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HIGH-ORDER METHODS FOR SEMILINEAR SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS[1]

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Abstract. We considered finite difference methods of higher order for semilinear singularly perturbed boundary value problems, consisted of constructing difference schemes on nonuniform meshes. Construction of schemes is presented and convergence uniform in perturbation parameter for one method is shown on Bakhvalov's type of mesh. Numerical experiments demonstrated influence of different meshes on developed schemes.

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1. Introduction

Our aim in this paper is to construct the difference schemes that have higher order of convergence uniform in small parameter ε combined with appropriate nonuniform mesh. In the paper [1] the idea was developed by Clavero, Gracia and Lisbona for the linear problem on the Shishkin mesh. By including more coefficients in the schemes we get one degree of freedom for their determination when we obtain the expected order of convergence that will be used when trying to provide stability and consistency of the method. We generalized the method for the semilinear problem of the form:

$$\begin{aligned} T_\varepsilon u = -\varepsilon u''(x) + a(x)u'(x) + b(x,u(x)) = 0, \qquad x \in (0,1), \\ Ru = (u(0), u(1)) = (0,0), \end{aligned} \tag{1}$$

where $0 < \varepsilon << 1$, a and b are the functions satisfying the following conditions

$$\begin{aligned} a(x) \geq \alpha > 0, \qquad x \in (0,1) \\ 0 \leq b_u(x,u) \leq G(x), \quad (x,u) \in (0,1) \times \mathbb{R} \end{aligned} \tag{2}$$

$$a \in C^k([0,1]), \qquad b \in C^k([0,1] \times \mathbb{R}), \qquad k \in \mathbb{N}. \tag{3}$$

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The condition (2) is the standard stability condition, which implies that both (1) and the reduced problem, $a(x)u'(x) + b(x, u(x)) = 0$, have unique smooth solutions u_ε and u_0, respectively, and the conditions (2, 3) provides us with useful bounds of the solution u_ε and its derivatives [16]:

$$\left|u_\varepsilon^{(i)}(x)\right| \leq M(1 + \varepsilon^{-i} \exp(-\alpha \frac{1-x}{\varepsilon})), \qquad x \in (0,1), \tag{4}$$

for all $i \in \{0, 1, 2, ..., k+1\}$.

In the following sections we develop the schemes of order two and three combined with Bakhvalov's type of meshes and for these methods prove convergence, uniform in small parameter, with some restrictions that will be emphasized. Numerical results in the last section confirmed the theoretical findings. The accuracy obtained on Bakhvalov's type of meshes is better than on Shishkin's mesh.

2. Difference schemes

For $x \in (0,1)$, we define the operator

$$T_\varepsilon^h w(x) = \sum_{k=1}^{n} r_k \; w(x + d_k h) + \sum_{k=n_1}^{N_1} q_k \; b(x + d'_k h, w(x + d'_k h)), \tag{5}$$

where d_k, $k = 1, 2, ..., N$ and d'_k, $k = n_1, ..., N_1$, $1 \leq n_1 \leq N_1$, are real numbers such that $x + d_k h$, $x + d'_k h \in (0,1)$. The coefficients d_k, and d'_k differ from each other. The unknown coefficients r_k and q_k are determined so that $T_\varepsilon^h w(x) = 0$ for all $w \in P_s[x]$, (the space of polynomials of degree not greater than s), including the normalization condition $\sum_{k=n_1}^{N_1} q_k = 1$.

Let $s \geq 2$. Depending on the base of the polynomial space $P_s[x]$, we get the system of equations for the determination of the parameters r_k and q_k. If we choose the base $\{1, x, x^2, ..., x^s\}$ we get the following system of linear equations:

$$\sum_{k=1}^{N} r_k = 0$$

$$\sum_{k=1}^{N} r_k (x + d_k h) - \sum_{k=n_1}^{N_1} q_k a(x + d'_k h) = 0$$

$$\sum_{k=1}^{N} r_k (x + d_k h)^2 + \sum_{k=n_1}^{N_1} q_k (2\varepsilon - 2(x + d'_k h) a(x + d'_k h)) = 0 \tag{6}$$

$$....$$

$$\sum_{k=1}^{N} r_k (x + d_k h)^s + \sum_{k=n_1}^{N_1} q_k (\varepsilon s(s-1)(x + d'_k h)^{s-2} - s(x + d'_k h)^{s-1} a(x + d'_k h)) = 0$$

$$\sum_{k=n_1}^{N_1} q_k = 1.$$

2.1. Scheme 2

For $s = 2$, $N = 3$, $n_1 = 1$, $N_1 = 2$ and $d_1 = d_1' = -\frac{h_i}{h}$, $d_2 = d_2' = 0$, $d_3 = \frac{h_{i+1}}{h}$, the system (6), expressed in the matrix form, using $x = x_i$ and the notation $a(x_i) = a_i$, $r_j = r_j(i)$ and $q_j = q_j(i)$, as follows:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ -h_i & 0 & h_{i+1} & -a_{i-1} & -a_i \\ h_i^2 & 0 & h_{i+1}^2 & 2h_i a_{i-1} & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} r_1(i) \\ r_2(i) \\ r_3(i) \\ q_1(i) \\ q_2(i) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2\varepsilon \\ 1 \end{bmatrix}. \tag{7}$$

The rank of the coefficient matrix is 4, so the system has one degree of freedom. The value $q_1(i)$ will be chosen freely.

The solution of the system is

$$r_1(i) = \frac{-2\varepsilon - q_1(i)(2h_i + h_{i+1})a_{i-1} - q_2(i)h_{i+1}a_i}{h_i(h_i + h_{i+1})},$$

$$r_3(i) = \frac{-2\varepsilon + h_i a_i - q_1(i)h_i(a_{i-1} + a_i)}{h_{i+1}(h_i + h_{i+1})},$$

$$r_2(i) = -r_1(i) - r_2(i),$$

$$q_2(i) = 1 - q_1(i).$$

Because of the boundary conditions, it holds that $w_0 = w_n = 0$, so we will observe the discrete problem for $w^h := (w_1, w_2, ..., w_{n-1})^T \in \mathbb{R}^{n-1}$, using the nodes $x_i,\ i = 0, 1, ..., n$:

$$\begin{aligned} F_1 w^h &:= r_2(1)w_1 + r_3(1)w_2 + q_1(1)b(x_0, w_0) + q_2(1)b(x_1, w_1) \\ F_i w^h &:= r_1(i)w_{i-1} + r_2(i)w_i + r_3(i)w_{i+1} + q_1(i)b(x_{i-1}, w_{i-1}) \\ &\quad + q_2(i)b(x_i, w_i) \qquad\qquad i = 2, 3, ..., n-2 \\ F_{n-1} w^h &:= r_1(n-1)w_{n-2} + r_2(n-1)w_{n-1} \\ &\quad + q_1(n-1)b(x_{i-1}, w_{i-1}) + q_2(n-1)b(x_{n-1}, w_{n-1}). \end{aligned} \tag{8}$$

The Jacobian matrix of the mapping $F = (F_1, F_2, ..., F_{n-1})$ is a tridiagonal matrix of the form $F'(w^h) = tridiag\{A_1^i, A_2^i, A_3^i\}$, where for $i = 1, 2, ..., n-1$,

$$A_1^i = \frac{-2\varepsilon - q_1(i)(2h_i + h_{i+1})a_{i-1} - q_2(i)h_{i+1}a_i}{h_i(h_i + h_{i+1})} + q_1(i)b_u(x_{i-1}, w_{i-1}),$$

$$A_2^i = -(r_1(i) + r_2(i)) + q_2(i)b_u(x_i, w_i),$$

$$A_3^i = \frac{-2\varepsilon + h_i a_i - q_1(i)h_i(a_{i-1} + a_i)}{h_{i+1}(h_i + h_{i+1})}.$$

In order to show the stability of the method we will determine the coefficient $q_1(i)$ so that the matrix $F'(w^h)$ becomes an $M-$matrix. We will prove the following theorem:

Theorem 2.1. *Let $n_0 \in \mathbb{N}$, so that*

$$\frac{M\,\|a'\|_\infty}{n_0} < \alpha, \qquad \frac{3M(\|a'\|_\infty + \|G\|_\infty)}{n_0} < \alpha \tag{9}$$

and the mesh $I_h = \{x_i; i = 0, 1, ..., n\}$ has the property

$$h_i \leq \frac{M}{n_0}, \quad i = 1, 2, ..., n. \tag{10}$$

If we choose $q_1(i)$ for all $i = 1, 2, ...n-1$, so that

$$0 \leq q_1(i) \leq 1, \tag{11}$$

and for i for which stands that $-2\varepsilon + h_i a_i \geq 0$,

$$q_1(i) = \frac{a_i}{a_i + a_{i-1}}, \tag{12}$$

then for all $n \geq n_0$ the matrix $F'(w^h)$ is an $M-$matrix.

