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A NOTE ON FUZZY CHARACTERISTIC AND FUZZY DIVISOR OF ZERO OF A RING

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Abstract. In this note, we introduce the notions of fuzzy characteristic and fuzzy divisor of zero of a ring, and investigate some properties of these notions in connection with fuzzy ideals of a ring.

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1. Introduction

In [1], the notion of fuzzy orders of elements of a group was introduced, and it was shown that some basic properties of orders of elements in the group theory are valid in the fuzzy subgroup theory when orders of elements are replaced by fuzzy orders of elements.

In this short communication, we have introduced the concepts of fuzzy characteristic and fuzzy divisor of zero of a ring. By using these concepts we have shown that some basic properties in the ring theory relating to the characteristic, divisor of zero and orders of elements in an additive group of a ring are valid in the fuzzy ideal theory when characteristic of the ring is replaced by fuzzy characteristic, divisor of zero is replaced by fuzzy divisor of zero and orders of elements are replaced by fuzzy orders of elements.

2. Fuzzy characteristic and Fuzzy divisor of zero

First we recall some basic definitions

Definition 2.1. [2] *A fuzzy subset A of a ring R is called a fuzzy ideal of R iff for all $x, y \in R$,*

i) $A(x-y) \geq \min(A(x), A(y))$,

ii) $A(xy) \geq \max(A(x), A(y))$.

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Example of a fuzzy ideal.

Let Z be the ring of integers. Define a fuzzy subring A of Z as follows: $A(x) = 1$ if $x \in \langle 6 \rangle$, $A(x) = .5$ if $x \in \langle 2 \rangle - \langle 6 \rangle$, and $A(x) = 0$ otherwise, where $\langle x \rangle$ denotes the ideal of Z generated by $x \in Z$. It can be easily verified that A is a fuzzy ideal of Z.

Definition 2.2. [1] *Let A be a fuzzy subgroup of a group G with the identity e. For each $x \in G$, the least positive integer n such that $A(x^n) = A(e)$ is called the fuzzy order of x with respect to A (briefly, $FO_A(x)$). If no such n exists, x is said to have infinite fuzzy order with respect to A.*

Definition 2.3. *Let A be a fuzzy subring of a ring R. The least positive integer m satisfying $A(mx) = A(o)$ for all x in R is said to be fuzzy characteristic of the ring R with respect to A (briefly, $FC_A(R)$), where o is the zero of the ring. R is said to be of fuzzy characteristic* 0 *(or infinity) with respect to A iff $m = 0$ is the only integer such that $A(mx) = A(o)$ for all x in R.*

If $FC_A(R)$ is m, then $FO_A(x)$ is finite for any element x in the additive group of R. In case $FC_A(R)$ is 0, then for any positive integer n there exists an element x in the additive group of R such that $FO_A(x)$ is greater than n.

In Definition 2.3, if $\{x \in R : A(x) = A(o)\} = \{o\}$ then $FC_A(R)$ is the characteristic of R. Thus the notion of fuzzy characteristic of a ring is the generalized and fuzzified notion of ordinary characteristic of the ring.

Theorem 2.1. *Let A be a fuzzy ideal of the ring R with an identity e. Then $FC_A(R)$ is equal to the $FO_A(e)$ in the additive group of R.*

Proof. First, let us assume that $FO_A(e)$ is finite and equal to m. Then we have $A(me) = A(o)$.

Let $x \in R$. We find

$$A(mx) = A((me)x) \geq \max(A(me), A(x)) = \max(A(o), A(x)) = A(o)$$

which implies $A(mx) = A(o)$ for all $x \in R$. Hence $FC_A(R)$ is m. On the other hand, it is obvious that $FC_A(R)$ is 0 in case $FO_A(e)$ is infinite.

This proves the theorem. □

Definition 2.4. *Let A be a fuzzy subring of a ring R. Further, let $a, b \in R$ such that $A(ab) \neq A(o)$ and $A(a) \neq A(o)$, $A(b) \neq A(o)$. Then a is called a left fuzzy divisor of zero and b is called a right fuzzy divisor of zero with respect to A.*

In Definition 2.4, if $\{x \in R : A(x) = A(o)\} = \{o\}$ then the fuzzy divisor of zero with respect to A is the divisor of zero. Thus the notion of fuzzy divisor of zero of a ring is the generalized and fuzzified notion of ordinary divisor of zero of the ring.

