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ON q- ITERATIVE METHODS FOR SOLVING EQUATIONS AND SYSTEMS*

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Abstract. We construct q-Taylor formula for the functions of several variables and develop some new methods for solving equations and systems of equations. They are much easier for application than the well-known ones. We introduce some values for measuring their accuracy, such as $(r;q)$-order of convergence. We made some analogue of known methods, such as q-Newton method.

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1. Introduction

In the last quarter of XX century, q-calculus appeared as a connection between mathematics and physics (see [7], [8]). It has a lot of applications in different mathematical areas, such as: number theory, combinatorics, orthogonal polynomials, basic hyper-geometric functions and in other sciences: quantum theory, mechanics and theory of relativity.

Let $q \in (0,\ 1)$. A q-natural number $[n]_q$ is defined by

$$[n]_q := 1 + q + \cdots + q^{n-1}, \quad n \in \mathbb{N}.$$

Generally, a q-complex number $[a]_q$ is

$$[a]_q := \frac{1-q^a}{1-q}, \quad a \in \mathbb{C}.$$

The factorial of a number $[n]_q$ and q-binomial coefficient, we define by

$$[0]_q! := 1, \quad [n]_q! := [n]_q[n-1]_q \cdots [1]_q, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

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Also, q-Pochammer symbol is

$$(z-a)^{(0)}=1, \quad (z-a)^{(k)}=\prod_{i=0}^{k-1}(z-aq^i) \quad (k\in\mathbb{N}). \tag{1.1}$$

2. On q-partial derivatives and differential

Let $f(\vec{x})$, where $\vec{x}=(x_1,x_2,\dots,x_n)$ be a multivariable real continuous function. We introduce an operator $\varepsilon_{q,i}$ which multiplies a coordinate of the argument by

$$(\varepsilon_{q,i}f)(\vec{x})=f(x_1,\dots,x_{i-1},qx_i,x_{i+1},\dots,x_n).$$

Furthermore,

$$(\varepsilon_q f)(\vec{x}):=(\varepsilon_{q,1}\cdots\varepsilon_{q,n}f)(\vec{x})=f(q\vec{x}).$$

We define q-partial derivative of a function $f(\vec{x})$ to a variable x_i by

$$D_{q,x_i}f(\vec{x}):=\frac{f(\vec{x})-(\varepsilon_{q,i}f)(\vec{x})}{(1-q)x_i} \qquad (x_i\neq 0),$$

$$D_{q,x_i}f(\vec{x})\Big|_{x_i=0}=\lim_{x_i\to 0}D_{q,x_i}f(\vec{x}).$$

In a similar way, high q-partial derivatives are

$$D^n_{q,x_i^n}f(\vec{x}):=D_{q,x_i}\Big(D^{n-1}_{q,x_i^{n-1}}f(\vec{x})\Big), \quad D^{m+n}_{q,x_i^m,x_j^n}f(\vec{x}):=D^m_{q,x_i^m}\Big(D^n_{q,x_j^n}f(\vec{x})\Big).$$

Obviously,

$$D^{m+n}_{q,x_i^m,x_j^n}f(\vec{x})=D^{m+n}_{q,x_j^n,x_i^m}f(\vec{x}) \quad (i,j=1,2\dots,n)\ (m,n=0,1,\dots).$$

Also, for an arbitrary $\vec{a}=(a_1,a_2,\dots,a_n)\in\mathbb{R}^n$, we can introduce q-differential

$$d_qf(\vec{x},\vec{a}):=(x_1-a_1)D_{q,x_1}f(\vec{a})+(x_2-a_2)D_{q,x_2}f(\vec{a})+\cdots+(x_n-a_n)D_{q,x_n}f(\vec{a}),$$

and high q-differentials:

$$\begin{aligned} d_q^kf(\vec{x},\vec{a}) &:= \Big((x_1-a_1)D_{q,x_1}+(x_2-a_2)D_{q,x_2}+\cdots+(x_n-a_n)D_{q,x_n}\Big)^{(k)}f(\vec{a}) \\ &= \sum_{\substack{i_1+\cdots+i_n=k\\ i_j\in\mathbb{N}_0}}\frac{[k]_q!}{[i_1]_q![i_2]_q!\cdots[i_n]_q!}D^k_{q,x_i^{i_1},\dots,x_n^{i_n}}f(\vec{a})\prod_{j=1}^n(x_j-a_j)^{(i_j)}. \end{aligned}$$

Notice that a continuous function $f(\vec{x})$ in a neighborhood that does not include any point with a zero coordinate, has also continuous q-partial derivatives.