Proof. Using (11), it follows that $0 \leq q_2(i) \leq 1$, $i = 1, 2, ..., n$. From the conditions (9), (2) and (10) we have

$$\begin{aligned} &-(2h_i + h_{i+1})a_{i-1} + h_i(h_i + h_{i+1})b_u(x_{i-1}, w_{i-1}) \\ \leq\ & h_i \frac{3M}{n_0}(\|a'\|_\infty + \|G\|_\infty) - h_i a_{i-1} < h_i(\alpha - a_{i-1}) < 0, \end{aligned}$$

so the coefficients $A_1^i < 0$, for all $i = 1, 2, ..., n-1$. Let $i \in \{1, 2, ..., n-1\}$, then for $-2\varepsilon + h_i a_i < 0$, we get $A_3^i < 0$, and if the mentioned condition is not satisfied, because of (12) we have

$$A_3^i = \frac{-2\varepsilon}{h_{i+1}(h_i + h_{i+1})} < 0.$$

Then $r_1(i), r_2(i) < 0, \quad i = 1, 2, ..., n-1$, and because of (2) it is true that $A_2^i > 0$, for all $i = 1, 2, ..., n-1$. Hence, $F'(w^h)$ is an $L-$matrix.

If we introduce the vector $v = (x_1, x_2, ..., x_{n-1})^T$, where $x_i \in I_h$, $i = 1, 2, ..., n-1$, we know that $v > 0$ and we will prove that $(F'(w^h)v) > 0$. For $i = 2, 3, ..., n-2$

$$A_1^i x_{i-1} + A_2^i x_i + A_3^i x_{i+1} = -h_i r_1(i) + h_{i+1} r_3(i) + R,$$

with $R = x_{i-1} q_1(i) b_u(x_{i-1}, w_{i-1}) + x_i q_2(i) b_u(x_i, w_i) \geq 0$.

Because of (7) and (9) we have

$$\begin{aligned} -h_i r_1(i) + h_{i+1} r_3(i) &= q_1(i) a_{i-1} + q_2(i) a_i \\ &= q_1(i)(a_i - h_i a'(\theta)) + q_2(i) a_i \\ &\geq \alpha - \frac{M}{n_0} \|a'\|_\infty = c > 0. \end{aligned} \tag{13}$$

So $(F'(w^h)v)_i > 0, \quad i = 2, 3, ..., n-2$. Since $x_0 = 0$ and $x_n = 1$, it follows that

$$\begin{aligned} A_2^1 x_1 + A_3^1 x_2 &= A_1^1 x_0 + A_2^1 x_1 + A_3^1 x_2, \\ A_1^{n-1} x_{n-2} + A_2^{n-1} x_{n-1} &= A_1^{n-1} x_{n-2} + A_2^{n-1} x_{n-1} \pm r_3(i) x_n, \end{aligned}$$

and using the fact that $r_3(i) < 0$ it leads to the conclusion

$$(F'(w^h)v)_1, (F'(w^h)v)_{n-1} > 0. \qquad \square$$

Nonzero components of the truncating error vector are

$$\begin{aligned} \tau_i[u_\varepsilon] \;=\; & \frac{1}{6}\,(2\varepsilon(h_i - h_{i+1}) + h_i h_{i+1} a_i \\ & . - q_1(i) h_i (6\varepsilon + h_{i+1} a_i + a_{i-1}(h_i + h_{i+1})))\;\; u_\varepsilon^{'''}(x_i) \\ & - \frac{h_i^3 (2\varepsilon - (1 - q_1(i)) h_{i+1} a_i + q_1(i) a_{i-1}(2h_i + h_{i+1}))}{24(h_i + h_{i+1})}\;\; u_\varepsilon^{IV}(\theta_{1,0}) \\ & - \frac{h_{i+1}^3 (2\varepsilon + q_1(i) h_i a_{i-1} - (1 - q_1(i)) h_i a_i)}{24(h_i + h_{i+1})}\;\; u_\varepsilon^{IV}(\theta_{3,0}) \\ & + \frac{1}{6} h_i^3 q_1(i) a_{i-1}\;\; u_\varepsilon^{IV}(\theta_{1,1}) + \frac{1}{2} h_i^2 q_1(i) \varepsilon\;\; u_\varepsilon^{IV}(\theta_{1,2}), \\ i \;=\; & 1, 2, ..., n-1, \end{aligned}$$

where $\theta_{1,0},\ \theta_{1,1},\ \theta_{1,2} \in (x_{i-1}, x_i)$ i $\theta_{3,0} \in (x_i, x_{i+1})$.

2.2. Scheme 3

Let $s = 3$, $N = 3$, $n_1 = 1$, $N_1 = 3$ and $d_1 = d_1' = -\frac{h_i}{h}, \quad d_2 = d_3' = 0, \quad d_3 = \frac{h_{i+1}}{h}, \quad d_2' = -\frac{h_i}{2h}$.

We introduce $\{1, x(x+h_i), x(x-h_{i+1}), x^3\}$ the base of the space $P_3[x]$ for the determination of the coefficients $r_j, q_j,\ j = 1, 2, 3$. If $x = x_i$, we use the notation

$x_i - \frac{h_i}{2} = x_{i-1/2}, \quad a(x_{i-1/2}) = a_{i-1/2}, \quad w(x_{i-1/2}) = w_{i-1/2}$. For $w(x) = 1$ we get the equation

$$T_\varepsilon^h w(x) = r_1(i) + r_2(i) + r_3(i) = 0, \tag{14}$$

for $w(x) = x(x + h_i)$

$$T_\varepsilon^h w(x) = r_3(i)h_{i+1}(h_i + h_{i+1}) + q_1(i)h_i a_{i-1} - q_3(i)h_i a_i + 2\varepsilon = 0.$$

So

$$r_3(i) = \frac{-2\varepsilon - q_1(i)h_i a_{i-1} + q_3(i)h_i a_i}{h_{i+1}(h_i + h_{i+1})}. \tag{15}$$

For $w(x) = x(x - h_{i+1})$

$$T_\varepsilon^h w(x) = r_1(i)h_i(h_i + h_{i+1}) + q_1(i)(2h_i + h_{i+1})a_{i-1}(h_i + h_{i+1})$$
$$+q_2(i)(h_i + h_{i+1})a_{i-1/2} + q_3(i)h_{i+1}a_i + 2\varepsilon = 0,$$

it follows that

$$r_1(i) = \frac{-2\varepsilon - q_1(i)(2h_i + h_{i+1})a_{i-1} - q_2(i)(h_i + h_{i+1})a_{i-1/2} - q_3(i)h_{i+1}a_i}{h_i(h_i + h_{i+1})}. \tag{16}$$

For $w(x) = x^3$, we get the additional condition for the determination $q_j = q_j(i),\ j = 1, 2, 3$:

$$\begin{aligned} T_\varepsilon^h w(x) &= -h_i^3 r_1(i) + h_{i+1}^3 r_3(i) - 3h_i(q_1(i)(2\varepsilon + h_i a_{i-1}) \\ &+q_2(i)(\varepsilon + 3\frac{h_i^2}{4} a_{i-1/2})) = 0. \end{aligned} \tag{17}$$

Applying the normalization condition we have

$$q_1(i) + q_2(i) + q_3(i) = 1, \tag{18}$$

The system for the determination of coefficients has five linearly independent equations, so we can choose one unknown freely, let it be $q_2(i)$.

