NOVI SAD J. MATH.
VOL. 34, NO. 1, 2004, 33-37

NONCOMMUTING f–CONTRACTION MAPPINGS

Tayyab Kamran[1]

Abstract. Some fixed and coincidence point theorems for R-weakly commuting mappings are obtained. Our results generalize some recent results of Daffer and Kaneko [2] and many of the others.

AMS Mathematics Subject Classification (2000): 54H25

Key words and phrases: metric space, fixed point, commuting map

1. Introduction and preliminaries

The study of fixed points of multivalued contraction mappings was initiated by Nadler [10] in 1969. Since then there has been a lot of activity in this area and a number of generalization of Nadler's contraction principle have appeared (see, for example, the work of Dube and Singh [3], Iseki [5], Ray [12], Itoh and Takahashi [6], Aubin and siegel [1], Hu [4], Massa [9], Kaneko [7], etc). Most of the theorems deal with commuting or with noncommuting mappings. Recently Pant [11] introduced the notion of R-weak commutativity and proved two common fixed point theorems for a pair of mappings. The purpose of this paper is to extend the idea of Pant to multivalued mappings and obtain some coincidence and fixed point theorems for R-weakly commuting multivalued mappings. These theorems generalize many existing fixed point theorems.

Throughout this paper X denotes a metric space with metric d, f a continuous self map of X. For $x \in X$ and $A \subseteq X$, $d(x, A) = \inf\{d(x, y) : y \in A\}$. A subset A of X is said to be proximal [2] if, for each $x \in X$, there exists an element $a \in A$ such that $d(x, a) = d(x, A)$. We denote by $P(X)$ the class of nonempty proximal subset of X, by $F(X)$ the class of all nonempty closed subsets of X and by $K(X)$ the class of all nonempty compact subsets of X. Let H be the Hausdorff metric with respect to d, that is, $H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}$ for every $A, B \in F(X)$. Let $T : X \to F(X)$ be a mapping. Then a sequence $\{x_n\}$ in X is said to be an f-orbit of x under T [2] if, $f(x_n) \in Tx_{n-1}$, $x = x_0$. We assumed that $TX \subseteq fX$. An f-orbit of x under T is said to be (i) regular [7] provided that for each $n, d(fx_n, fx_{n+1}) \leq H(Tx_{n-1}, Tx_n)$, and $d(fx_n, fx_{n+1}) \leq d(fx_{n-1}, fx_n)$; (ii) strongly regular [4] if, T maps x into a proximinal set Tx, and for each n, $d(fx_n, fx_{n+1}) = d(fx_n, Tx_n)$. A point $p \in X$ is said to be a fixed point of $T : X \to F(X)$ if $p \in Tp$. The point p is called a coincidence point of f and T

[1]Department of Mathematics, University of Manchester, Oxford Road, Manchester, M13 9PL, United Kingdom

if $fp \in Tp$. A mapping $T : X \to F(X)$ is said to be an f-contraction [2], where $f : X \to X$ is continuous, if for $x, y \in X$, $H(Tx, Ty) \leq d(fx, fy)$ whenever $fx \neq fy$. Note that this definition alone is not sufficient to ensure that T is continuous [2] (cf. Remark 1).

Definition 1.1. *The mappings $f : X \to X$ and $T : X \to F(X)$ are said to be R-weakly commuting if, for all $x \in X, fTx \in F(X)$ and there exists some positive real number R such that*

$$H(fTx, Tfx) \leq Rd(fx, Tx).$$

It is clear that weakly commuting mappings are R-weakly commuting. Nevertheless, R-weakly commuting maps are weakly commuting only when $R \leq 1$. If T is a single valued self mapping of X, this definition of R-weak commutativity reduces to that of Pant [11].

2. Main results

The following contains generalization of coincidence point theorem of Daffer and Kaneko [2], and those contain therein.

