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ON A FOURTH-ORDER FINITE DIFFERENCE METHOD FOR NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEMS [1]

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Abstract. We consider a finite difference method of order four for nonlinear two-point boundary value problems. In linear case the finite difference schemes lead to a tridiagonal linear system. Numerical experiments support the theoretical results.

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1. Introduction

This paper is concerned with the construction of finite difference approximations for the boundary value problem:

$$-y'' + f(x,y) = 0, \quad x \in I = [0,1], \qquad y(0) = y(1) = 0. \tag{1}$$

For simplicity, we shall assume that $f \in C^\infty(I \times \mathbb{R})$, and

$$0 < \gamma^2 \le f_y(x,y), \quad x \in I,\ y \in \mathbb{R}. \tag{2}$$

The condition (2) is the standard stability condition, which implies that (1) has an unique solution y, which is in $C^\infty(I)$.

In Section 2 we discuss a method for obtaining three-point finite difference approximations for the differential equation. These approximations involve derivatives of f. Assuming f to be sufficiently differentiable, the derivatives of f can be expressed in terms of y'. Appropriate approximations for y' at the mesh points are obtained for the use in particular formulas.

In Section 3 some difference schemes are derived and described and consistency errors are estimated. Numerical results are given to illustrate the order of accuracy achieved.

Throughout the paper, M, sometimes subscripted, denotes a generic positive constant, indepedent of number n of discretization subintervals that will be used to solve (1) numerically.

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2. Finite difference approximations

Let us introduce the following notation. Let n be a positive integer, x_k, $k = 0, 1, \dots, n$, be the mesh points,

$$0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1,$$

and

$$h_k = x_k - x_{k-1}, \qquad k = 1, 2, \dots, n.$$

From now on we shall assume that our mesh has the following properties:

$$H_{\max} \le h_{\min} \left(1 + M h_{\min}\right)$$

where $\quad H_{\max} = \max\left\{h_k : k = 1, 2, \dots, n\right\}, \quad h_{\min} = \min\left\{h_k : k = 1, 2, \dots, n\right\}.$

Such a mesh is called almost equidistant, see [7].

At mesh points x_k, we set $y_k = y\left(x_k\right), y''\left(x_k\right) = y_k'' = f_k$, $f_k' = \frac{\partial}{\partial x} f(x, y(x))$, $f_k'' = \frac{\partial^2}{\partial x^2} f(x, y(x))$, etc. In the following, we consider the obtaining of three-point finite difference approximations for the differential equation at a fixed point x_k, $k \in \{1, 2, \dots, n-1\}$. For simplicity, we define for a fixed k

$$h = x_k - x_{k-1}, \qquad H = x_{k+1} - x_k.$$

Since our mesh is almost equidistant, it then holds

$$|H - h| \le M h^2.$$

Let w^h be a mesh function. Mesh functions will be defined with the $\mathbb{R}^{n+1}$ column vectors

$$w^h = \left[w_0, w_1, \dots, w_n\right]^\top$$

(for simplicity, the superscript h is omitted in the components). In particular,

$$u^h = \left[u\left(x_0\right), u\left(x_1\right), \dots, u\left(x_n\right)\right]^\top.$$

The standard maximum norm will be used:

$$\left|\left|w^h\right|\right|_\infty = \max\left\{\left|w_i\right| : i = 0, 1, \dots, n\right\}.$$

$||\cdot||$ will also denote the matrix norm induced by the maximum vector norm.

