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NOTE ON THE p-NILPOTENCY IN FINITE GROUPS

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Abstract. Using some properties of nilpotent Hall subgroups, we establish a splitting criterion that is a generalization of the splitting criterion due to Carter.

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Let π be a set of primes and π' it's complement in the set of all primes. With $O_\pi(G)$ and $O^\pi(G)$ we shall denote, as it is usual, the largest normal π–subgroup of G and the subgroup of G generated with all π'–subgroups, respectively.

Let S be a finite p-group. We shall say that S is L-local if the local theorem holds, i.e., if the following is true: if Sylow p-subgroup of some group G is isomorphic to S, then holds $G/O^p(G) \cong N/O^p(N)$, where $N = N_G(S)$. Two examples of the L-local groups are:

1) regular p-groups (the local theorem proved by Wielandt)

2) let S be a p-group and $\Omega = \{A | A < S, A \text{ is Abelian and } |A| = n\}$ where n is the maximum of the orders of the Abelian sugroups of S. If $S =< \Omega >$, then S is L-local (the local theorem proved by Glauberman).

This paper is inspired by the following theorem due to Wielandt.

Theorem 1.(Wielandt) *Let G be a finite group and H its nilpotent Hall subgroup. If $N_G(S) = H$ for every Sylow subgroup S of H, then H has a normal complement in G.*

We use the above theorem (in fact, we use the idea of its proof) to obtain some criterions for p-nilpotency when Sylow p-subgroup is L-local. The main result is a generalization of the following theorem due to Carter:

Theorem 2.(Carter) *Let G be a finite group and H its nilpotent Hall subgroup. If H is self-normalizing and its Sylow subgroups are regular, then H has a normal complement in G.*

We are going to prove the following:

Theorem 3. *Let G be a finite group and H its nilpotent Hall subgroup. If H is self-normalizing and its Sylow subgroups are L-local, then H has a normal*

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complement in G.

The following criterion for p-nilpotency is well-known:

Theorem 4. *Let G be a finite group and S its Sylow p-subgroup. Group G is p-nilpotent iff the following holds: any two elements of S that are conjugated in G are conjugated in S.*

We begin with a proposition for supersoluble Hall subgroups:

Lemma 1. *Let G be a finite group and let H and K be its supersoluble Hall subgroups. If $|K|$ divides $|H|$, then K is contained in some conjugate of H.*

Proof. Proof goes by induction on the order of G. Let K_1 be a subgroup of H, with $|K| = |K_1|$. Let p be a maximal prime divisor of the order of K, and let S and S_1 be the Sylow p-subgroups of K and K_1 respectively. Then S and S_1 are normal subgroups of K and K_1. Also, S and S_1 are the Sylow subgroups in G and so $S = gS_1g^{-1}$ for some $g \in G$. In the group $L =< K, gK_1g^{-1} >$, its subgroup S is normal, because it is normal in K and gK_1g^{-1}. By the induction hypothesis K/S and gK_1g^{-1}/S are conjugated in L/S, which implies $K = hgK_1(hg)^{-1}$ for some $h \in G$ and we have $K \subseteq hgH(hg)^{-1}$. □

Corollary 1. *Let G be a finite group and H and K its supersoluble Hall subgroups. If $|K| = |H|$, then K and H are conjugated.*

Theorem 5. *Let G be a finite group and H its supersoluble Hall subgroup. If $N_G(H) = S \times H$ for some Sylow p-subgroup S of G, then G is p-nilpotent.*

Proof. Let $a, b \in S$ and $gag^{-1} = b$ for some $g \in G$. Then H and gHg^{-1} are contained in $C_G(b)$. By the corollary we have that $tHt^{-1} = gHg^{-1}$ for some $t \in C_G(b)$ and so $g^{-1}t \in N(H)$. Since $N(H) = S \times H$ it follows that $g^{-1}t = sh$, $s \in S$ and $h \in H$, which implies $a = g^{-1}bg = sbs^{-1}$. By Theorem 4 we conclude that G is p-nilpotent. □

Proof of Theorem 3. It is clearly enough to prove that G is p-nilpotent for any p that divides the order of H. Let N be a normalizer of S, where S is a Sylow p-subgroup of H. Then $H < N$. If Q is a p-complement of S in H, then H is a normalizer of Q in N. Really, if $gQg^{-1} = Q$ for some $g \in N$, then $gHg^{-1} = H$ and so $g \in H$. By Theorem 5 we have that N is p-nilpotent and therefore (since S is L-local) G is p-nilpotent too. □

We shall now give a criterion for non-simplicity, based on the following theorem:

Theorem 6.(Glauberman) *Let G be a finite group and S its Sylow p-subgroup for $p > 5$. If $N_G(S)/C_G(S)$ is a p-group then $O^p(G) \neq G$.*

We prove the following:

Theorem 6'. *Let G be a finite group, S its Sylow p-subgroup, and let H be*

a supersoluble Hall subgroup of G such that $\pi(H) \subseteq p'$ and $[S,H] = \{1\}$. If $N_G(S \times H) = S \times H$ and $p > 5$ then $O^p(G) \neq G$.
Proof. If $L = N_G(S)$, then $H < L$ and $N_L(H) = S \times H$. By Theorem 5 L is p-nilpotent, so, $N_G(S)/C_G(S)$ is a p-group. Then the theorem follows from Theorem 6. □

Corollary 2: *Let H be a nilpotent, self-normalizing, Hall subgroup of a finite group G. If p is a prime divisor of $|H|$ and $p > 5$ then $O^p(G) \neq G$.*

Let G be a finite soluble group. Then G contains self-normalizing nilpotent subgroup known as the Carter subgroup. We are going to prove a theorem anologous to Theorem 3 in which group H (from Theorem 3) is not necesserely Hall subgroup of G. We need the following result (see [3]):

Theorem 7. *Let G be a p-soluble group, and Q its p'-subgroup. If Q is centralized with some p-Sylow sugroup of G, then $Q < O_{p'}(G)$.*

Theorem 8. *Let G be a finite soluble group and C its Carter subgroup. If S is L-local Sylow p-subgroup of C, which is also a Sylow subgroup of G, then G is p-nilpotent.*

Proof. Let $N = N_G(S)$. We use induction on the order of G. If $N = S = C$ the theorem follows immediately from the local theorem. If $N \neq S$ then p-complement of S in C is not trivial and is contained in $O_{p'}(G)$ (Theorem 7). Hence, $O_{p'}(G)$ is not trivial. Applying the induction hypothesis on the group $G/O_{p'}(G)$, we obtain a group $K < G$ such that $K/O_{p'}(G)$ is a normal p-complement of $G/O_{p'}(G)$. But then K is a normal p-complement in G and the theorem is proved. □

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