Theorem 2.1. *Let X be a connected metric space, $T : X \to F(X), f : X \to X$, are R-weakly commuting mappings, where T is a point closed mapping which is an f-contraction, f is continuous such that $T(X) \subseteq f(X)$. If for some $x \in X$ an f-orbit of x is regular and contains a subsequence $\{x_{n_k}\}$ such that $fx_{n_k} \to t_0$ and $fx_{n_k+1} \to t_1$, then $t_0 = t_1$ and $ft_0 \in Tt_0$.*

Proof. Suppose that $t_0 \neq t_1$. Then it follows from Corollary 1, and Lemma 7 of Daffer and Kaneko [2] that there is a constant $h < 1$ and a neighborhood U of $f^{-1}t_0 \times f^{-1}t_1$ such that, for all $(x, y) \in U$,

$$H(Tx, Ty) \leq hd(fx, fy)$$

and

$$d(fx, fy) > \frac{1}{2}d(t_0, t_1). \tag{1}$$

Since f is continuous, $(x_{n_k}, x_{n_k+1}) \in U$ for all sufficiently large k; thus for some N, $k \geq N$ implies that

$$d(fx_{n_k}, fx_{n_k+1}) > \frac{1}{2}d(t_0, t_1)$$

and

$$H(Tx_{n_k}, Tx_{n_k+1}) \leq hd(fx_{n_k}, fx_{n_k+1}).$$

Since the orbit of x is regular, we get $d(fx_{n_k+1}, fx_{n_k+2}) \leq hd(fx_{n_k}, fx_{n_k+1})$. Again, using regularity, if $i > j > N$, then

$$d(fx_{n_i}, fx_{n_i+1}) \leq d(fx_{n_i-1}, fx_{n_i}) \leq d(fx_{n_i-1}, fx_{n_{i-1}+1}).$$

Inductively, we obtain $d(fx_{n_i}, fx_{n_i+1}) \leq h^{i-j} d(fx_{n_j}, fx_{n_j+1})$, so that upon fixing j, we get $d(fx_{n_i}, fx_{n_i+1}) \to 0$ as $i \to \infty$, contradicting (1). Hence $t_0 = t_1$. Since $fx_{n_k} \in Tx_{n_k}$, T is closed valued, f and T are R-weakly commuting therefore we have,

$$\begin{aligned} d(Tfx_{n_k}, ffx_{n_k+1}) &\leq H(Tfx_{n_k}, fTx_{n_k}) \\ &\leq Rd(fx_{nk}, Tx_{n_k}) \leq Rd(fx_{n_k}, fx_{n_k+1}) \to 0, \text{as } n \to \infty. \end{aligned}$$

Thus $d(Tt_0, ft_0) = 0$. This implies $ft_0 \in Tt_0$. □

Remark 2.2. **[2]** *Conclusion of Theorem 2.1 remains valid if we replace regular f-orbit by strongly regular f-orbit, and $F(X)$ by $P(X)$.*

Lemma 2.3. **[13]** *If $T : X \to K(X)$ is upper semi-continuous, $x_n \to x_0$ and $y_n \in Tx_n$ for each n, then there exists a subsequence $\{y_{n_k}\}$ which converges to a point in Tx_0.*

Lemma 2.4. **[13]** *If $T : X \to K(X)$ is an f-contractive, then T is upper semi-continuous.*

Lemma 2.5. *Let $f : X \to X$, $T : X \to F(X)$ are R-weakly commuting mappings, where T is an f contraction. Suppose given $x, s \in X$, that $f^n x \to s$ and $fx \in TX$. Then $s \in Ts$.*

Proof. $d(s, Ts) \leq d(s, f^n x) + d(f^n x, Ts)$ and $d(s, f^n x) \to 0$. Since f is continuous we have $fs = s$, and for given $x \in X$, we have $fx \in Tx$ this implies, $f^2 x \in fTx$. Since f, T are R-weakly commuting we have

$$d(Tfx, f^2 x) \leq H(Tfx, fTx) \leq Rd(Tx, fx) = 0,$$

this implies $f^2 x \in Tfx$. It further implies that,

$$d(Tf^2 x, f^3 x) \leq H(Tf^2 x, fTfx) \leq Rd(Tfx, ffx) = 0,$$

this gives $f^3 x \in Tf^2 x$. Continuing in this manner we have $f^n x \in Tf^{n-1} x$, for all $n \in N$. This yields

$$d(f^n x, Ts) \leq H(Tf^{n-1} x, Ts) \leq d(f^n x, fs) = d(f^n x, s) \to 0,$$

as $n \to \infty$. Hence $d(s, Ts) = 0$. □

Theorem 2.6. *Let X be a connected metric space, $T : X \to K(X)$, $f : X \to X$, are R-weakly commuting mappings, where T is a point compact mapping which is an f-contraction, f is continuous such that $T(X) \subseteq f(X)$. Assume that there exists $x \in X$ with a strongly regular f-orbit possessing a cluster point. Moreover, assume that $fx \in Tx$ implies that $\lim_{n \to \infty} f^n x$ exists. Then T has a fixed point, which is also a fixed point of f.*