For $w^h := (w_1, w_2, ..., w_{n-1})^T \in \mathbb{R}^{n-1}$ $(w_0 = w_n = 0)$ using the nodes

$x_i,\ i = 0, 1, ..., n$, instead of a discrete problem of the form

$$F_1 w^h := r_2(1)w_1 + r_3(1)w_2 + q_1(1)b(0,0) + q_2(1)b(x_{1/2}, w_{1/2})$$
$$+q_3(1)b(x_2, w_2)$$
$$F_i w^h := r_1(i)w_{i-1} + r_2(i)w_i + r_3(i)w_{i+1} + q_1(i)b(x_{i-1}, w_{i-1})$$
$$+q_2(i)b(x_{i-1/2}, w_{i-1/2}) + q_3(i)b(x_i, w_i)$$
$$i = 2, 3, ..., n-2$$
$$F_{n-1} w^h := r_1(n-1)w_{n-2} + r_2(n-1)w_{n-1}$$
$$+q_1(n-1)b(x_{n-1}, w_{n-1}) + q_2(n-1)b(x_{n-1/2}, w_{n-1/2})$$
$$+q_3(n-1)b(1,0),$$

we will form another one, when we use the Taylor expansion

$$(19)\quad w_{i-1/2} - \frac{h_i + 2h_{i+1}}{4(h_i + h_{i+1})} w_{i-1} - \frac{h_i + 2h_{i+1}}{4h_{i+1}} w_i + \frac{h_i^2}{4h_{i+1}(h_i + h_{i+1})} w_{i+1} = \widetilde{R}_i(w),$$

where

$$(20)\quad \begin{aligned} \widetilde{R}_i(w) &= \frac{1}{3!}\frac{1}{8} h_i^2 (h_i + 2h_{i+1}) w'''(x_i) + \frac{h_i^4 (h_i + 2h_{i+1})}{96(h_i + h_{i+1})} w^{IV}(\alpha_1^i) \\ &- \frac{h_i^4}{384\ h_{i+1}} w^{IV}(\alpha_2^i) + \frac{h_i^2 {h_{i+1}}^3}{96(h_i + h_{i+1})} w^{IV}(\alpha_3^i), \end{aligned}$$

with $\alpha_1^i \in (x_{i-1}, x_i)$, $\alpha_2^i \in (x_{i-1/2}, x_i)$, $\alpha_3^i \in (x_i, x_{i+1})$. Let

$$\widetilde{w}_{i-1/2} = \frac{h_i + 2h_{i+1}}{4(h_i + h_{i+1})} w_{i-1} + \frac{h_i + 2h_{i+1}}{4h_{i+1}} w_i - \frac{h_i^2}{4h_{i+1}(h_i + h_{i+1})} w_{i+1},$$

then

$$(21)\quad \begin{aligned} b(x_{i-1/2}, w_{i-1/2}) &= b(x_{i-1/2}, \widetilde{w}_{i-1/2}) + \widetilde{R}_i(w) b_u(x_{i-1/2}, \widetilde{w}_{i-1/2}) \\ &+ \frac{\widetilde{R}_i^2(w)}{2} b_{uu}(x_{i-1/2}, \theta_i'), \end{aligned}$$

for $\theta_i' \in (\widetilde{w}_{i-1/2}, w_{i-1/2})$.

Now, the discrete problem we are going to analyze has the following form

$$\widetilde{F}_1 w^h := r_2(1)w_1 + r_3(1)w_2 + q_1(1)b(0,0) + q_2(1)b(x_{1/2},\widetilde{w}_{1/2}) \\ +q_3(1)b(x_2,w_2)$$

$$\widetilde{F}_i w^h := r_1(i)w_{i-1} + r_2(i)w_i + r_3(i)w_{i+1} + q_1(i)b(x_{i-1},w_{i-1}) \\ +q_2(i)b(x_{i-1/2},\widetilde{w}_{i-1/2}) + q_3(i)b(x_i,w_i) \tag{22}$$

$$i = 2,3,...,n-2$$

$$\widetilde{F}_{n-1} w^h := r_1(n-1)w_{n-2} + r_2(n-1)w_{n-1} + q_1(n-1)b(x_{n-1},w_{n-1}) \\ +q_2(n-1)b(x_{n-1/2},\widetilde{w}_{n-1/2}) + q_3(n-1)b(1,0),$$

where $r_j(i),\ q_j(i),\ j = 1,2,3,\ i = 1,2,3,...,n-1$ are given by (14, 15, 16, 18), and because of our approximation (19), instead of equation (17), we get

$$-h_i^3 r_1(i) + h_{i+1}^3 r_3(i) - 3h_i(q_1(i)(2\varepsilon + h_i a_{i-1}) + q_2(i)(\varepsilon + 3\tfrac{h_i^2}{4}a_{i-1/2})) \\ -q_2(i)\frac{1}{8}h_i^2(h_i + 2h_{i+1})b_u(x_{i-1/2},\widetilde{w}_{i-1/2}) = 0.$$

The Jacobian matrix of the mapping $\widetilde{F} = (\widetilde{F}_1, \widetilde{F}_2, ..., \widetilde{F}_{n-1})$ is a tridiagonal matrix of the form $\widetilde{F}'(w^h) = tridiag\{A_1^i, A_2^i, A_3^i\}$, where for $i = 1,2,...,n-1$,

$$A_1^i = r_1(i) + q_1(i)b_u(x_{i-1},w_{i-1}) + q_2(i)\frac{h_i + 2h_{i+1}}{4(h_i + h_{i+1})}b_u(x_{i-1/2},\widetilde{w}_{i-1/2}),$$
$$A_2^i = -(r_1(i) + r_3(i)) + q_3(i)b_u(x_i,w_i) + q_2(i)\frac{h_i + 2h_{i+1}}{4h_{i+1}}b_u(x_{i-1/2},\widetilde{w}_{i-1/2}),$$
$$A_3^i = r_3(i) - q_2(i)\frac{h_i^2}{4h_{i+1}(h_i + h_{i+1})}b_u(x_{i-1/2},\widetilde{w}_{i-1/2}).$$

We shall choose the coefficient $q_2(i)$ in the appropriate way so that the matrix $\widetilde{F}'(w^h)$ becomes an $L-$matrix. For that we will use the following lemma:

Lemma 2.1 *Let $n_0 \in \mathbb{N}$, so that for all $n \geq n_0$ it stands that*

$$\frac{3M\,\|a'\|_\infty}{n_0} < \alpha \tag{23}$$

and the mesh $I_h$is chosen so that $h_{i+1} \leq h_i$ for all $i \in \{0,1,...n-1\}$. In the case when $2h_i\,\|a\|_\infty < 3\varepsilon$ we define

$$q_1(i) = \frac{2h_i^3 + 2h_{i+1}^3 - 3q_2(i)h_i^2(h_i + h_{i+1})}{12h_i^2(h_i + h_{i+1})} \tag{24}$$

and

$$q_2(i) = \frac{2}{3h_i^2}\Big(6\varepsilon(h_i^2 - h_ih_{i+1} - h_{i+1}^2) - a_{i-1}(h_i^3 + h_{i+1}^3) + 5a_ih_i^2h_{i+1}$$

$$+a_ih_{i+1}^2(h_i - h_{i+1})\Big) \,/\, \Big(6\varepsilon - a_{i-1}(h_i + h_{i+1}) - a_{i-1/2}h_i$$

$$+3a_ih_{i+1} + \frac{3}{2}h_i^2b_u(x_{i-1/2}, \widetilde{w}_{i-1/2})\Big).$$

Then

$$-\frac{2}{3} < q_2(i) < 1, \quad 1 - q_2(i) - 2q_1(i) = \frac{4}{3} - \delta > 0,$$

where $\delta > 0$ is the constant independent of ε.