3. About q-Taylor formula for a multivariable function

Now we can discuss a new expansion of the function whose variable is from $\mathbb{R}^n$. First of all, we need the next lemma.

Lemma 3.1. *It is valid*

$$D_{q,x}(x-\alpha)^{(n)} = [n]_q (x-\alpha)^{(n-1)} \quad (x,\alpha \in \mathbb{R},\ n \in \mathbb{N}).$$

Proof. For the proof see, for example, J. Cigler [2]. □

Theorem 3.2. *Suppose that all q-differentials of $f(x,y)$ exist in some neighborhood of (a,b). Then*

$$f(x,y) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{D^n_{q,x^i,y^{n-i}} f(a,b)}{[i]_q! [n-i]_q!} (x-a)^{(i)} (y-b)^{(n-i)}.$$

Proof. Suppose that the function can be written in the following form

$$f(x,y) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} c_{n,i} (x-a)^{(i)} (y-b)^{(n-i)}.$$

Application of q-partial derivative operators $D_{q,x}$ and $D_{q,y}$ gives

$$D^{k+m}_{q,x^k,y^m} f(x,y) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} c_{n,i} D^{k+m}_{q,x^k,y^m} (x-a)^{(i)} (y-b)^{(n-i)}.$$

On the basis of the previous lemma we conclude

$$D^{k+m}_{q,x^k,y^m} (x-a)^{(i)} (y-b)^{(n-i)} = 0 \quad (k > i \ \wedge \ m > n-i).$$

In other cases, we have

$$\begin{aligned} &D^{k+m}_{q,x^k,y^m} (x-a)^{(i)} (y-b)^{(n-i)} \\ &= [i]_q \cdots [i-k+1]_q (x-a)^{(i-k)} \ [n-i]_q \cdots [n-i-m+1]_q (y-b)^{(n-i-m)}. \end{aligned}$$

Supposed expansion is valid in some neighborhood of (a,b). Putting $x=a$ and $y=b$, all members of the sum vanish, except for $i=k$ and $n-i=m$. Hence,

$$D^{k+m}_{q,x^k,y^m} f(a,b) = c_{k+m,k} \ [k]_q! \ [m]_q!. \ \square$$

In the same manner we can prove the analogous theorem for the general case.

Theorem 3.3. *Suppose that there exist all q-differentials of $f(\vec{x})$ in some neighborhood of $\vec{a}$. Then*

$$f(\vec{x}) = \sum_{k=0}^{\infty} \frac{d^k_q f(\vec{x}, \vec{a})}{[k]_q!}.$$

4. On q-Newton-Kantorovich method

We consider a system of nonlinear equations

$$\vec{f}(\vec{x}) = 0,$$

where $\vec{f}(\vec{x}) = \big(f_1(\vec{x}), f_2(\vec{x}), \dots f_n(\vec{x})\big)$ with $\vec{x} = (x_1, x_2, \dots x_n)$, $n \in \mathbb{N}$. We will suppose that this system has an isolated real solution $\vec{\xi}$. Using q-Taylor series of the function $\vec{f}(\vec{x})$ around some value $\vec{x}^{(m)} \approx \vec{\xi}$, we have

$$f_i(\vec{\xi}) \approx f_i(\vec{x}^{(m)}) + \sum_{j=1}^{n} D_{q,x_j} f_i(\vec{x}^{(m)})(\xi_j - x_j^{(m)}) \quad (i = 1, 2, \dots, n).$$

In the matrix form, we rewrite

$$\vec{f}(\vec{\xi}) \approx \vec{f}(\vec{x}^{(m)}) + W_q(\vec{x}^{(m)})(\vec{\xi} - \vec{x}^{(m)}),$$

where

$$W_q(\vec{x}) = D_q \vec{f}(\vec{x}) = \big[D_{q,x_j} f_i(\vec{x})\big]_{n\times n}$$

is the Jacobi matrix of q-partial derivatives. If the matrix W_q is regular, there exists the inverse matrix W_q^{-1}, so that we can formulate the q-Newton-Kantorovich method in the form

$$\vec{x}^{(m+1)} = \vec{x}^{(m)} - W_q^{-1}(\vec{x}^{(m)})\vec{f}(\vec{x}^{(m)}).$$

5. On q-Newton method

If in the previous speculation we took $n = 1$, the system of equations reduced to the equation $f(x) = 0$, and the main objects of the work are functions of one variable.