Let us define the operators δ, μ and ψ:

$$\delta y_k = -2y_k + \frac{2H}{h+H} y_{k-1} + \frac{2h}{h+H} y_{k+1},$$

$$\psi y_k = hH\left(f_k + A\delta f_k + D\left(H - h\right) f_k' + ChH f_k''\right).$$

By Taylor's expansion we obtain

$$\delta y_k = 2hH\left(\sum_{j=1}^{\infty} \frac{H^j - (-h)^j}{(j+1)!\,(H+h)} f_k^{(j-1)}\right) \tag{3}$$

$$\delta f_k = 2hH\left(\sum_{j=1}^{\infty}\frac{H^j-(-h)^j}{(j+1)!\,(H+h)}f_k^{(j+1)}\right) \tag{4}$$

Now, we can form various three-point approximations for the differential equation by using the terms δy_k, $\frac{h+H}{2}\mu f_k'$ and hHf_k'' for approximation of 1. However, we shall focus our attention here on obtaining approximations for constructing methods of order four.

For this purpose we define

$$\delta y_k = \psi y_k + \tau_k\left(h,H\right). \tag{5}$$

With the help of (3) and (4) we obtain

$$\tau_k\left(h,H\right) = \delta y_k - \psi y_k = hH\left(E_1+E_2+E_3+E_4+E_5\right)$$

where

$$E_1 = \left(D-\frac{1}{3}\right)\left(h-H\right)f_k', \qquad E_2 = \left(\frac{h^3+H^3}{12\left(h+H\right)} - AhH - ChH\right)f_k'',$$

$$E_3 = \frac{1}{120}\left(h-H\right)\left(-2\left(h^2+H^2\right)+40AhH\right)f_k^{(3)},$$

$$E_4 = \frac{1}{360\left(h+H\right)}\left(h^5+H^5-30AHh\left(H^3+h^3\right)\right)f_k^{(4)},$$

$$E_5 = \left(h-H\right)\mathcal{O}\left(H_{\max}^2\right)+\mathcal{O}\left(H_{\max}^4\right).$$

We first obtain approximations for f_k' and f_k''. We easily find that

$$f_k' = y_k'f_k^y + f_k^x, \qquad f_k'' = y_k''f_k^y + 2y_k'f_k^{x,y} + (y_k')^2 f_k^{y,y} + f_k^{x,x}.$$

Since

$$y_k' = \frac{y_{k+1}-y_{k-1}}{h+H} + \frac{1}{2}\left(h-H\right)y_k'' + \mathcal{O}\left(H_{\max}^3\right), \qquad y_k'' = f_k,$$

we get

$$f_k' = \frac{y_{k+1}-y_{k-1}}{h+H}f_k^y + f_k^x + \mathcal{O}\left(H_{\max}^2\right),$$

and

$$f_k'' = f_kf_k^y + 2\frac{y_{k+1}-y_{k-1}}{h+H}f_k^{x,y} + \left(\frac{y_{k+1}-y_{k-1}}{h+H}\right)^2 f_k^{y,y} + f_k^{x,x} + \mathcal{O}\left(H_{\max}^2\right).$$

Now, because of $(h-H)=\mathcal{O}\left(H_{\max}^2\right)$ and $hH=\mathcal{O}\left(H_{\max}^2\right)$, we have

$$\begin{aligned}(H-h)\,f_k' &= (H-h)\left(\frac{y_{k+1}-y_{k-1}}{h+H}f_k^y + f_k^x + \mathcal{O}\left(H_{\max}^2\right)\right)\\ &= (H-h)\left(\frac{y_{k+1}-y_{k-1}}{h+H}f_k^y + f_k^x\right) + \mathcal{O}\left(H_{\max}^4\right)\end{aligned}$$

and

$$hHf_k'' = hH\left(f_k f_k^y + 2\frac{y_{k+1}-y_{k-1}}{h+H}f_k^{x,y} + \left(\frac{y_{k+1}-y_{k-1}}{h+H}\right)^2 f_k^{y,y} + f_k^{x,x} + \mathcal{O}\left(H_{\max}^2\right)\right)$$
$$= hH\left(f_k f_k^y + 2\frac{y_{k+1}-y_{k-1}}{h+H}f_k^{x,y} + \left(\frac{y_{k+1}-y_{k-1}}{h+H}\right)^2 f_k^{y,y} + f_k^{x,x}\right) + \mathcal{O}\left(H_{\max}^4\right).$$