Theorem 2.2. *Let A be a fuzzy ideal of a ring R with no fuzzy divisor of zero with respect to A and $a, b \in R$ be such that $A(a) \neq A(o)$ and $A(b) \neq A(o)$. Then $FO_A(a) = FO_A(b)$ in the additive group of R.*

Proof. Let us assume that $FO_A(a)$ is finite and equal to m. It suffices to show that m is the least positive integer such that $A(mb) = A(o)$. For this purpose we have

$$\begin{aligned} A(a(mb)) &= A(m(ab)) = A((ma)b) \\ &\geq \max(A(ma), A(b)) = \max(A(o), A(b)) = A(o)\,, \end{aligned}$$

which shows $A(a(mb)) = A(o)$. Since R has no fuzzy divisor of zero with respect to A and $A(a) \neq A(o)$, this implies $A(mb) = A(o)$.

On the other hand, if $A(nb) = A(o)$ for some positive integer $n < m$, we find

$$\begin{aligned} A((na)b) &= A(n(ab)) = A(a(nb)) \\ &\geq \max(A(a), A(nb)) = \max(A(a), A(o)) = A(o). \end{aligned}$$

This gives $A((na)b) = A(o)$. Since R has no fuzzy divisor of zero with respect to A and $A(b) \neq A(o)$, we must have $A(na) = A(o)$. This leads to a contradiction. Hence $FO_A(b) = m$. This completes the proof. □

Theorem 2.3. *Let A be a fuzzy ideal of the ring R with no fuzzy divisor of zero with respect to A and $FC_A(R) \neq 0$. If $A(a) \neq A(o)$ for some $a \in R$, then $FC_A(R)$ is a prime number.*

Proof. Let $FC_A(R)$ be m and $A(a) \neq A(o)$ for $a \in R$. Then by Theorem 2.2 and definition of fuzzy characteristic, $FO_A(a)$ is m. Let m be represented as the product $p\,q$ of two positive integers p and q. Then we have

$$\begin{aligned} A((pa)(qa)) &= A(pqa^2) = A((ma)a)) \\ &\geq \max(A(ma), A(a)) = \max(A(o), A(a)) = A(o) \end{aligned}$$

which implies $A((pa)(qa)) = A(o)$. Since R has no fuzzy divisor of zero with respect to A, we must have $A(pa) = A(o)$ or $A(qa) = A(o)$. Since $FO_A(a)$ is m, one of the two integers p and q must be m, and hence other must be one. This shows that m is a prime. □

The following corollary is immediate consequence of Theorems 2.2 and 2.3.

Corollary 2.1. *Let A be a fuzzy ideal of the ring R with no fuzzy divisor of zero with respect to A. Then $FC_A(R)$ is 0 iff $FO_A(a)$ is infinite for every $a \in R$ with $A(a) \neq A(o)$; otherwise, $FC_A(R)$ is a prime p and $FO_A(a)$ is p for every $a \in R$ with $A(a) \neq A(o)$.*

Definition 2.5. *Suppose A be a fuzzy subring of a ring R with identity e such that $0 \neq A(e) \neq A(o)$. An element $a \in R$ with $A(a) \neq A(o)$ is said to be a fuzzy unit of R with respect to A (briefly, $FU_A(R)$) if there exists $b \in R$ with $A(b) \neq A(o)$ such that $A(ab) = A(ba) = A(e)$.*

Definition 2.6. [5] *Let $'\bullet'$ be a binary operation in a nonempty set X. A fuzzy subset P of X is said to be translational invariant with respect to $'\bullet'$ if for any $x, y \in X$, $P(x) = P(y) \Rightarrow P(x \bullet a) = P(y \bullet a)$ and $P(a \bullet x) = P(a \bullet y)$ for all $a \in X$.*

Theorem 2.4. *Suppose A is a fuzzy ideal of the ring R with identity e such that $0 \neq A(e) \neq A(o)$ and A is translational invariant with respect to multiplication. If the set of all elements of R which are not $FU_A(R)$ form an additive subgroup of $(R, +)$, then $FC_A(R)$ is either 0 or else a power of prime.*

Proof. We assume that the set of all elements of R which are not $FU_A(R)$ form an additive subgroup of $(R, +)$. Suppose $FC_A(R) = n \neq 0$ and n has two distinct prime divisors p, q. Now $A(pe) \neq A(o)$ and $A(qe) \neq A(o)$. We claim that pe is not a $FU_A(R)$. For, if pe is a $FU_A(R)$ then there exists $u \in R$ such that $A(u) \neq A(o)$ and $A(pe)u) = A(e)$. Then $A(((mq)e)(pe)u) = A(((mq)e)e) = A(mq(e))$ since A is translational invariant with respect to multiplication, where m is a natural number such that $n = mpq$.

