,k} - \Psi_{2,j,k}/6) \\ \Phi_{1,j,k} = \Psi_{1,j,k} + h_j(\eta_{j-1,k}/\sigma_{j-1,k} + \eta_{j,k}/\sigma_{j,k} - \Psi_{2,j,k}) \\ \Psi_{b,j,k} = y^{(4)}(\Theta_{b,j}, t_k)\ h_j^{4-b}/(4-b)!,\ x_{j-1} \le \Theta_{b,j} \le x_j, \\ \eta_{j,k} = (\varepsilon - \sigma_{j,k})y''_{j,k}. \end{cases}$$

By analyzing the values $\Delta_{j,k} = |R^c_{j,k}| - |R^-_{j,k}| - |R^+_{j,k}|$ we obtain

$$(9)\qquad \Delta_{j,k} \ge \begin{cases} \delta^{-1}_{j,k} = \dfrac{M}{\tau\varepsilon n^{-1}\ln n},\ j = 1,2,...,i_0-1, j = n-i_0+1,...,n, \\ \delta^{-1}_{j,k} = M\,\dfrac{n^{-1}\tau + \varepsilon n^{-1}\ln n}{\tau\varepsilon n^{-2}\ln n},\ j = i_0, n-i_0, \\ \delta^{-1}_{j,k} = \dfrac{M}{\tau n^{-1}},\ j = i_0+1,...,n-i_0-1. \end{cases}$$

Now, we multiply the equations of the scheme by the corresponding $\delta_{j,k}$. Let A be the matrix of the system which corresponds to new scheme, then

$$(10)\qquad \|z_{j,k}\| \le \|A^{-1}\|\ (\|\delta_{j,k}\ Q\ z_{j,k-1}\| + \|\delta_{j,k}\ \tau_{j,k}(y)\|),$$

and $\|A^{-1}\| \le M$. For $k=1$, $z_{j,k} = 0$ and $\|z_{j,1}\| \le \|A^{-1}\|\ \|\delta_{j,k}\ \tau_{j,k}(y)\|$. Since $\|\delta_{j,k}\ \tau_{j,k}(y)\| \le \|\delta_{j,k}\ \tau_{j,k}(g)\| + \|\delta_{j,k}\ \tau_{j,k}(v)\|$ from (7),(8) we obtain

$$\begin{aligned} \|\delta_{j,k}\ \tilde{\tau}_{j,1}(g)\| &\le M\tau\max(\varepsilon^2, h_j^2), \\ \|\delta_{j,k}\ \tilde{\tau}_{j,1}(v)\| &\le M\tau\ n^{-2}\ln^2 n,\ j \ne i_0, n-i_0. \end{aligned}$$

For $j = i_0$ and $j = n - i_0$ we use the truncation error in the form

$$\tilde{\tau}_{j,1}(v) = rv_{j,k} - q(-\varepsilon v''_{j,k} + c_{j,k}v_{j,k}),$$

and we have that

$$\|\delta_{j,k}\ \tilde{\tau}_{j,1}(v)\| \leq M\tau n^{-2}.$$

Thus we have the following theorem.

Theorem 3.1 *Let $N_0\tau = T$ and the mesh $\bar{Q}_h^*$ be defined by (6). Let $\tau \geq n^{-1}$. Then for $0 \leq j\tau \leq T$ and the solution $u_{j,k}$ of the scheme (4), the estimate*

$$|y(x_j, t_k) - u(x_j, t_k)| \leq M(n^{-2}\ln^2 n + \tau) \tag{11}$$

is valid.