3. Difference scheme

In order to form a discretization of the problem (1) we approximate the differential equation of (1) by considering (5) . After division by hH we obtain

$$-\frac{1}{hH}\delta y_k + \frac{1}{hH}\psi y_k + E_1 + E_2 + E_3 + E_4 + E_5 = 0.$$

It is easy to see that

$$\frac{1}{hH}\psi y_k = f_k + A\delta f_k + D(H-h)f_k' + ChHf_k''$$
$$= f_k + A\delta f_k + D(H-h)\left(\frac{y_{k+1}-y_{k-1}}{h+H}f_k^y + f_k^x\right)$$
$$+ChH\left(f_k f_k^y + 2\frac{y_{k+1}-y_{k-1}}{h+H}f_k^{x,y} + \left(\frac{y_{k+1}-y_{k-1}}{h+H}\right)^2 f_k^{y,y} + f_k^{x,x}\right) + E_6,$$

where $E_6 = \mathcal{O}\left(H_{\max}^4\right)$. In the equations above we neglect the terms $E_1, E_2, \ldots, E_6$ and get

$$-\frac{1}{hH}\delta w_k + \frac{1}{hH}\psi w_k = 0, \tag{6}$$

where $w_k \approx y_k = y(x_k)$. We shall use

$$-\frac{1}{hH}\delta w_k = a_1(k)\,w_{k-1} + a_0(k)\,w_k + a_2(k)\,w_{k+1},$$

where

$$a_1(k) = \tfrac{-2}{h(h+H)}, \quad a_0(k) = \tfrac{2}{hH}, \quad a_2(k) = \tfrac{-2}{H(h+H)}.$$

and

$$\frac{1}{hH}\psi w_k = b_1(k)\,f(x_{k-1}, w_{k-1}) + b_0(k)\,f(x_k, w_k) + b_2(k)\,f(x_{k+1}, w_{k+1}),$$

where b_0, b_1 and b_2 depend only on x_{i-1}, x_i, x_{i+1}, A, C and D. Now, we conclude that

$$-\frac{1}{hH}\delta y_k + \frac{1}{hH}\psi y_k = -\frac{1}{hH}\delta w_k + \frac{1}{hH}\psi w_k + \mathcal{O}\left(H_{\max}^4\right),$$

if $E_i = \mathcal{O}\left(H_{\max}^4\right)$, $i = 1, 2, \ldots, 5$.

Using this, from (6) we obtain the following approximation of the differential equation (1) at $x_i \in I_h$, $i = 1, 2, \ldots, n-1$:

$$\begin{aligned} F_i := \quad & a_1(i)\, w_{i-1} + a_0(i)\, w_i + a_2(i)\, w_{i+1} \\ & + b_1(i)\, c(x_{i-1}, w_{i-1}) + b_0(i)\, c(x_i, w_i) + b_2(i)\, c(x_{i+1}, w_{i+1}) = 0. \end{aligned}$$

We form a discrete analogue of problem (1) in the form $F(w) = 0$, where $F = (F_0, F_1, \ldots, F_n)$, and

$$F_0 := w_0 = 0, \qquad F_n := w_n = 0.$$

The solution $w^* = [w_0^*, w_1^*, \ldots, w_n^*]^\top$ to $F(w) = 0$, is an approximation to the exact solution y of (1).