The q-derivative of a function $f(x)$ is

$$(D_q f)(x) := \frac{f(x) - f(qx)}{x - qx} \quad (x \neq 0), \quad (D_q f)(0) := \lim_{x\to 0}(D_q f)(x), \tag{5.1}$$

and the high q-derivatives $D_q^0 f := f, \quad D_q^n f := D_q(D_q^{n-1} f), \quad n = 1, 2, 3, \dots .$

From the above definition it is obvious that a continuous function on an interval, which does not include 0 is continuous q-differentiable.

In the q-analysis, q-integral is defined by

$$I_q(f) = \int_0^a f(t) d_q(t) := a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n.$$

Notice that according to [6] it holds

$$I(f) = \int_0^a f(t)\ dt = \lim_{q\uparrow 1} I_q(f).$$

Also,

$$\int_a^b f(t)d_q(t) := \int_0^b f(t)d_q(t) - \int_0^a f(t)d_q(t).$$

The next q-Taylor formula with the remainder term

$$f(x) = \sum_{k=0}^{n-1} \frac{\left(D_q^k f\right)(a)}{[k]_q!}(x-a)^{(k)} + R_n(f,x,a,q),$$

where

$$R_n(f,x,a,q) = \int_{t=a}^{t=x} \frac{(x-t)^{(n)}}{x-t} \frac{\left(D_q^n f\right)(t)}{[n-1]_q!} d_q(t). \tag{5.2}$$

is given in [3] (see also [5]).

Suppose that an equation $f(x) = 0$ has the unique isolated solution $x = \xi$. If x_n is an approximation to the exact solution ξ, by using Jackson's q-Taylor formula we have

$$0 = f(\xi) \approx f(x_n) + (D_q f)(x_n)(\xi - x_n),$$

hence

$$\xi \approx x_n - \frac{f(x_n)}{(D_q f)(x_n)}.$$

So, we can construct the *q-Newton method*

$$x_{n+1} = x_n - \frac{f(x_n)}{(D_q f)(x_n)}.$$

According to (5.1), we can rearrange the above expression to the form

$$x_{n+1} = x_n \left\{ 1 - \frac{1-q}{1 - \frac{f(qx_n)}{f(x_n)}} \right\}.$$

This method written in the form

$$x_{n+1} = x_n - \frac{x_n - qx_n}{f(x_n) - f(qx_n)} f(x_n)$$

resembles the method of chords (secants).

The next theorem is a q-analogue of the well-known statement about convergence (see [1]).

Theorem 5.1. *Let the equation $f(x) = 0$ has a unique isolated root $x = \xi$ and $a > 0,\ 1 \le p \le 2$. Let the function $f(x)$ satisfies*

(1) $\quad |(D_q f)(x)| \ge M_1^{p-1} > 0,$

(2) $\quad |f(x) - f(y) - (D_q f)(y)(x-y)| < L^{p-1}|x-y|^p,$

where M_1 *and* L *are positive constants. Then, for all initial values* $x_0 \in (\xi - b, \xi + b)$, *where* $b = \min\{a, M_1/L\}$, *the q-Newton method converges to exact solution of the equation* $f(x) = 0$ *and it is valid*

$$|\xi - x_n| \le \left(\frac{L}{M_1}\right)^{p^n - 1} |\xi - x_0|^{p^n}.$$

Proof. We can write the q-Newton method in the form

$$(D_q f)(x_n)(x_{n+1} - x_n) = -f(x_n).$$

From the condition (2), we have

$$|f(\xi) - f(x_n) - (D_q f)(x_n)(\xi - x_n)| < L^{p-1}|\xi - x_n|^p.$$

Hence, using $f(\xi) = 0$, we get

$$|(D_q f)(x_n)(\xi - x_{n+1})| < L^{p-1}|\xi - x_n|^p.$$

By the condition (1) we have

$$|\xi - x_{n+1}| < \frac{L^{p-1}}{|(D_q f)(x_n)|}|\xi - x_n|^p < \left(\frac{L}{M_1}\right)^{p-1}|\xi - x_n|^p.$$