Proof. Using conditions (2) and $2h_i\left\|a\right\|_\infty < 3\varepsilon$ it follows that

$$3h_i^2(6\varepsilon - a_{i-1}(h_i + h_{i+1}) - a_{i-1/2}h_i + 3a_ih_{i+1} + \tfrac{3}{2}h_i^2b_u(x_{i-1/2}, \widetilde{w}_{i-1/2})) > 0, \tag{25}$$

We can prove that $q_2(i) < 1$ from the fact that $q_2(i) < 1$ if and only if

$$2(6\varepsilon(h_i^2 - h_ih_{i+1} - h_{i+1}^2) - a_{i-1}(h_i^3 + h_{i+1}^3) + 5a_ih_i^2h_{i+1}$$

$$+a_ih_{i+1}^2(h_i - h_{i+1})) - 3h_i^2(6\varepsilon - a_{i-1}(h_i + h_{i+1}) - a_{i-1/2}h_i$$

$$+3a_ih_{i+1} + \tfrac{3}{2}h_i^2b_u(x_{i-1/2}, \widetilde{w}_{i-1/2})) < 0.$$

Because of (25) we can prove that $q_2(i) > -\frac{2}{3}$, using the fact that $q_2(i) > -\frac{2}{3}$, if and only if

$$6\varepsilon(h_i^2 - h_ih_{i+1} - h_{i+1}^2) - a_{i-1}(h_i^3 + h_{i+1}^3) + 5a_ih_i^2h_{i+1} + a_ih_{i+1}^2(h_i - h_{i+1})$$

$$+h_i^2(6\varepsilon - a_{i-1}(h_i + h_{i+1}) - a_{i-1/2}h_i + 3a_ih_{i+1} + \tfrac{3}{2}h_i^2b_u(x_{i-1/2}, \widetilde{w}_{i-1/2}))$$

$$> 0.$$

The form (24) leads to

$$1 - q_2(i) - 2q_1(i) = -\frac{1}{2}q_2(i) + \frac{2h_i^2 + h_ih_{i+1} - h_{i+1}^2}{3h_i^2},$$

and using the bounds for the coefficients $q_1(i)$ and $q_2(i)$ we have

$$-\frac{1}{2}q_2(i) + \frac{2h_i^2 + h_ih_{i+1} - h_{i+1}^2}{3h_i^2} < \frac{1}{3} + \frac{2h_i^2 + h_ih_{i+1}}{3h_i^2} \le \frac{4}{3}$$

and

$$-\frac{1}{2}q_2(i) + \frac{2h_i^2 + h_ih_{i+1} - h_{i+1}^2}{3h_i^2} > -\frac{1}{2} + \frac{2}{3} + \frac{h_ih_{i+1} - h_{i+1}^2}{3h_i^2} \ge 0.$$

So, there exists a constant $\delta > 0$ such that the following stands:

$$1 - q_2(i) - 2q_1(i) = \frac{4}{3} - \delta > 0.$$

□

Theorem 2.2. *Let $i \in \{1, 2, ...n-1\}$. If $2h_i \|a\|_\infty < 3\varepsilon$, $q_1(i)$ and $q_2(i)$ be defined as in the previous lemma, and for i for which it holds that $2h_i \|a\|_\infty \geq 3\varepsilon$, the coefficients are given in the form*

$$q_1(i) = \frac{a_i - q_2(i)(a_i + \frac{h_i}{4} b_u(x_{i-1/2}, \widetilde{w}_{i-1/2}))}{a_i + a_{i-1}} \tag{26}$$

and

$$\begin{aligned} q_2(i) \quad = \quad & \left(2\varepsilon(\frac{h_{i+1}}{h_i} - 1)(a_i + a_{i-1}) + a_i(6\varepsilon + h_i a_{i-1})\right) / \\ & \left((-3\varepsilon + \frac{h_i a_{i-1/2}}{4} - \frac{h_i^2 b_u(x_{1/2}, \widetilde{w}_{i-1/2})}{8})(a_i + a_{i-1}) \right. \\ & \left. +(a_i + \frac{h_i b_u(x_{1/2}, \widetilde{w}_{i-1/2})}{4})(6\varepsilon + h_i a_{i-1})\right). \end{aligned}$$

Let $n_0 \in \mathbb{N}$ be the number for which the following conditions are satisfied

$$\frac{\max\{\gamma, 1\} M(4\|a'\|_\infty + 3\|G\|_\infty)}{n_0} < \min\{1, \delta\}\alpha, \tag{27}$$

$$\frac{2\gamma M(\|a'\|_\infty + \|G\|_\infty)\|a\|_\infty}{n_0} < \alpha^2, \tag{28}$$

and

$$\frac{\gamma M^2(\|a'\|_\infty^2 + 4\|G\|_\infty^2)}{n_0^2} < \alpha^2, \tag{29}$$

let δ be determined in the previous lemma, and $\gamma = \max\{|q_j(i)|; j = 1, 2, 3\}$. Then for $n \geq n_0$ it follows that $F'(w^h)$ is an $L-$matrix.

Proof. Let $i \in \{1, 2, ...n-1\}$ and $2h_i \|a\|_\infty < 3\varepsilon$, from the previous lemma and (27) it follows that

$$\begin{aligned} & -2\varepsilon - q_1(i)h_i a_{i-1} + q_3(i)h_i a_i - q_2(i)\tfrac{h_i^2}{4} b_u(x_{i-1/2}, \widetilde{w}_{i-1/2}) \\ & \leq -2\varepsilon + h_i(\tfrac{4}{3} - \delta)\|a\|_\infty + h_i \tfrac{1}{3}\min\{1, \delta\}\alpha \\ & \leq -2\varepsilon + h_i \tfrac{4}{3}\|a\|_\infty < 0, \end{aligned}$$

that is $A_3^i < 0$. We have

$$\begin{aligned}
& -2\varepsilon - q_1(i)(2h_i + h_{i+1})a_{i-1} - q_2(i)(h_i + h_{i+1})a_{i-1/2} - q_3(i)h_{i+1}a_i \\
& +q_1(i)h_i(h_i + h_{i+1})b_u(x_{i-1}, w_{i-1}) \\
& +q_2(i)\frac{h_i(h_i + 2h_{i+1})}{4}b_u(x_{i-1/2}, \widetilde{w}_{i-1/2}) \\
= \quad & -2\varepsilon - h_{i+1}a_i(1 - 2q_1(i) - q_2(i)) - 2q_1(i)h_ia_i - q_2(i)h_ia_i \pm h_ia_i \\
& +q_1(i)a'(\eta_1)h_i(2h_i + h_{i+1}) + \frac{h_i}{2}a'(\eta_2)q_2(i)(h_i + h_{i+1}) \\
& +q_1(i)h_i(h_i + h_{i+1})b_u(x_{i-1}, w_{i-1}) \\
& +q_2(i)\frac{h_i(h_i + 2h_{i+1})}{4}b_u(x_{i-1/2}, \widetilde{w}_{i-1/2}) \\
< \quad & -2\varepsilon + \frac{4}{3}h_ia_i - h_ia_i + h_i\left\|a'\right\|_\infty \gamma\frac{4M}{n_0} + h_i\left\|G\right\|_\infty \gamma\frac{3M}{n_0} \\
< \quad & -2\varepsilon + \frac{4}{3}h_ia_i - h_ia_i + h_i\alpha < 0,
\end{aligned}$$

so $A_1^i < 0$. It only remains to show that $A_2^i > 0$, which follows from

$$\begin{aligned}
& q_3(i)a_i(h_i - h_{i+1}) - q_2(i)h_{i+1}a_{i-1/2} - q_1(i)(h_i + h_{i+1})a_{i-1} \\
& -q_3(i)h_ih_{i+1}b_u(x_i, w_i) - q_2(i)\tfrac{h_i(h_i+2h_{i+1})}{4}b_u(x_{i-1/2}, \widetilde{w}_{i-1/2}) \\
& < 2\varepsilon.
\end{aligned}$$

If $2h_i\left\|a\right\|_\infty \geq 3\varepsilon$, then because of (26) we have

$$A_3^i = \frac{-2\varepsilon}{h_{i+1}(h_i + h_{i+1})} < 0.$$

In this case $q_1(i)$ is of the form (26), so using the condition (28) it follows that

$$\begin{aligned}
& -2\varepsilon - q_1(i)(2h_i + h_{i+1})a_{i-1} - q_2(i)(h_i + h_{i+1})a_{i-1/2} - q_3(i)h_{i+1}a_i \\
& +q_1(i)h_i(h_i + h_{i+1})b_u(x_{i-1}, w_{i-1}) \\
& +q_2(i)\frac{h_i(h_i + 2h_{i+1})}{4}b_u(x_{i-1/2}, \widetilde{w}_{i-1/2}) \\
\leq \quad & -2\varepsilon - \frac{2(h_i + h_{i+1})a_{i-1}a_i + h_{i+1}a_i^2}{a_i + a_{i-1}} + \frac{q_2(i)}{a_i + a_{i-1}}(a_i(a_{i-1} - a_{i-1/2}) \\
\leq \quad & -2\varepsilon + \frac{1}{a_i + a_{i-1}}\big(-2(h_i + h_{i+1})\alpha^2 - h_{i+1}\alpha^2 \\
& +\gamma h_i(h_i + h_{i+1})\left\|a\right\|_\infty\left\|a'\right\|_\infty + \gamma(3h_i^2 + 4h_ih_{i+1})\left\|a\right\|_\infty\left\|G\right\|_\infty\big)
\end{aligned}$$