Let

$$y^h = [y(x_0), y(x_1), \ldots, y(x_n)]^\top,$$

be the restriction of y on the discretization mesh. Our aim is to prove that there holds

$$\left\| y^h - w^* \right\|_\infty \le M H_{\max}^4, \tag{7}$$

for the following five choices of A, C and D. In each case different values for A and C are given. D always equals $\frac{1}{3}$ and because of that, $E_1 = 0$ in all cases. Also, in all cases $E_4 = \mathcal{O}\left(H_{\max}^4\right)$. Since $(h - H) = \mathcal{O}\left(H_{\max}^2\right)$, $E_5 = \mathcal{O}\left(H_{\max}^4\right)$. Terms E_2 and E_3 are different for each case:

3.1 Case 1. $A = \frac{1}{12}, C = 0$

$$\begin{aligned} E_2 &= \frac{1}{12}(h-H)^2 = \mathcal{O}\left(H_{\max}^4\right), \\ E_3 &= \frac{h-H}{360}\left(10hH - 6\left(h^2 + H^2\right)\right) = \mathcal{O}\left(H_{\max}^4\right). \end{aligned}$$

3.2 Case 2. $A = \frac{-h^2+3hH-H^2}{12hH}, C = 0$

$$\begin{aligned} E_2 &= \frac{1}{6}(h-H)^2 = \mathcal{O}\left(H_{\max}^4\right), \\ E_3 &= \frac{H-h}{180}\left(8h^2 - 15hH + 8H^2\right) = \mathcal{O}\left(H_{\max}^4\right). \end{aligned}$$

3.3 Case 3. $A = \frac{-h^2+2hH-H^2}{12hH}, C = \frac{2h^2-3hH+2H^2}{12hH}$

$$\begin{aligned} E_2 &= 0, \\ E_3 &= \frac{H-h}{90}\left(4h^2 - 5hH + 4H^2\right) = \mathcal{O}\left(H_{\max}^4\right). \end{aligned}$$

3.4 Case 4. $A = \frac{1}{12}, C = \frac{h^2 - 2hH + H^2}{12hH}$

$$\begin{aligned} E_2 &= 0, \\ E_3 &= \frac{H-h}{360}\left(6h^2 - 10hH + 6H^2\right) = \mathcal{O}\left(H_{\max}^4\right). \end{aligned}$$

3.5 Case 5. $A = \frac{-2h^2 + 5hH - 2H^2}{60hH}, C = \frac{h^2 + H^2}{20hH}$

$$\begin{aligned} E_2 &= 0, \\ E_3 &= 0. \end{aligned}$$

In an equidistant case, i.e. if $h_{\min} = H_{\max} = h$ we obtain

$$\tau_k\left(h, h\right) = h^2\left(\left(\frac{1}{12} - A - C\right)h^2 f_k'' + \frac{1}{360}\left(1 - 30A\right)h^4 f_k^{(4)} + \mathcal{O}\left(h^4\right)\right).$$

Parameter D does not appear here. If $A = C = 0$, then we obtain a well-known approximation

$$\delta y_k = h^2 f_k + \frac{h^4}{12} f_k'' + \mathcal{O}\left(h^6\right).$$

As a special case, our schemes contain the fourth-order scheme from [1] when the mesh is equidistant. (Cases 1, 2 and 4.)

The main result of this paper can be summarized in the following theorem.

Theorem 3.1. *Let* $w^* = \left[w_0^*, w_1^*, \ldots, w_n^*\right]^\top$ *be the solution of* $F\left(w\right) = 0$, *and let* y *be the exact solution of* (1), *and*

$$y^h = \left[y\left(x_0\right), y\left(x_1\right), \ldots, y\left(x_n\right)\right]^\top,$$

be the restriction of y *on the discretization mesh. There exists an* n_0 *such that for* $n \geq n_0$ *there holds*

$$\left\|y^h - w^*\right\|_\infty \leq M H_{\max}^4.$$

Proof. As we have already shown, our discretization error is $\mathcal{O}\left(H_{\max}^4\right)$. It remains to be proved that the Frechet derivative of F is uniformly bounded for a sufficiently small $H_{\max}$:

$$\left\|\left(F'\left(u\right)\right)^{-1}\right\|_\infty \leq M_0, \qquad u \in \left\{z \in \mathbb{R}^{n+1} : \left\|y^h - z\right\|_\infty \leq M_1 H_{\max}^4\right\}$$

with some suitable M_0.

