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PARAMETER ESTIMATION FOR UNIFORM MAXIMUM PROCESS

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Abstract. Lewis and McKenzie have described the maximum process with marginal distribution $\mathcal{U}(\prime,\infty)$. In this paper, we discuss some properties of this process. We also apply some estimation methods for estimating the parameter of the process. It is shown that the conditional least squares estimator is strongly consistent and asymptotically normal.

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1. Introduction

The uniform maximum process was introduced by Lewis and McKenzie [4]. The process is defined by the equation

$$X_n = \alpha \max\{X_{n-1}, Z_n\}, \tag{1}$$

where $0 < \alpha < 1$, $\{Z_n\}$ is an innovation process of independent and identically distributed (i.i.d.) random variables chosen to ensure that $\{X_n\}$ is a stationary sequence whose marginal distribution is $\mathcal{U}(\prime,\infty)$ and the sequences $\{X_n\}$ and $\{Z_n\}$ are semi-independent, i.e. the random variables X_m and Z_n are independent iff is $m < n$.

The innovation process $\{Z_n\}$ is given by

$$Z_n = \begin{cases} 0, & \text{w.p. } \alpha, \\ 1 + \frac{1-\alpha}{\alpha} U_n, & \text{w.p. } 1-\alpha, \end{cases}$$

where $\{U_n\}$ is the sequence of i.i.d. random variables with $\mathcal{U}(\prime,\infty)$ distribution and the sequences $\{X_n\}$ and $\{Z_n\}$ are semi-independent. We can now write equation (1) as

$$X_n = \begin{cases} \alpha X_{n-1}, & \text{w.p. } \alpha, \\ \alpha + (1-\alpha) U_n, & \text{w.p. } 1-\alpha, \end{cases} \tag{2}$$

and conclude that the process $\{X_n\}$ is the first-order Markovian.

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2. Some properties of the uniform maximum process

In this section, we discuss some properties of the uniform maximum process, as the autocovariance and the autocorrelation functions are. We also discuss the regression and the conditional distribution function (transition distribution function).

Theorem 2.1. *The uniform maximum process has:*

(i) the real-valued absolutely summable autocovariance function

$$\gamma_X(j) = \alpha^{2j}/12, \ j = 0, 1, \ldots$$

(ii) the real-valued absolutely summable autocorrelation function

$$\rho_X(j) = \alpha^{2j}, \ j = 0, 1, \ldots.$$

By using (2) and the Markovian properties of the process $\{X_n\}$, the joint Laplace-Stieltjes transform of X_n and X_{n-1} can be obtained as

$$\begin{aligned} \Phi_{X_n, X_{n-1}}(s, t) &\equiv E\left(e^{-sX_n - tX_{n-1}}\right) \\ &= \alpha \Phi_X(\alpha s + t) + (1-\alpha) e^{-\alpha s} \Phi_U((1-\alpha)s) \Phi_X(t) \\ &= \alpha \frac{1 - e^{-(\alpha s + t)}}{\alpha s + t} + \frac{e^{-\alpha s} - e^{-s}}{s} \cdot \frac{1 - e^{-t}}{t}, \end{aligned}$$

which is not symmetrical in s and t. As a consequence, the process $\{X_n\}$ is not a time-reversible one.

The process $\{X_n\}$ has a conditional distribution function

$$P(X_n \le u | X_{n-1} = x) = \alpha I_{\{x \le u/\alpha\}} + (1-\alpha) \max\left(\frac{u-\alpha}{1-\alpha}, 0\right), \ 0 < u < 1.$$

The regression of X_n on $X_{n-1} = x$ and of X_{n-1} on $X_n = x$ follow the theorem:

Theorem 2.2. *Let $\{X_n\}$ be the uniform maximum process defined by* (2)*.*

(i) The regression of X_n on $X_{n-1} = x$ is

$$E(X_n | \ X_{n-1} = x) = \alpha^2 x + \frac{1-\alpha^2}{2}, \ x \in (0, 1).$$

(ii) The regression of X_{n-1} on $X_n = x$ is

$$E(X_{n-1} | \ X_n = x) = I^{-1}_{(0,1)}(x) \left[\frac{1}{\alpha} x I_{(0,\alpha)}(x) + \frac{1}{2} I_{(\alpha,1)}(x)\right].$$

(iii) From (i) and (ii) follows that the process $\{X_n\}$ is not time-reversible.

Proof. (i) The best linear regression of X_n on X_{n-1} by means of the conditional expectation follows

$$\begin{aligned} E(X_n|\ X_{n-1}=x) &= \alpha\cdot\alpha x+[\alpha+(1-\alpha)E(U_n)]\cdot(1-\alpha) \\ &= \alpha^2 x+\frac{1-\alpha^2}{2}. \end{aligned}$$

(ii) To obtain the regression of X_{n-1} on $X_n = x$ we differentiate (3) with respect to t, set $t \to 0+$, invert with respect to s and then divide by $-I_{(0,1)}(x)$. □

3. Random coefficient representation and conditional least squares estimation

Random coefficient representation gives linear form to the model (2). The uniform maximum process given by equation (2) can be well represented by the random coefficient model

$$X_n = A_n X_{n-1} + B_n W_n, \tag{3}$$

where the following conditions are satisfied:

(A_1) $\{W_n\}$ is the sequence of i.i.d. random variables with $\mathcal{U}(\infty, \infty/\alpha)$ distributions and W_n is independent of A_i and B_j for every n, i and j,

(A_2) $\{(A_n, B_n)\}$ is the sequence of i.i.d. random vectors distributions given by $P(A_n = \alpha, B_n = 0) = \alpha$ and $P(A_n = 0, B_n = \alpha) = 1-\alpha$.