Now, if $x_n \in (\xi - b, \xi + b)$, then

$$|\xi - x_{n+1}| < \left(\frac{L}{M_1}\right)^{p-1} b^p = \left(\frac{L}{M_1}\right)^{p-1} b^{p-1} b \le b.$$

Denote by $c = L/M_1$. Now

$$|\xi - x_{n+1}| < c^{p-1}|\xi - x_n|^p \quad \Rightarrow \quad c\,|\xi - x_{n+1}| < c^p|\xi - x_n|^p,$$

wherefrom we get the final conclusion. □

6. Analysis of the convergence and error estimation

Our purpose is to formulate and prove the theorem for scanning the convergence of an iterative process

$$x_{k+1} = \Phi(x_k) \quad (k = 0, 1, 2, \ldots),$$

by q-analysis.

Theorem 6.1. *Suppose that* $\Phi(x)$ *is a continuous function on* $[a, b]$ $(0 \notin [a, b])$, *which satisfies the following conditions:*

$$\text{(1)} \qquad \Phi: \ [a, b] \mapsto [a, b],$$

$$\text{(2)} \qquad \Big(\forall q \in (\min\{a, b\}/\max\{a, b\}, 1)\Big)\Big(\forall x \in (a, b)\Big): \ \big|(D_q f)(x)\big| \le \lambda < 1.$$

Then the iterative process $x_{k+1} = \Phi(x_k),\ k = 0, 1, 2, \ldots,$ *with the initial value* $x_0 \in [a, b]$, *is converging to the fixed point of* $\Phi(x)$, *i.e.,*

$$\lim_{k\to\infty} x_k = \xi, \quad \Phi(\xi) = \xi.$$

Proof. Notice that for a continuous function $\Phi(x)$ on $[a, b]$ $(0 \notin [a, b])$, for all x and y such that $a < x < y < b$, it is valid

$$\Phi(y) - \Phi(x) = \big(D_{x/y}\Phi\big)(y)(y - x), \quad \Phi(y) - \Phi(x) = \big(D_{y/x}\Phi\big)(x)(y - x).$$

Consider

$$\xi = x_0 + \sum_{k=0}^{\infty}(x_{k+1} - x_k).$$

Let $x_k^{(M)} = \max\{x_k, x_{k-1}\}$, $x_k^{(m)} = \min\{x_k, x_{k-1}\}$ and $q = x_k^{(m)}/x_k^{(M)}$. Now, we have

$$\Phi(x_k) - \Phi(x_{k-1}) = (D_q\Phi)(x_k^{(M)})(x_k - x_{k-1}).$$

So, it is valid

$$\big|x_{k+1} - x_k\big| = \big|\big(D_q\Phi\big)(x_k^{(M)})\big|\ |x_k - x_{k-1}| \le \lambda\ |x_k - x_{k-1}|.$$

Since

$$\big|x_{k+1} - x_k\big| \le \lambda^k |x_1 - x_0|,$$

we get

$$\sum_{k=0}^{\infty} |x_{k+1} - x_k| \le |x_1 - x_0| \sum_{k=0}^{\infty} \lambda^k = \frac{|x_1 - x_0|}{1 - \lambda}.$$

Hence, the series S converges and

$$\xi = \lim_{n\to\infty} S_n = \lim_{n\to\infty} x_{n+1}.$$

Since $\Phi(x)$ is a continuous function we have

$$\xi = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \Phi(x_n) = \Phi(\lim_{n\to\infty} x_n) = \Phi(\xi). \ \Box$$

Definition 6.1 An iterative method $x_{n+1} = \Phi(x_n)$ $(n = 0, 1, 2, \ldots)$ with the fixed point ξ, has the $(r; q)$-order of convergence if there exists $C_r \in \mathbb{R}^+$ such that for a large enough n it is valid

$$|\xi - x_{n+1}| < C_r |(\xi - x_n)^{(r)}|,$$

where the last exponent (r) is defined by (1.1).