Proof. From the above analysis we have

$$|z_{j,k}| \leq Mk\tau(n^{-2}\ln^2 n + \tau).$$

Since $k\tau \leq T$, the theorem holds. □

4. Numerical example

Let us consider the boundary value problem

$$\begin{cases} Lu(x,t) \equiv -\varepsilon^2 y_{xx}(x,t) + y(x,t) + u_t(x,t) = 0, \ (x,t) \in Q \\ y(x,t) = W(x,t), \quad (x,t) \in S = \bar{Q}/Q. \end{cases} \tag{12}$$

where

$$\begin{aligned} W(x,0) &= \sin(\pi x) + 1/2\sin(3\pi x), \quad 0 \leq x, t \leq 1, \\ W(0,t) &= 0, \quad W(1,t) = 0, \\ y(x,t) &= \exp(-(1+\varepsilon^2\pi^2 t))\sin(\pi x) + 1/2\exp(-(1+9\varepsilon^2\pi^2 t))\sin(3\pi x). \end{aligned}$$

Using the solutions of the difference scheme (5) on grid (6) we calculated the values

$$E_n = \max_{\bar{Q}_h} |y(x_j, t_k) - u(x_j, t_k)|,$$

for various values of $\varepsilon = 2^{-k}$, $N = N_0$ and $T = 1$.

Table 1 contains results when the fitting factor is used at all points of the grid.

k	n							
	16	32	64	128	256	512	1024	
15	2.91(-2)	1.00(-2)	3.61(-3)	1.82(-3)	1.02(-3)	5.38(-4)	2.77(-4)	En_n
		1.54	1.48	.987	.840	.918	.960	Ord
16	2.91(-2)	1.00(-2)	3.61(-3)	1.82(-3)	1.02(-3)	5.38(-4)	2.77(-4)	E_n
		1.54	1.48	.988	.840	.918	.960	Ord
20	2.91(-2)	1.00(–2)	3.61(–3)	1.82(-3)	1.02(-3)	5.38(-4)	2.77(-4)	E_n
		1.54	1.48	.988	.840	.918	.960	Ord
29	2.91(-2)	1.00(-2)	3.44(-3)	1.82(-3)	1.02(-3)	5.38(-4)	2.77(-4)	E_n
		1.54	1.48	.988	8.40	.918	.960	Ord

Table 1.

Table 2 contains results when the fitting factor is used only outside the boundary layers, i.e. for $h_j = h_2$.

k	n							
	16	32	64	128	256	512	1024	
15	2.92(-2)	1.05(-2)	3.62(-3)	1.82(-3)	1.02(-3)	5.38(-4)	2.77(-4)	E_n
		1.53	1.48	.987	.840	.918	.960	Ord
16	2.92(-2)	1.05(-2)	3.68(-3)	1.82(-3)	1.02(-3)	5.39(-4)	2.77(-4)	E_n
		1.54	1.48	.988	.840	.918	.960	Ord
20	2.91(-2)	1.00(-2)	3.61(-3)	1.82(-3)	1.02(-3)	5.39(-4)	2.77(-4)	E_n
		1.54	1.48	.988	.840	.918	.960	Ord
29	2.91(-2)	1.00(-2)	3.44(-3)	1.82(-3)	1.02(-3)	5.39(-4)	2.77(-4)	E_n
		1.54	1.48	.988	8.40	.918	.960	Ord

Table 2.

References

[1] Farel, A. P., Hemker, W.P., Shishkin, G. I., Discrete approximations for singularly perturbed boundary value problems with parabolic layers. CWI, Report NM-R9502, February 1995.

[2] Friedman, A., Partial differential equations of parabolic type. Prentice-Hall, Englewood Cliffs, 1964.

[3] Shishkin, G. I., A Discrete Approximation of Singularly Perturbed Elliptic and Parabolic Equations. Russian Academy of Sciences, Ural Section, Ekaterinburg, 1992.

[4] Shishkin, G. I., Increasing accuracy of the solutions of difference schemes for parabolic equations with small parameter by the highest derivative. J. Num. Mat. Mat. Fiz. 24(6)(1984), 864–875 (in Russian).

[5] Surla, K., Vukoslavčević, V., On Stability of Some Difference Schemes for Parabolic Diffusin Equation, Novi Sad J. Math. 31(2)(2001), 27–34.

[6] Surla, K., On Modelling Semilinear Singularly Perturbed Reaction–Diffusion Problem. Nonlinear Analysis, Theory,Methods and Appl. 30(1)(1997), 61–66.

Received by the editors December 22, 2002