$$\begin{aligned} &= -2\varepsilon + \frac{1}{a_i + a_{i-1}} \Big(h_i(-2\alpha^2 + \gamma\frac{M}{n_0}\|a\|_\infty \|a'\|_\infty) \\ &\quad +3\gamma\frac{M}{n_0}\|a\|_\infty \|G\|_\infty + h_{i+1}(-3\alpha^2 + \gamma\frac{M}{n_0}\|a\|_\infty \|a'\|_\infty \\ &\quad +4\gamma\frac{M}{n_0}\|a\|_\infty \|G\|_\infty) \Big) \\ &< 0 \end{aligned}$$

and $A_1^i < 0$. It can be shown that $A_2^i > 0$, using the fact that

$$-q_3(i)a_i(h_i - h_{i+1}) + q_2(i)a_{i-1/2}h_{i+1} + q_1(i)a_{i-1}(h_i + h_{i+1})$$

$$+q_3(i)h_ih_{i+1}b_u(x_i, w_i) + q_2(i)\tfrac{h_i(h_i+2h_{i+1})}{4}b_u(x_{i-1/2}, \widetilde{w}_{i-1/2}) \geq 0.$$

considering that $q_1(i)$ is of the form (26). Hence, $F'(w^h)$ is an $L-$matrix. □

For the coefficients $q_j(i), j = 1, 2, 3$ we do not have the nonnegativity property so we shall prove that $\widetilde{F}'(w^h)$ is an $M-$matrix only for the case when $b(x, u)$ is the linear function in u, that is for

$$b(x, u) = \widetilde{b}(x)u - f(x), \tag{30}$$

with $\widetilde{b}$ and f are functions smooth enough. From (2) it follows that $\widetilde{b}(x) \geq 0$, for $x \in (0, 1)$.

Theorem 2.3. *Let all conditions from the previous theorem using the function $b(x, u)$ be of the form (30), for all $i \in \{1, \frac{3}{2}, 2, ..., n-1\}$*

$$\widetilde{b}(x_i)\frac{3}{2}\frac{M}{n_0}\gamma \left\|\widetilde{b}'\right\|_\infty \leq \widetilde{b}(x_i)^2, \tag{31}$$

then for $n \geq n_0$ the matrix $F'(w^h)$ is an $M-$matrix.

Proof. Using the vector $v = (x_1, x_2, ..., x_{n-1})^T$, with $x_i \in I_h$, $i = 1, 2, ..., n-1$, we know that $v > 0$ and we can show that $(F'(w^h)v) > 0$, so the theorem holds.□

The truncating error is

$$\begin{aligned} \tau_i[u_\varepsilon] = \frac{1}{48}&((4q_1(i)a_{i-1} - q_2(i)a_{i-1/2})h_i^3 \\ &-2(q_1(i)a_{i-1} - q_3(i)a_i)(h_{i+1}^2h_i - h_{i+1}h_i^2) \\ &+4\varepsilon h_{i+1}(h_i - h_{i+1}) + 2\varepsilon h_i^2(3q_2(i) + 12q_1(i) - 2))\ u_\varepsilon^{IV}(x_i) \\ &-b_u(x_{i-1/2}, \widetilde{u}_\varepsilon(x_{i-1/2}))\left(\frac{h_i^4(h_i + 2h_{i+1})}{96(h_i + h_{i+1})}u_\varepsilon^{IV}(\alpha_1^i)\right. \end{aligned}$$

$$
\begin{aligned}
&-\frac{h_i^4}{384\ h_{i+1}}u_\varepsilon^{IV}(\alpha_2^i)+\frac{h_i^2 {h_{i+1}}^3}{96(h_i+h_{i+1})}u_\varepsilon^{IV}(\alpha_3^i)\Big) \\
&-\frac{\widetilde{R}_i^2(u_\varepsilon)}{2}b_{uu}(x_{i-1/2},\theta_i')+\frac{h_i^4}{120(h_i+h_{i+1})}(2\varepsilon+q_3(i)a_ih_{i+1} \\
&+q_2(i)a_{i-1/2}(h_i+h_{i+1})+q_1(i)a_{i-1}(2h_i+h_{i+1}))\ \overset{V}{u_\varepsilon}(\theta_{1,0}) \\
&-\frac{h_{i+1}^4(2\varepsilon+q_1(i)h_ia_{i-1}-q_3(i)h_ia_i)}{120(h_i+h_{i+1})}\ \overset{V}{u_\varepsilon}(\theta_{3,0}) \\
&-\frac{1}{24}h_i^4q_1(i)a_{i-1}\ \overset{V}{u_\varepsilon}(\theta_{1,1})-\frac{1}{384}h_i^4q_2(i)a_{i-1/2}\overset{V}{u_\varepsilon}(\theta_{2,1}) \\
&-\frac{1}{6}q_1(i)\varepsilon h_i^3\ \overset{V}{u_\varepsilon}(\theta_{1,2})-\frac{1}{48}q_2(i)\varepsilon h_i^3\ \overset{V}{u_\varepsilon}(\theta_{2,2}), \\
i\ &=\ 1,2,...,n-1,
\end{aligned}
$$

where $\theta_{1,0},\ \theta_{1,1},\ \theta_{1,2},\ \alpha_1^i \in (x_{i-1},x_i),\ \theta_{2,1},\ \theta_{2,2},\ \alpha_2^i \in (x_{i-1/2},x_i),\ \theta_{3,0},\ \alpha_3^i \in (x_i,x_{i+1})$ i $\theta_i' \in (\widetilde{u}_\varepsilon(x_{i-1/2}),u_\varepsilon(x_{i-1/2}))$, and $\widetilde{R}_i$ is given with (20).

3. Meshes

In order to obtain a good approximation for the exact solution of the problem (1) we use the nonuniform meshes that are dense in the neighborhood of the point $x=1$, where the boundary layer appears. We considered two types of meshes, Bakhvalov's and Shiskin's. Because of getting better numerical results when applying Bakhvalov's type of meshes, we are going to prove the uniform convergence of the method obtained on a mesh of type, constructed by Vulanović ([16], [**?**]). The mesh, further on called H-mesh, is generated by the function

$$
\lambda(t)=\begin{cases}\lambda_1(t)=\lambda_2'(\tau)\,t, & t\in[0,\tau]\\ \lambda_2(t)=1-\frac{A\varepsilon(1-t)}{q-(1-t)}, & t\in[\tau,1]\end{cases} \tag{32}
$$

with

$$
\tau=1-\frac{q-\sqrt{Aq\varepsilon(1-q+A\varepsilon)}}{1+A\varepsilon},
$$

and the constants A and q satisfy

$$
q\in(0,1),\quad A\in(0,q/\varepsilon), \tag{33}
$$

so that the transition point has the property $\tau\in(1-q,1)$. The mesh points are

$$
x_i=\lambda\left(\frac{i}{n}\right),\quad i=0,1,\ldots,n.
$$

The Shiskin mesh we use in numerical experiments has a generating function of the form

$$
\lambda(t)=\begin{cases}\lambda_1(t)=2(1-\tau)t, & t\in[0,0.5]\\ \lambda_2(t)=1-\tau+2\tau(t-0.5), & t\in[0.5,1]\end{cases},
$$

with the transition point $\tau = \min\{0.5, \varepsilon\alpha \ln n\}$. We have to emphasize the following property of the nodes of the H-mesh where $h_i = x_i - x_{i-1}$

Lemma 3.1 *For $i \in \{1, 2, ..., n-1\}$, it holds true that $h_i \geq h_{i+1}$ and $h_i \leq M\frac{1}{n}$.*

4. Convergence

4.1. Scheme 2 and H-mesh

Lemma 4.1 *For the discrete problem (8) applied on an H-mesh when a, b are functions smooth enough, then for $n \geq n_0$, and $i \in \{1, 2, ..., n-1\}$ for which holds $-2\varepsilon + h_i a_i < 0$, the coefficient $q_1(i)$ is of the form*

$$q_1(i) = \frac{h_i - h_{i+1}}{3h_i}, \tag{34}$$

otherwise is of the form (12). If the constants of the mesh satisfy (33) and additionally $q > \frac{3}{n}$, then

$$|\tau_i[u_\varepsilon]| \leq \begin{cases} M(h_i^2 + \frac{1}{h_i}\exp(-\alpha\frac{1-x_{i+1}}{\varepsilon})), & \frac{i+1}{n} \leq \tau, \\ M(h_i^2 + \frac{1}{\varepsilon}n^{-2}), & \text{otherwise.} \end{cases}$$