The next theorem we proved in [9].

Theorem 6.2. *Let $f(x)$ be a continuous function on $[a,b]$ and $R_n(f,z,c,q)$, $(z,c\in(a,b))$ be the remainder term (5.2) in the q-Taylor formula. Then there exists $\hat{q}\in(0,1)$ such that for all $q\in(\hat{q},1)$, there can be found $\tau\in(a,b)$ between c and z which satisfies*

$$R_n(f,z,c,q)=\frac{(D_q^n f)(\tau)}{[n]_q!}(z-c)^{(n)}.$$

Now, we are ready to prove the main theorem of this section.

Theorem 6.3. *Suppose that a function $f(x)$ is continuous on a segment $[a,b]$ and that the equation $f(x)=0$ has a unique isolated solution $\xi\in(a,b)$. If the conditions*

$$|(D_q f)(x)|\geq M_1, \qquad |(D_q^2 f)(x)|\leq M_2,$$

are satisfied for some positive constants M_1 and M_2 and all $x\in(a,b)$, then there exists $\hat{q}\in(0,1)$ such that for all $q\in(\hat{q},1)$, the iterations obtained by the q-Newton method satisfy

$$|\xi-x_{k+1}|\leq\frac{M_2}{(1+q)M_1}|(\xi-x_k)^{(2)}|,$$

i.e., the q-Newton method has the $(2;q)$-order of convergence.

Proof. From the formulation of the q-Newton method we have

$$x_{k+1}-\xi=x_k-\xi-\frac{f(x_k)}{(D_q f)(x_k)},$$

hence

$$f(x_k)+(D_q f)(x_k)(\xi-x_k)=(D_q f)(x_k)(\xi-x_{k+1}).$$

By using the q-Taylor formula of order $n=2$ at the point x_k for $f(\xi)$, we have

$$f(\xi)=f(x_k)+(D_q f)(x_k)(\xi-x_k)+R_2(f,\xi,x_k,q).$$

Since $f(\xi)=0$, we get

$$(D_q f)(x_k)(\xi-x_{k+1})=-R_2(f,\xi,x_k,q),$$

i.e.

$$|\xi-x_{k+1}|=\frac{|R_2(f,\xi,x_k,q)|}{|(D_q f)(x_k)|}.$$

According to Theorem 6.2, there exists $\hat{q}\in(0,1)$ such that for all $q\in(\hat{q},1)$ it can be found $\xi\in(a,b)$ such that

$$R_2(f,\xi,x_k,q)=\frac{(D_q^2 f)(\xi)}{[2]_q}(\xi-x_k)^{(2)}.$$

Now,

$$|\xi - x_{k+1}| = \frac{|(D_q^2 f)(\xi)|}{|(D_q f)(x_k)|} \, \frac{|(\xi - x_k)^{(2)}|}{1+q}.$$

Using the conditions that satisfy the function $f(x)$ and its q-derivatives, we get the statement of the theorem. □

7. Examples

Example 7.1 Let us consider the next system of nonlinear equations

$$x_1^2 + 7x_2 - x_3^4 = 2, \quad x_1^2 - 49x_2^2 + x_3^2 = 6, \quad x_1^2 + 7(x_2 - 1) - x_3^2 = -3.$$

If we use the q-method, we get the following Jacobi matrix

$$W_q = \begin{bmatrix} (1+q)x_1 & 7 & -(1+q)(1+q^2)x_3^3 \\ (1+q)x_1 & -49(1+q)x_2 & (1+q)x_3 \\ (1+q)x_1 & 7 & -(1+q)x_3 \end{bmatrix}.$$

Using $q = 0.9$, we find the solutions ($x_1 = \sqrt{5},\ x_2 = 1/7,\ x_3 = \sqrt{2}$), with an accuracy on five decimal digits after $n = 7$ iterations.

$$\vec{x}^{(k)} : \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1.613 \\ 0.705 \\ 2.299 \end{bmatrix}, \begin{bmatrix} 2.199 \\ 0.353 \\ 1.747 \end{bmatrix}, \begin{bmatrix} 2.1794 \\ 0.1937 \\ 1.4633 \end{bmatrix}, \begin{bmatrix} 2.2331 \\ 0.1450 \\ 1.4078 \end{bmatrix} \to \begin{bmatrix} 2.23607 \\ 0.142871 \\ 1.41427 \end{bmatrix}.$$

The next example will show the advantages of the q-Newton-Kantorovich method over the classical one.