Proof. By assumption, we may choose a strongly regular f-orbit $\{x_n\}$ under T having a cluster point, say x^*. Then there exist a subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} x_{n_k} = x^*$. By definition of an f-orbit, $fx_{n_k} \subseteq Tx_{n_k}$. Using Lemmas 2.3 and 2.4, we may find a further subsequence $fx_{n_{k_j}} \subseteq fx_{n_k}$ such that $\lim_{j\to\infty} fx_{n_{k_j}+1} \equiv y^* \in Tx^*$. Since f is continuous, $\lim_{k\to\infty} fx_{n_k} = fx^*$ and consequently $\lim_{j\to\infty} fx_{n_{k_j}} = fx^*$. By virtue of Remark 2.2, $y^* = fx^*$ giving $fx^* \in Tx^*$. Since f and T are R-weakly commuting, therefore it follows from Lemma 2.5, that $f^n x^* \in Tf^{n-1}x^*$, by taking $\lim_{n\to\infty}$, we get $\lim_{n\to\infty} f^n x^* = s \in Ts$. □

Remark 2.7. *Theorem 2.6 generalizes many important fixed point theorems (see for instance, Kaneko [7], Daffer and Kaneko [2], Smithson [13]).*

Example 2.8. *Let $X = [1,\infty)$ and d the usual metric on X. Define $f : X \to X$ and $T : X \to F(X)$ by $fx = 2x^3 - 1$, $Tx = [1, 2x-1]$ for all $x \in X$. Then, for any $x \in X$,*

$$d(fx, Tx) = 2x(x-1)(x+1) \quad H(fTx, Tfx) = 12x(x-1)^2;$$

that is,

$$H(fTx, Tfx) \leq 6d(fx, Tx).$$

Thus the mappings $f : X \to X$ and $T : X \to CB(X)$ are R-weakly commuting with $R = 6$ but they are not weakly commuting (e.g., at $x = 2$). Now it is easily seen that f and T satisfy all the conditions of Theorem 2.6 and have a common fixed point $x = 1$. Note that f and T do not satisfy the conditions of theorems in [2], [7], and [13].

Remark 2.9. **[2]** *If T is assumed to be continuous, then the metric space X need not be connected.*

References

[1] Aubin, P., Siegel, J., Fixed points and statinary points of dissipative multivalued maps. Proc. Amer. Math. Soc. 78(1980), 391–398.

[2] Daffer, P., Kaneko, H., Multivalued f–contractive mappings. Boll. U. M. I. (7)8-A(1994), 233–241.

[3] Dube, L. S., Singh, S. P., On multivalued contraction mappings. Bull. Math. Soc. Sci. Math. R. S. Roumanie, 14(62)3(1970), 307–310.

[4] Hu, T., Fixed point theorems for multivalued mappings. Canad. Math. Bull. 23(1980), 193–197.

[5] Iseki, K., Multivalued contraction mappings in complete metric spaces. Rend. Sem. Math. Univ. Padova, 53(1975), 15–19.

[6] Itoh, S., Takahashi, W., Single valued mappings, multivalued mappings, and fixed point theorems. J. Math. Anal. and Appl. 59(1977), 514–521.

[7] Kaneko, H., Single valued and multivalued f–contractions. Boll. U. M. I. 4A(1985), 29–33.

[8] Kaneko, H., Fixed points for contractive multivalued mappings. Bull. Inst. Math. Acad. Sinica, 14(1986), 141–145.

[9] Massa, S., Multi-applications du-type de-Kannan, Fixed point theory. Lect. Notes in Math. (Springer Verlag) 886(1981), 265–269.

[10] Nadler, Jr., S. B., Multivalued contraction mappings. Pacific, J. Math. 30(1969), 475–488.

[11] Pant, R. P., Common fixed points of noncommuting mappings. J. Math. Anal. Appl. 188(1994), 436–440.

[12] Ray, B. K., On Ciric's fixed point theorem. Fund. Math. 94(1977), 221–229.

[13] Smithson, R., Fixed points for contractive multifunctions. Proc. Amer. Math. Soc. 27(1971), 192–194.

Received by the editors May 1, 2002.