Proof. If we denote the exact solution of the problem (1) by u_ε then we observe the truncating error given earlier for Scheme 2. Let $i \in \{1, 2, ..., n-1\}$. In the case when $-2\varepsilon + h_i a_i < 0$ and $-2\varepsilon + h_i a_i \geq 0$ we have different forms of the coefficient $q_1(i)$. In both cases we have

$$|\tau_i[u_\varepsilon]| \leq M\ (h_i^2\ \left|u_\varepsilon^{'''}(x_i)\right| + \max\{\varepsilon, h_i\}(h_i^2\left|u_\varepsilon^{IV}(\theta_{1,0})\right|$$
$$+\frac{h_{i+1}^3}{h_i + h_{i+1}}\left|u_\varepsilon^{IV}(\theta_{3,0})\right|) + h_i^3\left|u_\varepsilon^{IV}(\theta_{1,1})\right| + \varepsilon h_i^2\left|u_\varepsilon^{IV}(\theta_{1,2})\right|..$$

Using (4) it follows that

$$\begin{aligned} |\tau_i[u_\varepsilon]| \leq M\ (h_i^2\ (1 + \varepsilon^{-3}\exp(-\alpha\frac{1-x_i}{\varepsilon})) \\ +\max\{\varepsilon, h_i\}(h_i^2(1 + \varepsilon^{-4}\exp(-\alpha\frac{1-x_i}{\varepsilon})) \\ +\frac{h_{i+1}^3}{h_i + h_{i+1}}\left|u_\varepsilon^{IV}(\theta_{3,0})\right|) + h_i^3(1 + \varepsilon^{-4}\exp(-\alpha\frac{1-x_i}{\varepsilon})) \\ +\varepsilon h_i^2(1 + \varepsilon^{-4}\exp(-\alpha\frac{1-x_i}{\varepsilon})). \end{aligned} \tag{35}$$

So,we will consider two cases:

1. Let $\frac{i+1}{n} \leq \tau$ Then $h_i = h_{i+1}$ and $\max\{\varepsilon, h_i\} = h_i$, because of the condition $\varepsilon < \frac{1}{n}$. If we use the integral form of the error in the Taylor expansion of the function $u_\varepsilon^{IV}(x)$, it follows that

$$u_\varepsilon^{IV}(\theta_{3,0}) = \frac{4}{h_{i+1}^4}\int_{x_i}^{x_{i+1}}(x_{i+1}-s)^3 u_\varepsilon^{IV}(s)ds.$$

Using (4) we have

$$\begin{aligned}\left|u_\varepsilon^{IV}(\theta_{3,0})\right| &\leq \frac{4}{h_{i+1}^4}\int_{x_i}^{x_{i+1}}(x_{i+1}-s)^3 M(1+\varepsilon^{-4}\exp(-\alpha\frac{1-s}{\varepsilon}))ds\\ &\leq M + \frac{M}{\varepsilon^4 h_{i+1}^4}\int_{x_i}^{x_{i+1}}(x_{i+1}-s)^3\exp(-\alpha\frac{1-s}{\varepsilon})ds,\end{aligned}$$

that is

$$\begin{aligned}|\tau_i[u_\varepsilon]| \leq M\ (h_i^2\ &+ (h_i^2\varepsilon^{-3} + h_i^3\varepsilon^{-4} + h_i^3\varepsilon^{-4} + h_i^2\varepsilon^{-3})\exp(-\alpha\frac{1-x_i}{\varepsilon})\\ &+\frac{1}{h_i}\exp(-\alpha\frac{1-x_{i+1}}{\varepsilon})).\end{aligned}$$

For $s \geq 0$ it holds true that $s^k\exp(-s) \leq M_1$, $k \in \mathbb{N}$, using $x_i = x_{i+1} - h_{i+1}$ it follows

$$\begin{aligned}\frac{h_i^{k-1}}{\varepsilon^k}\exp(-\alpha\frac{1-x_i}{\varepsilon}) &= \frac{1}{h_i}\exp(-\alpha\frac{1-x_{i+1}}{\varepsilon})\frac{h_i^k}{\varepsilon^k}\exp(-\alpha\frac{h_i}{\varepsilon})\\ &\leq M_1\frac{1}{h_i}\exp(-\alpha\frac{1-x_{i+1}}{\varepsilon}).\end{aligned}$$

So,

$$|\tau_i[u_\varepsilon]| \leq M\ (h_i^2\ + \frac{1}{h_i}\exp(-\alpha\frac{1-x_{i+1}}{\varepsilon}))$$

2. Let $\tau < \frac{i+1}{n}$. Then

$$\begin{aligned}|\tau_i[u_\varepsilon]| \leq M\ (h_i^2 &+ (h_i^2\varepsilon^{-3} + h_i^2\max\{\varepsilon, h_i\}\varepsilon^{-4} + h_i^3\varepsilon^{-4}\\ &+h_i^2\varepsilon^{-3})\exp(-\alpha\frac{1-x_i}{\varepsilon})) + h_i^2\varepsilon^{-3}\exp(-\alpha\frac{1-x_{i+1}}{\varepsilon})).\end{aligned}$$

Now, we have two possibilities

   (a) $1 - q + \frac{3}{n} < \frac{i+1}{n}$. Then

$$\begin{aligned}\frac{h_i^2}{\varepsilon^2}\exp(-\alpha\frac{1-x_{i+1}}{\varepsilon}) &= \frac{h_i^2}{\varepsilon^2}\exp\left(-\alpha Aq\frac{1-\frac{i+1}{n}}{q-1+\frac{i+1}{n}}\right)\\ \leq \frac{1}{n^2}\left(\frac{Aq}{(q-1+\frac{i-1}{n})^2}\right)^2 &\exp\left(-\frac{\alpha Aq(1-\frac{i+1}{n})}{q-1+\frac{i+1}{n}}\right) \leq M_1 n^{-2}.\end{aligned}$$

Because of

$$\frac{h_i^3}{\varepsilon^3}\exp(-\alpha\frac{1-x_i}{\varepsilon}) \le \frac{h_i^3}{\varepsilon^3}\exp(-\alpha\frac{1-x_{i+1}}{\varepsilon}) \le M_1 n^{-2},$$

we have the statement.

(b) $\tau < \frac{i+1}{n} < 1-q+\frac{3}{n}$. Then

$$\frac{h_i^2}{\varepsilon^2}\exp(-\alpha\frac{1-x_{i+1}}{\varepsilon})$$
$$\le \frac{1}{n^2}\left(\frac{Aq}{(q-1+\tau)^2}\right)^2 \exp\left(-\alpha\frac{1-\lambda_2(1-q+\frac{3}{n})}{\varepsilon}\right) \le Mn^{-2},$$

follows from

$$\left(\frac{Aq}{(q-1+\tau)^2}\right)^2 exp(-Mn) \le \left(\frac{Aq}{(q-1+\tau)^2}\right)^2 M_1 n^{-k} \le M,$$

for $k \in \mathbb{N}$ big enough.