Example 7.2 Let us consider the following system of nonlinear equations

$$|x_1^2 - 4| + e^{7x_2 - 36} = 2, \quad \log_{10}\Big(\frac{12x_1^2}{x_2} - 6\Big) + x_1^4 = 10.$$

If we use the q-method for $q = 0.9$, we get the following iterations for the exact solutions $(x_1, x_2) = (\sqrt{3},\ 36/7)$:

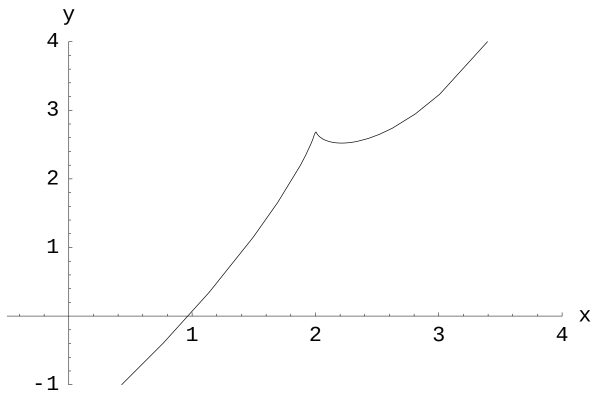
$$\vec{x}^{(k)} : \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1.78067 \\ 5.29844 \end{bmatrix}, \begin{bmatrix} 1.73405 \\ 5.20213 \end{bmatrix}, \begin{bmatrix} 1.73208 \\ 5.15274 \end{bmatrix}, \begin{bmatrix} 1.73205 \\ 5.14302 \end{bmatrix} \to \begin{bmatrix} 1.73205 \\ 5.14286 \end{bmatrix}.$$

The classical Newton-Kantorovich method with initial values $x_1 = 2,\ x_2 = 5$ can not be used in this case because the partial derivative of the first function with respect to the first variable does not exist.

Example 7.3. Let us consider the equation

$$f(x) \equiv \sqrt[3]{x^3 - 9x^2 + 24x - 20} + e^{x/2} = 0.$$

The function $f(x)$ is not differentiable at the point $x = 2$. However, it is not problem for our q-Newton method. Really, starting with the initial value $x_0 = 2$, we find the solution with six exact digits after five iterations.



iteration	value
0	2.000000
1	1.592916
2	1.043515
3	0.970143
4	0.969425
5	0.969426

Figure 7.1. The function is not differentiable at the initial point, but this does not influence convergence

Example 7.4. The advantages of the q-Newton method over the classical Newton method can be seen in the case of the equations with multiple zeros. So, for solving the equation

$$f(x) \equiv x^6 - 5x^5 + 8.25x^4 - 10x^3 + 13.5x^2 - 5x + 6.25 = 0, \quad x_0 = 2,$$

the classical Newton method has to be replaced by the special Newton method for multiple zeros ($\xi = 2.5$ is a double root). But, the q-Newton method has large enough intervals of convergence, which can be seen Figure 7.2.

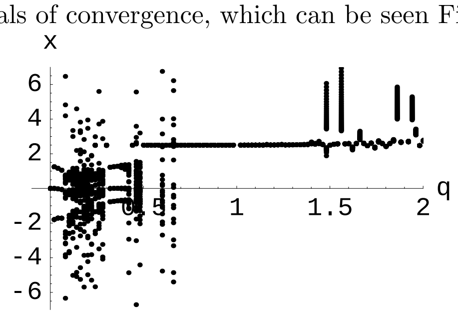


Figure 7.2. Solving the equation with multiple roots. The values of the iterations from $n = 100$ to $n = 140$.

Remark. All examples were evaluated by the software package MATHEMATICA.

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