Thus we find

$$\begin{aligned} A((mq)e) &= A(((mpq)e)u) = A((ne)u) \\ &\geq \max(A(ne), A(u)) = \max(A(o), A(u)) \,. \end{aligned}$$

This shows that $A((mq)e) = A(o)$ which is a contradiction since $mq < n$. Hence pe is not a $FU_A(R)$. Similarly qe is not a $FU_A(R)$. Since pe and qe are not $FU_A(R)$, any integral linear combination of them is not a $FU_A(R)$ by assumption. But now we have $ps + qt = 1$ for some integers s and t since p, q are coprimes and hence

$$A((s(pe) + t(qe))e) = A(e) \neq A(o) \,.$$

This shows that $s(pe) + t(qe)$ is a $FU_A(R)$, which is a contradiction. Hence n must be a power of a prime. This completes the proof. □

Definition 2.7. [4] *Let A and B be fuzzy subsets of the sets G and H, respectively. The product of A and B, denoted by $A \times B$, is the function defined by setting for all x in G and y in H,*

$$(A \times B)(x, y) = \min(A(x), A(y)) \,.$$

We state the following theorem whose proof is straightforward.

Theorem 2.5. *If A and B are fuzzy subrings of the rings R and S, respectively, then*

$$\begin{aligned} FC_{A\times B}(R \times S) &= 0\,, \text{ if } FC_A(R) \text{ is } 0 \text{ and } FC_B(S) \text{ is } 0\,, \\ &= p\,, \text{ if } FC_A(R) \neq 0 \text{ and } FCB(S) \neq 0\,, \end{aligned}$$

where $p = lcm(FC_A(R), FC_B(S))$.

3. Fuzzy Homomorphism and Fuzzy Isomorphism

Mordeson and Malik [3] introduced the concepts of *fuzzy homomorphisms* and fuzzy isomorphisms of *rings* as follows:

Let $\theta \in [0,1)$.

Let R and S be nonempty sets. Consider the following conditions on a *fuzzy subset* f of the *Cartesian product* $R \times S$:

(1) $\forall x \in R$, $\exists y \in S$ such that $f(x,y) > \theta$;

(2) $\forall y \in S$, $\exists x \in R$ such that $f(x,y) > \theta$;

(3) $\forall x \in R$, $\forall y_1, y_2 \in S$, $f(x,y_1) > \theta$ and $f(x,y_2) > \theta$ implies $y_1 = y_2$;

(4) $\forall x_1, x_2 \in R$, $\forall y \in S$, $f(x_1,y) > \theta$ and $f(x_2,y) > \theta$ implies $x_1 = x_2$.

Let f be a *fuzzy subset* of $R \times S$. If conditions (1) and (3) hold, then f is called a *fuzzy function* of R into S. If conditions (1), (2) and (3) hold, then f is called a *fuzzy function* of R onto S. If conditions (1), (3) and (4) hold, then f is called a *one-to-one fuzzy function* of R into S.

A fuzzy function f from a ring R into a ring S is called fuzzy homomorphism if and only if

$$\begin{aligned}
&\forall x_1, x_2 \in R\,,\ \forall y \in S\\
&f(x_1 + x_2, y) \geq \sup\{\min(f(x_1,y_1), f(x_2,y_2) : y_1, y_2 \in S, y = y_1 + y_2)\};\\
&f(x_1 x_2, y) \geq \sup\{\min(f(x_1,y_1), f(x_2,y_2) : y_1, y_2 \in S, y = y_1 y_2)\}.
\end{aligned}$$

If a fuzzy homomorphism from R onto S is one–to–one, then it is called a fuzzy isomorphism of R onto S.

Definition 3.1. *Suppose f is a fuzzy homomorphism from a ring R onto a ring S and A is a fuzzy subset of R, then f is called a fuzzy isomorphism with respect to A iff*

$$\begin{aligned}
&\forall x_1, x_2 \in R\,,\ \forall y \in S\\
&f(x_1, y) > 0 \ \ \textit{and}\ f(x_2, y) > 0 \ \ \textit{implies}\ A(x_1) = A(x_2)\,.
\end{aligned}$$

Theorem 3.1. *Suppose A is a fuzzy ideal of a ring R with identity e such that $A(e) > 0$. If $FC_A(R)$ is 0, then the additive subgroup of R generated by the identity e is fuzzy isomorphic to the integral domain* **Z** *of all integers.*

Proof. Let $FC_A(R)$ be 0. Then $FO_A(e)$ in the additive group of R is infinite. Let R^* denote the additive subgroup of R generated by e. Since A is a fuzzy ideal of R we have $0 < A(e) \leq A(x) \leq A(o)$ for all $x \in R$. We also note that

$A(ne) < A(o)$ for any $n \in \mathbf{Z}$ by assumption. We define a fuzzy subset f of the Cartesian cross-product $R^* \times \mathbf{Z}$ as follows:

$$\begin{aligned} f(ne, n) &= A(ne) \text{ for all integers } n \in \mathbf{Z}, \\ f(ne, m) &= 0 \text{ for all integers } n, m \in \mathbf{Z} \text{ with } n \neq m. \end{aligned}$$

From the definition of f it clearly follows that f is a one–to–one fuzzy function of R^* onto Z.