(A_3) $\{X_n\}$ and $\{W_n\}$ are semi-independent,

(A_4) $\{X_n\}$ and $\{A_n\}$ are semi-independent,

(A_5) $\{X_n\}$ and $\{B_n\}$ are semi-independent.

Let $\mathcal{F}_\backslash$ be the σ-field generated by the set of vectors $\{(A_s, B_s, W_s),\ s \le n\}$. The following lemma will be needed to prove Theorem 3.1.

Lemma 3.1. *Under the conditions $(A_1)-(A_5)$, the random difference equation (3) has a unique, weakly and strictly stationary, $\mathcal{F}_\backslash$-measurable and ergodic solution of the form*

$$X_n = \sum_{i=0}^{\infty}\left(\prod_{j=0}^{i-1} A_{n-j}\right) B_{n-i} W_{n-i} + B_n W_n.$$

Proof. The Proof follows from Nicholls and Quinn [5] and Doob [3], page 458. □

We can now estimate the parameter α^2 using conditional least squares method. The equation (3) can be written as

$$Y_n = \alpha^2 Y_{n-1} + \epsilon_n, \tag{4}$$

where $Y_n = X_n - 1/2$ and $\epsilon_n = (A_n - \alpha^2)Y_{n-1} + B_n W_n + (A_n - 1)/2$.

Let $(X_1, \dots, X_N)$ be a sample of size N. If we translate each observation of this sample in the following way $Y_n = X_n - 1/2$, we obtain the sample $(Y_1, \dots, Y_N)$.

The conditional least squares estimator $\hat{\alpha}_N^2$ of the parameter α^2 is obtained by minimizing the function

$$S(\alpha) = \sum_{n=1}^{N} \left\{Y_n - \alpha^2 Y_{n-1}\right\}^2$$

with respect to α^2. So, it is of the form

$$\hat{\alpha}_N^2 = \frac{\sum\limits_{n=1}^{N} Y_n Y_{n-1}}{\sum\limits_{n=1}^{N} Y_{n-1}^2}.$$

The following theorem gives the limit distribution of the conditional least squares estimator $\hat{\alpha}_N^2$.

Theorem 3.1. *If the conditions* $(A_1)-(A_5)$ *are satisfied, then* $\hat{\alpha}_N^2$ *is a strongly consistent estimator for* α^2 *and* $\sqrt{N-1}(\hat{\alpha}^2 - \alpha^2)$ *has asymptotically normal distribution with zero mean and variance* $(5 + 4\alpha^3 - 9\alpha^4)/5)$, *i.e.* $\{\hat{\alpha}_N^2\}$ *is asymptotically normal with mean* α^2 *and variance* $(5 + 4\alpha^3 - 9\alpha^4)/(5(N-1))$.

Proof. It follows from Nicholls and Quinn [5] that $\hat{\alpha}_N^2$ is consistent and asymptotically normal estimate of α^2. □

The asymptotic distribution of the estimator $\hat{\alpha}_N$ follows from Theorem 3.2 and Proposition 6.4.1 (Brockwell and Davis [2]). It follows that $\hat{\alpha}_N$ is asymptotically normal with mean α and variance $(5 + 4\alpha^3 - 9\alpha^4)/(20\alpha^2(N-1))$.

4. Other estimation methods

In this section we give some other estimation methods to estimate the unknown parameter α.

Consider the probability $p = P\{X_n < X_{n-1}\}$. After a calculation we obtain that $p = (1 + \alpha^2)/2$. Let $\tilde{p}_N$ be the estimator of p given by

$$\tilde{p}_N = \frac{1}{N-1} \sum_{n=2}^{N} I\{X_n < X_{n-1}\}, \quad I\{X_n < X_{n-1}\} = \begin{cases} 1, & X_n < X_{n-1}, \\ 0, & X_n \geq X_{n-1}. \end{cases}$$

It is not hard to show that $\tilde{p}_N$ is an unbiased estimator for p. Also, using the Chebychev's inequality we can prove that $\tilde{p}_N$ is a consistent estimator for p. Finally, from Proposition 6.1.4 (Brockwell and Davis [2]), we have that $\tilde{\alpha}_N$, given by $\tilde{\alpha}_N = \sqrt{2\tilde{p}_N - 1}$, is a consistent estimator for α.

The fact that the transition distribution function has points of mass, which vary with the parameter α, is an important observation: It shows that the Fisher Information is ∞ and superfast estimators of α exist. Since $X_{i-1} < 1$, it will be $\alpha X_{i-1} < \alpha + (1-\alpha)U_i$ and $\min\{\alpha X_{i-1}, \alpha + (1-\alpha)U_i\} = \alpha X_{i-1}$. It implies that we can use

$$\tilde{\alpha} = \min_{1 \le i \le N} \left\{ \frac{X_i}{X_{i-1}} \right\}$$

as the estimator for α. It satisfies

$$P(\tilde{\alpha} \neq \alpha) = (1-\alpha)^n.$$

This is about the fastest convergence one can think of: The probability that the estimate is not equal to the true value decreases exponentially! A similar phenomenon can be found in situations where the unknown parameter may take only finitely many values.

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