So the theorem is proven.□

If we define the functions for $i = 1, 2, ..., n-1$ and $t > 0$

$$\phi_i(t) = \prod_{j=i+1}^{n} \frac{1}{1+t\frac{h_j}{\varepsilon}}$$

with $\phi_n(t) = 1$, we can prove the following lemma:

Lemma 4.2 *Let*

(36) $$0 < \alpha_0 < \alpha, \quad t \le \alpha_0/2 \quad \textit{and} \quad b(x,u) \ge 0$$

for $(x, u) \in (0, 1) \times \mathbb{R}$ *under the conditions of Theorem 2.1. Then*

$$T_\varepsilon^h \phi_i(t) \ge \frac{C(t)}{\varepsilon + th_i}\phi_i(t).$$

We shall use also the result of the following lemma:

Lemma 4.3 *For* $t \in (0, \alpha_0)$ *and* $i = 0, 1, ..., n$ *it holds true that*

$$\exp(-\alpha\frac{1-x_i}{\varepsilon})) \le \phi_i(t)$$

Using the results from the previous section and the previous lemma we obtain the main conclusion:

Theorem 4.1. *If we denote the solution of the discrete problem (8) by* w^* *applied on an H-mesh and if* u_ε^h *is the discrete exact solution of the problem (1), then for the functions* a *and* b *smooth enough, under the conditions (2), (36), and also under conditions of Theorem 2.1, and previous lemmas, then for* $n \leq 1/\sqrt{\varepsilon}$*, it follows that*

$$\left\| u_\varepsilon^h - w^* \right\|_\infty \leq Ch^2.$$

Proof. Let w^h and v^h be the mesh functions. Using the results of Theorem 2.1, we have that $F'(w^h + s(v^h - w^h))$ is $M-$matrix. For some $s \in (0,1)$, from

$$w_0^h \geq u_0^h, \quad w_n^h \geq u_n^h \quad i \quad T_\varepsilon^h w_i^h \geq T_\varepsilon^h u_i^h, \quad i = 1, 2, ..., n-1,$$

it follows that

$$w^h - u^h = F'(w^h + s(v^h - w^h))^{-1}(F(w^h) - F(u^h)) \geq 0,$$

so the operator T_ε^h satisfies the discrete maximum principle. Defining the barrier function

$$\psi_i(t) = C\left((1 + x_i)n^{-2} + (1 + t\frac{h_{i+1}}{\varepsilon})\phi_i(t)\right)$$

for $i = 0, 1, ...n$, and h_{n+1} some positive number, we conclude that

$$\psi_0(t) \pm (u_\varepsilon^h - w^*)_0 \geq 0 \quad \psi_n(t) \pm (u_\varepsilon^h - w^*)_n \geq 0,$$

and for $i = 1, 2, ...n-1$ using previous lemmas it follows that

$$\begin{aligned} & T_\varepsilon^h(\psi_i(t) \pm (u_\varepsilon^h - w^*)_i) \\ & \geq T_\varepsilon^h(C((1 + x_i)n^{-2} + \phi_{i+1}(t))) - |\tau_i[u_\varepsilon]| \\ \geq C & \left((q_1(i)a_{i-1} + q_2(i)a_i)n^{-2} + \frac{C(t)}{\varepsilon + th_{i+1}}\phi_{i+1}(t)\right) - |\tau_i[u_\varepsilon]| \\ & > C_1 n^{-2} + \frac{C_2}{\varepsilon + th_i}\phi_{i+1}(t) - |\tau_i[u_\varepsilon]| \geq 0. \end{aligned}$$

Using the discrete maximum principle for the observed operator, we have

$$\left|(u_\varepsilon^h - w^*)_i\right| \leq \psi_i(t). \tag{37}$$

Let $k \in \{0, 1, ..., n-1\}$ be the number that

$$\frac{k+1}{n} \leq \tau < \frac{k+2}{n}. \tag{38}$$

We will show that for all $i \leq k+1$ is satisfied $\phi_{i+1}(t) \leq Mn^{-2}$. It stands that

$$\phi_{i+1}(t) \leq \prod_{j=k+3}^{n} \frac{1}{1 + t\frac{h_j}{\varepsilon}}$$

Because of (38) we have

$$k+3 \le 2+n-n\frac{q-\sqrt{Aq\varepsilon(1-q+A\varepsilon)}}{1+A\varepsilon} = t_n.$$

Using

$$\frac{h_j}{\varepsilon} = \frac{Anq}{(j-1+n(q-1))(j+n(q-1))} \ge \frac{Anq}{(-\frac{1}{2}+j+n(q-1))^2},$$

and $n\sqrt{\varepsilon} \le 1$ it follows that

$$\phi_{i+1}(t) \le \prod_{j=k+3}^{n} \frac{1}{1+t\frac{h_j}{\varepsilon}} \le \prod_{j=\lfloor t_n \rfloor}^{n} \frac{1}{1+t\frac{Anq}{(-\frac{1}{2}+j+n(q-1))^2}}$$
$$\le \prod_{j=1}^{n-\lfloor t_n \rfloor} \frac{1}{1+\frac{4Anqt(1+A\varepsilon)^2}{(1+A(1+2nq)\varepsilon+2j(1+A\varepsilon)+2n\sqrt{Aq\varepsilon(1-q+A\varepsilon)})^2}}$$
$$\le \prod_{j=1}^{2} \frac{1}{\frac{4Anqt(1+A\varepsilon)^2}{(1+A2q\sqrt{\varepsilon}+4+5A\varepsilon+2\sqrt{Aq(1-q+A\varepsilon)})^2}}$$
$$\le \frac{(5+2Aq+5A+2\sqrt{Aq(1-q+A)})^2}{4Anqt} \le Mn^{-2}.$$

So, for all $i \le k+1$ from (37) and the previous conclusions it follows that

$$\left|(u_\varepsilon^h - w^*)_i\right| \le Mn^{-2}.$$

Now, we define a new barrier function

$$\varphi_i(t) = C\left((1+x_i)n^{-2} + n^{-2}\phi_i(t)\right)$$

for $i = k+1, k+2, ...n$. Then

$$\varphi_{k+1}(t) \pm (u_\varepsilon^h - w^*)_{k+1} \ge 0, \varphi_n(t) \pm (u_\varepsilon^h - w^*)_n \ge 0,$$

and

$$T_\varepsilon^h(\varphi_i(t) \pm (u_\varepsilon^h - w^*)_i)$$
$$\ge T_\varepsilon^h(C((1+x_i)n^{-2} + n^{-2}\phi_i(t))) - |\tau_i[u_\varepsilon]|$$
$$> C_1 n^{-2} + n^{-2}\frac{C_2}{\varepsilon + th_i}\phi_i(t) - |\tau_i[u_\varepsilon]| \ge 0.$$

So,

$$\left|(u_\varepsilon^h - w^*)_i\right| \le Mn^{-2}$$

for $i \in \{0, 1, ..., n\}$ and the theorem is proven.□

4.2. Scheme 3 and H-mesh

In a similar way as in the previous theorem, we can get the following conclusion:

Theorem 4.2. *If we denote the solution of the discrete problem (22) by* w^* *applied on an H-mesh and if* u_ε^h *is the discrete exact solution of the problem (1), then for the functions a and b smooth enough, under the conditions (2), (36), and also under the conditions of Theorems 2.2 and 2.3, then for* $n \leq 1/\sqrt{\varepsilon}$*, it follows that*

$$\left\| u_\varepsilon^h - w^* \right\|_\infty \leq Ch^3.$$

5. Numerical results

The obtained theoretical results are confirmed by numerical experiments. Exact solutions of the tested examples are known, so the error is measured by $E_n = \left\| u_\varepsilon^h - w^* \right\|_\infty$, where w^* is the solution of the discrete problem, whereas $u_\varepsilon^h = (u_\varepsilon(x_0), ..., u_\varepsilon(x_n))^T$, for u_ε exact solution of the observed problem. The order of convergence is calculated with

$$Ord_n = \frac{\ln E_n - \ln E_{2n}}{\ln 2}.$$

The approximations, obtained from (8) and 22 applied on an H-mesh and Shiskin (S) mesh are tested for the different values of ε and n. The results confirmed the order of convergence of the methods, but the error E_n was smaller for H-mesh, which is a consequence of the greater number of nodes in the boundary layer. Newton's method is used for solving the nonlinear system of equations $F(w^h) = 0$ with the initial approximation $w^0 = (u_0(x_0), ..., u_0(x_n))^T$, u_0 as the solution of the reduced problem. The stop criterion applied is

$$\max \left\{ \left\| w^k - w^{k-1} \right\|_\infty, \left\| F(w^k) \right\|_\infty \right\} < 10^{-3}.$$

Some of the tested problems are:

Example 1

$$(39) \qquad -\varepsilon u'' + (1 + x(1-x))\, u' = f(x), \qquad u(0) = u(1) = 0,$$

where $f(x)$ is the function for which

$$u_\varepsilon(x) = \frac{1 - e^{-(1-x)/\varepsilon}}{1 - e^{-1/\varepsilon}} - \cos\frac{\pi}{2}x,$$

is the exact solution.

Example 2

$$(40) \qquad -\varepsilon u'' + u' + u^2 + u = f(x), \qquad u(0) = u(1) = 0,$$

where $f(x)$ is the function for which

$$u_\varepsilon(x) = \frac{1 - e^{-x/\varepsilon}}{e^{1/\varepsilon} - 1} + x$$

is the exact solution.