Let $n_1e, n_2e \in R^*$ and $n \in \mathbf{Z}$. If $n = n_1 + n_2$, then

$$\begin{aligned} f(n_1e + n_2e, n) &= f((n_1 + n_2)e, n_1 + n_2) = A(n_1e + n_2e) \\ &\geq \inf(A(n_1e), A(n_2e)) = \inf(f(n_1e, n_1), f(n_2e, n_2)). \end{aligned}$$

If $n \neq n_1+n_2$, we suppose $n = m_1+m_2$, where $m_1, m_2 \in \mathbf{Z}$, then either $n_1 \neq m_1$ or $n_2 \neq m_2$. Let $n_1 \neq m_1$. In this case $f(n_1e+n_2e), n) = 0$ and $f(n_1e, m_1) = 0$ and so $\inf(f(n_1e, m_1), f(n_2e, m_2)) = 0$.

Thus we find that $\forall n_1e, n_2e \in R^*$ and $\forall n \in \mathbf{Z}$,

$$\begin{aligned} &f((n_1e)(n_2e), n) \geq \\ &\quad \geq \sup\{\inf(f(n_1e, m_1), f(n_2e, m_2)) : n = m_1 + m_2, m_1, m_2 \in \mathbf{Z}\}. \end{aligned}$$

Now if $n = n_1n_2$, then

$$\begin{aligned} f((n_1e)(n_2e), n) &= f((n_1n_2)e, n_1n_2) = A((n_1n_2)e) = A((n_1e)(n_2e)) \\ &\geq \sup(A(n_1e), A(n_2e)) = \sup(f(n_1e, n_1), f(n_2e, n_2)). \end{aligned}$$

If $n \neq n_1n_2$, suppose $n = m_1m_2$ where $m_1, m_2 \in \mathbf{Z}$, then either $n_1 \neq m_1$ or $n_2 \neq m_2$. In this case we observe $f((n_1e)(n_2e), n) = 0$ and $f(n_1e, m_1) = 0$, if $n_1 \neq m_1$. So $\inf(f(n_1e, m_1), f(n_2e, m_2)) = 0$. Thus we find that $\forall n_1e, n_2e \in R^*$ and $\forall n \in \mathbf{Z}$,

$$\begin{aligned} &f((n_1e)(n_2e), n) \geq \\ &\quad \geq \sup\{\inf(f(n_1e, m_1), f(n_2e, m_2)) : n = m_1m_2, m_1, m_2 \in \mathbf{Z}\}. \end{aligned}$$

Hence f is a fuzzy isomorphism of R^* onto Z. Thus the proof is completed. □

Theorem 3.2. *Suppose A is a fuzzy ideal of a ring R with identity e such that $A(e) > 0$ and $A(e) \neq A(o)$. If R contains no fuzzy divisor of zero with respect to A and $FC_A(R) \neq 0$, then R^* is fuzzy isomorphic with respect to A to the field $\mathbf{Z}p$ of all integers* mod p*, for some prime p.*

Proof. Suppose R contains no fuzzy divisors of zero with respect to A and $FC_A(R) \neq 0$. Then $FC_A(R) \neq 0$ is a prime p. Since $A(e) \neq A(o)$, $FO_A(e)$ is p. We define a fuzzy subset g of the Cartesian cross-product $R^* \times \mathbf{Z}p$ as follows:

$g(me,(n)) = A(ne)$, where n is the least no–negative integer obtained as remainder when m is divided by p and $(n) \in \mathbf{Z}p$, and $g(me, (n_1)) = 0$ if $n_1 \neq n$, $(n_1) \in \mathbf{Z}p$.

It can be easily verified that g is a fuzzy homomorphism from R^* onto $\mathbf{Z}p$.

If $f(m_1e,(n)) > 0$ and $f(m_2e,(n)) > 0$, then there exist $r_1, r_2 \in \mathbf{Z}$ such that $m_1 = r_1p + n$ and $m_2 = r_2p + n$. This implies that $A(m_1e - m_2e) = A(((r_1 - r_2)p)e) = A(o)$. Which implies that $A(m_1e) = A(m_2e)$. Hence f is a fuzzy isomorphism with respect to A of R^* onto $\mathbf{Z}p$. □

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