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# A PROBABILISTIC GENERALIZATION OF INTEGRABILITY FOR POSITIVE FUNCTIONS

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**Abstract.** The purpose of this paper is to develop an integration theory for positive functions with respect to a probabilistic measure in the meaning of Šerstnev, using submeasures with probabilistic structures and the topological ring associated to them. The probabilistic measures are introduced for modelling those situations in which we have only probabilistic information about the measure of a set [7]. The point of view adopted to define the integral and the set of integrable functions belongs to Sion, Bartle, Dunford and Schwartz. It is shown that the classical theory of integration with respect to a positive measure is naturally included in our general case.

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## 1. Preliminary concepts

This paper uses the terminology and notations from [7] and [8].

Let $D_+$ be the family of all distribution functions $F$ such that $F(0) = 0$ (recall that $F$ is nondecreasing, left continuous and $\sup_{x>0} F(x) = 1$). By $\varepsilon_0$ we denote the function of $D_+$ such that $\varepsilon_0(x) = 1$, for all $x > 0$.

The mapping

$$\tau_M : D_+ \times D_+ \to D_+;\ \tau_M(F, G)(x) = \sup_{u+v=x} \mathrm{Min}(F(u), G(v))$$

is a triangular function, and $(D_+, \tau_M)$ is a commutative semigroup with unit $\varepsilon_0$. We denote by $(D_+, \tau_M, d_L)$ the uniform topological semigroup with respect to the modified Levy's metric $d_L$.

If $j$ denotes the identity function on $[-\infty, \infty]$, then for any $F$ in $D_+$ and $a > 0$, the distribution function in $D_+$, whose value is $F\left(\frac{x}{a}\right)$, for any $x \geq 0$, may be conveniently denoted by $F\left(\frac{j}{a}\right)$.

In [11], it is shown that

$$\tau_M\left(F\left(\frac{j}{a}\right), F\left(\frac{j}{b}\right)\right) = F\left(\frac{j}{a+b}\right),\ \ F \in D_+,\ \ a, b > 0.$$

In the sequel we make the conventions : $F\left(\frac{j}{0}\right) = \varepsilon_0$, $F\left(\frac{j}{\infty}\right) = \varepsilon_0$ and we denote $\overset{n}{\underset{i=1}{\tau_M}} = \tau_M(F_i) = \tau_M(F_1, \tau_M(F_2, \ldots, \tau_M(F_{n-1}, F_n)\ldots))$.

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**Definition 1.** *Let $S$ be a nonempty set and $\mathcal{S} \subset \mathcal{P}(\mathcal{S})$ an algebra of subsets. A set function $\mu : \mathcal{S} \to D_+$ is called probabilistic measure (in the meaning of Šerstnev) on $\mathcal{S}$ if:*

*($m_1$)* $\mu(A) = \varepsilon_0$, *iff* $A = \emptyset$.

*($m_2$)* $A, B \in \mathcal{S}$, $A \cap B = \emptyset$ *implies* $\mu(A \cup B) = \tau_M(\mu(A), \mu(B))$.

We will denote $\mu(A) = \mu_A, \mu(A)(x) = \mu_A(x)$, for any $x > 0$.

**Remark 1.** *Every real measure may be regarded as a probabilistic measure of special kind. If $\nu : \mathcal{S} \to I\!R_+$ is a measure, then using the function $\varepsilon_0$, a probabilistic measure on $\mathcal{S}$ can be defined by $\mu_A(x) = \varepsilon_0(x - \nu(A))$. The number $\mu_A(x)$ is interpreted as the probability that the measure $\nu(A)$ is less than $x$.*

In [8] we defined a general form of submeasure with a probabilistic $\nu$-structure in such a way that the topological ring of sets induced is a uniform space.

Let $\mathcal{V}$ be a family of subsets of $D_+$ with the properties:

($\nu_1$) $a \in \mathcal{V} \Longrightarrow \varepsilon_0 \in a$,

($\nu_2$) $u \in \mathcal{V}, v \in \mathcal{V} \Longrightarrow \exists \omega \in \mathcal{V}, \omega \subset u \cap v$,

($\nu_3$) $[G \in D_+, G \geq F, F \in u \in \mathcal{V}] \Longrightarrow G \in u$,

that is, $\mathcal{V}$ is a filter base at $\varepsilon_0$ on $D_+$ compatible with the relation $\geq$.

**Definition 2.** *Let $\mathcal{S}$ be a ring of subsets of a fixed set $S$. A mapping $\gamma : \mathcal{S} \to D_+$ such that:*

*($m_1$)* $A = \emptyset \Longleftrightarrow \gamma_A(x) = \varepsilon_0(x), \forall x > 0$

*($m_2$)* $A \subset B \Longrightarrow \gamma_A(x) \geq \gamma_B(x), \forall x > 0$

*($PSm_3$)* $(\forall) v \in \mathcal{V}, (\exists) u \in \mathcal{V}; \gamma_A \in u, \gamma_B \in u) \Longrightarrow \gamma_{A \cup B} \in v$

*is called submeasure with the probabilistic $\nu$-structure and $(\mathcal{S}, \gamma, \nu)$ will be called a ring with probabilistic $\nu$-structure. In* [8] *there were defined the submeasures with probabilistic $\beta$-structure, probabilistic $H$-submeasure, probabilistic $f$-submeasure and the $\nu$-Šerstnev submeasure.*