Table 1:Example 1 (Scheme 2 and H-mesh with $A = 7$ and $q = 0.5$)

	n						
ε	64	128	256	512	1024	2048	
2^{-4}	4.60(-5)	1.10(-5)	2.68(-6)	6.61(-7)	1.64 (-7)	4.09(-8)	E_n
	2.07	2.03	2.02	2.01	2.00		Ord_n
2^{-6}	6.35(-5)	1.54(-5)	3.80(-6)	9.44(-7)	2.35(-7)	5.87(-8)	E_n
	2.04	2.02	2.01	2.00	2.00		Ord_n
2^{-8}	1.31(-4)	4.64(-5)	8.54(-6)	2.12(-6)	5.28(-7)	1.32(-7)	E_n
	1.50	2.44	2.01	2.00	2.00		Ord_n
2^{-10}	1.41(-4)	3.98(-5)	1.24(-5)	4.39(-6)	1.27(-6)	2.22(-7)	E_n
	1.83	1.68	1.50	1.79	2.52		Ord_n
2^{-12}	1.65(-4)	4.18(-5)	1.11(-5)	3.17(-6)	9.90(-7)	3.47(-7)	E_n
	1.98	1.91	1.81	1.68	1.51		Ord_n
2^{-14}	1.81(-4)	4.54(-5)	1.14(-5)	2.95(-6)	7.92(-7)	2.26(-7)	E_n
	2.00	1.99	1.95	1.90	1.81		Ord_n
2^{-16}	1.89(-4)	4.77(-5)	1.19(-5)	2.99(-6)	7.61(-7)	1.97(-7)	E_n
	1.98	2.00	1.99	1.98	1.95		Ord_n
2^{-18}	1.94(-4)	4.86(-5)	1.22(-5)	3.06(-6)	7.66(-7)	1.93(-7)	E_n
	2.00	1.99	2.00	2.00	1.99		Ord_n
2^{-20}	1.97(-4)	4.93(-5)	1.24(-5)	3.10(-6)	7.74(-7)	1.94(-7)	E_n
	2.00	2.00	2.00	2.00	2.00		Ord_n

Table 2: Example 1 (Scheme 2 and S-mesh with $\sigma_0 = 4$)

	n						
ε	64	128	256	512	1024	2048	
2^{-4}	4.55(-4)	1.08(-4)	2.65(-6)	6.54(-7)	1.62(-7)	4.05(-8)	E_n
	2.07	2.03	2.02	2.01	2.00		Ord_n
2^{-6}	1.28(-4)	2.58(-4)	5.11(-6)	9.9(-7)	1.88(-7)	3.49(-8)	E_n
	1.99	2.01	2.00	2.00	2.00		Ord_n
2^{-8}	2.6(-4)	8.83(-4)	3.16(-4)	4.3(-6)	1.03(-6)	2.47(-7)	E_n
	1.56	1.48	2.88	2.06	2.06		Ord_n
2^{-10}	2.19(-4)	6.18(-4)	1.91(-4)	6.6(-6)	2.56(-6)	3.51(-7)	E_n
	1.82	1.70	1.53	1.37	2.86		Ord_n
2^{-12}	2.06(-4)	5.32(-4)	1.43(-4)	4.05(-6)	1.26(-6)	4.37(-7)	E_n
	1.95	1.90	1.82	1.69	1.52		Ord_n
2^{-14}	2.02(-4)	5.1(-4)	1.3(-4)	3.37(-6)	9.04(-7)	2.57(-7)	E_n
	1.99	1.97	1.95	1.890	1.81		Ord_n
2^{-16}	2.01(-4)	5.04(-4)	1.27(-4)	3.2(-6)	8.15(-7)	2.11(-7)	E_n
	2.00	1.99	1.98	1.97	1.94		Ord_n
2^{-18}	2.01(-4)	5.03(-4)	1.26(-4)	3.15(-6)	7.92(-7)	2.(-7)	E_n
	2.00	2.00	2.00	1.99	1.98		Ord_n
2^{-20}	2.01(-4)	5.02(-4)	1.26(-4)	3.14(-6)	7.86(-7)	1.97(-7)	E_n
	2.00	2.00	2.00	2.00	2.00		Ord_n

Table 3: Example 3 (Scheme 2 and H-mesh with $A = 4$ and $q = 0.8$)

ε	n = 64	128	256	512	1024	2048	
2^{-4}	7.44(-5)	1.65(-5)	3.87(-6)	9.38(-7)	2.31(-7)	5.73(-8)	E_n
	2.18	2.09	2.04	2.02	2.01		Ord_n
2^{-6}	1.36(-4)	2.95(-5)	6.84(-6)	1.64(-6)	4.03(-7)	9.97(-8)	E_n
	2.20	2.11	2.06	2.03	2.01		Ord_n
2^{-8}	1.50(-4)	3.26(-5)	7.57(-6)	1.82(-6)	4.46(-7)	1.10(-7)	E_n
	2.20	2.11	2.06	2.03	2.01		Ord_n
2^{-10}	1.54(-4)	3.35(-5)	7.79(-6)	1.87(-6)	4.59(-7)	1.14(-7)	E_n
	2.20	2.11	2.06	2.03	2.01		Ord_n
2^{-12}	1.55(-4)	3.38(-5)	7.85(-6)	1.89(-6)	4.63(-7)	1.15(-7)	E_n
	2.20	2.11	2.06	2.03	2.00		Ord_n
2^{-14}	1.55(-4)	3.39(-5)	7.86(-6)	1.89(-6)	4.64(-7)	1.15(-7)	E_n
	2.20	2.11	2.06	2.03	2.01		Ord_n
2^{-16}	1.55(-4)	3.39(-5)	7.86(-6)	1.89(-6)	4.64(-7)	1.15(-7)	E_n
	2.20	2.11	2.06	2.03	2.01		Ord_n
2^{-18}	1.55(-4)	3.39(-5)	7.86(-6)	1.89(-6)	4.64(-7)	1.15(-7)	E_n
	2.20	2.11	2.06	2.03	2.01		Ord_n
2^{-20}	1.55(-4)	3.39(-5)	7.86(-6)	1.89(-6)	4.64(-7)	1.15(-7)	E_n
	2.20	2.11	2.06	2.03	2.01		Ord_n

Table 4: Example 1 (Scheme 3 and H-mesh with $A = 2$ and $q = 0.8$)

ε	n = 32	64	128	256	512	1024	
2^{-4}	3.56(-5)	6.30(-6)	1.40(-6)	2.11(-7)	2.86(-8)	3.71(-9)	E_n
	2.50	2.17	2.73	2.88	2.95		Ord_n
2^{-6}	2.97(-3)	1.29(-4)	1.11(-5)	1.82(-7)	3.27(-8)	4.71(-9)	E_n
	4.52	3.54	5.94	2.47	2.80		Ord_n
2^{-8}	3.27(-3)	1.30(-4)	4.75(-5)	7.35(-6)	2.47(-7)	4.66(-9)	E_n
	4.66	4.77	2.69	1.57	5.73		Ord_n
2^{-10}	3.36(-3)	1.31(-4)	3.55(-6)	2.46(-7)	5.04(-8)	1.46(-8)	E_n
	4.68	5.20	3.85	2.29	1.79		Ord_n
2^{-12}	3.38(-3)	1.31(-4)	3.21(-6)	1.39(-7)	1.67(-8)	3.32(-9)	E_n
	4.69	5.35	4.53	3.06	2.33		Ord_n
2^{-14}	7.44(-3)	3.08(-4)	9.23(-6)	3.09(-7)	1.77(-8)	1.65(-9)	E_n
	4.70	5.39	4.82	3.64	3.01		Ord_n
2^{-16}	3.39(-3)	1.31(-4)	3.09(-6)	1.02(-7)	6.77(-9)	5.64(-10)	E_n
	3.64	5.40	4.91	3.92	3.59		Ord_n
2^{-18}	3.39(-3)	1.31(-4)	3.08(-6)	1.00(-7)	6.25(-9)	4.38(-10)	E_n
	3.79	5.40	4.93	4.01	3.84		Ord_n
2^{-20}	3.39(-3)	1.31(-4)	3.08(-6)	1.00(-7)	6.15(-9)	4.43(-10)	E_n
	3.93	5.40	4.94	4.02	3.80		Ord_n

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