The rest of the proof can be carried out using the technique given in [2] and [9]. □

4. Numerical results

To illustrate computationally the fourth-order method we solved the following nonlinear two-point boundary value problem

$$y'' = \frac{1}{3}\left((2-x)\,e^{2(y-x\ln 2)} + \frac{1}{1+x}\right), \qquad y(0) = y(1) = 0,$$

with the exact solution $y(x) = \ln \frac{1}{1+x} + x\ln 2$. The discretization mesh was generated using the mesh generating function

$$\lambda(t) = \frac{1}{2}\left(1 - \sin\left(\frac{\pi}{2}\cos(\pi t)\right)\right),$$

and the mesh points are

$$x_i = \lambda\left(\frac{i}{n}\right), \qquad i = 0, 1, \ldots, n.$$

Our discrete analogue $F(w) = 0$ is a nonlinear system. We solve this system using the Newton-Raphson method, where a tridiagonal linear system is solved in each step. We performed the calculation in *Mathematica*.

The errors $E_n = \|u_{\varepsilon,h} - w^*\|_\infty$, where w^* is the numerical solution on a mesh with n subintervals, are given in the table. Also, we define in the usual way the order of convergence Ord for two successive values of n with respective errors E_n and E_{2n} :

$$Ord = \frac{\ln E_n - \ln E_{2n}}{\ln 2}.$$

We expect that $Ord = 4$.

n	**Case 1**	**Case 2**	**Case 3**	**Case 4**	**Case 5**
4	$3.09 \cdot 10^{-4}$ –	$8.27 \cdot 10^{-4}$ –	$4.68 \cdot 10^{-4}$ –	$3.98 \cdot 10^{-4}$ –	$4.21 \cdot 10^{-4}$ –
8	$6.22 \cdot 10^{-5}$ 2.310	$1.98 \cdot 10^{-4}$ 2.063	$8.33 \cdot 10^{-5}$ 2.492	$7.84 \cdot 10^{-5}$ 2.337	$7.56 \cdot 10^{-5}$ 2.476
16	$4.08 \cdot 10^{-6}$ 3.930	$1.36 \cdot 10^{-5}$ 3.865	$5.41 \cdot 10^{-6}$ 3.944	$5.83 \cdot 10^{-6}$ 3.751	$5.98 \cdot 10^{-6}$ 3.659
32	$2.59 \cdot 10^{-7}$ 3.977	$8.72 \cdot 10^{-7}$ 3.960	$3.43 \cdot 10^{-7}$ 3.979	$3.80 \cdot 10^{-7}$ 3.937	$3.93 \cdot 10^{-7}$ 3.930
64	$1.63 \cdot 10^{-8}$ 3.994	$5.53 \cdot 10^{-8}$ 3.980	$2.17 \cdot 10^{-8}$ 3.981	$2.41 \cdot 10^{-8}$ 3.980	$2.48 \cdot 10^{-8}$ 3.982
128	$1.02 \cdot 10^{-9}$ 3.998	$3.46 \cdot 10^{-9}$ 3.997	$1.36 \cdot 10^{-9}$ 3.999	$1.51 \cdot 10^{-9}$ 3.996	$1.56 \cdot 10^{-9}$ 3.996
256	$6.37 \cdot 10^{-11}$ 3.999	$2.17 \cdot 10^{-10}$ 3.999	$8.50 \cdot 10^{-11}$ 4.000	$9.45 \cdot 10^{-11}$ 3.998	$9.74 \cdot 10^{-11}$ 3.999
512	$3.97 \cdot 10^{-12}$ 4.002	$1.35 \cdot 10^{-11}$ 4.000	$5.32 \cdot 10^{-12}$ 3.999	$5.91 \cdot 10^{-12}$ 3.999	$6.09 \cdot 10^{-12}$ 3.998

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