**Definition 3.** *If $\gamma : \mathcal{S} \to \mathcal{D}_+$ is a submeasure with probabilistic $\nu$-structure, then the Jordan extension of $\gamma$ is defined by*

$$\begin{aligned} \gamma^* &: \mathcal{P}(\mathcal{S}) \to D_+, \\ \gamma^*_A(x) &= \sup\{\gamma_E(x), A \subseteq E \in \mathcal{S}\}, \quad A \subset S. \end{aligned}$$

**Remark 2.** *Let $\Gamma = \{\gamma^i\}_{i \in I}$ be a family of submeasures with probabilistic $\nu$-structures on $\mathcal{S}$ and consider the family $\beta_\Gamma = \{\mathcal{V}_{K,v} : K = \text{finite} \subset I, v \in \mathcal{V}\}$ where $\mathcal{V}_{K,v} = \{A \in \mathcal{P}(S), \gamma^i_A \in v, i \in K\}$.*

*Then there exists a unique topology $\tau_\Gamma$ on $\mathcal{P}(S)$ so that $[\mathcal{P}(S)](\Gamma) = (\mathcal{P}(S), \cap, \tau_\Gamma)$ is a topological ring and $\beta_\Gamma$ is a normal base of neighborhoods of $\emptyset$.*

*We denote by $E_\alpha \xrightarrow{\Gamma} E$ the convergence from the space $[\mathcal{P}(S)](\Gamma)$.*

*We say that the set $E \in \mathcal{S}$ is $\Gamma$-finite ($\Gamma$-negligible) if for any $i \in I$, $\sup_{x>0} \gamma^i_E(x) = 1$ $\left(\gamma^i_E(x) = \varepsilon_0(x), x > 0, \text{ respectively}\right)$.*

## 2. Basic assumptions

Let $[0,\infty]$ be endowed with the usual additive operation and the natural topology.

Then $([0,\infty],\Sigma)$ is a compact and uniform semigroup, where $\Sigma$ is the uniform structure on $[0,\infty]$.

We fix a probabilistic measure $\mu : \mathcal{S} \rightarrow \mathcal{D}_+$ and choose a family of submeasures with probabilistic $\mathcal{V}-$structures, $\Gamma_\mu$, so that the following continuity axioms are satisfied:

$C_1$) For every $A \in \mathcal{S}$ and every entourage $W$ from $(D_+, \tau_M, d_L)$ there exists the entourage $\sigma \in \Sigma$ with the following property: if the sequence $\{(a_i, b_i)\}_{i=1}^n$ is from $\sigma$ and if $\{E_i\}_{i=1}^n$ is a sequence of pairwise disjoint sets from $\mathcal{S}$, then

$$\left( \underset{i=1}{\overset{n}{\tau_M}} \mu_{E_i \cap A}\left(\frac{j}{a_i}\right), \underset{i=1}{\overset{n}{\tau_M}} \mu_{E_i \cap A}\left(\frac{j}{b_i}\right)\right) \in \mathcal{W}$$

$C_2$) For any $a \in [0,\infty]$,

$$\lim_{\substack{E \overset{\Gamma_\mu}{\to} \emptyset \\ E \in \mathcal{S}}} \mu_E\left(\frac{j}{a}\right) = \epsilon_0.$$

Generalizing the model defined in [4] we introduce a uniform structure on $[0,\infty]^S$ in the following way:

To a finite $K \subset I, v \in \mathcal{V}$ and the entourage $\sigma \in \Sigma$ we associate the set $\mathcal{W}_K(v,\sigma) = \{(f,g) \in [0,\infty]^S \times [0,\infty]^S ; \gamma^{i*}_{\{s \in S : (f(s),g(s)) \notin \sigma\}} \in v, i \in K\}$.

The family $\{\mathcal{W}_K(v,\sigma); v \in \mathcal{V}\ \sigma \in \Sigma\}$ forms a fundamental system of entourages for a uniform structure $\mathrm{U}_{\Gamma_\mu}$ on $[0,\infty]^S$.

We denote $[0,\infty]^S(\Gamma_\mu) = \left([0,\infty]^S, \mathrm{U}_{\Gamma_\mu}\right)$.

If $\{f_\alpha\}$ is a generalized sequence from $[0,\infty]^S$, and if $\{f_\alpha\}$ converges to $f$ in $[0,\infty]^S(\Gamma_\mu)$, we say that $\{f_\alpha\}$ converges to $f$ in $\Gamma_\mu$ submeasures and denote by $f_\alpha \overset{\Gamma_\mu}{\longrightarrow} f$.

In the sequel we define the integrability of functions from the space $[0,\infty]^S$.

## 3. Integrable functions

**Definition 4.** *A step function $f \in [0,\infty]^S$ is $\Gamma_\mu$-integrable if the following conditions are fulfilled:*

*a) $f$ takes a finite number of distinct values $x_1, x_2, \dots, x_n$ on the sets $E_1, E_2$, $\dots E_n \in \mathcal{S}$ respectively; (we denote $f = \sum\limits_{i=1}^n x_i \chi_{E_i}$, where $\chi_E$ is the characteristic function of $E$)*

*b) If $x_i = \infty$ it results that $E_i$ is $\Gamma_\mu$-negligible.*

*For $E \in \mathcal{S}$ the $\Gamma_\mu$-integral of $f$ on $E$ is defined by*

$$\int_E f d\mu = \underset{i=1}{\overset{n}{\tau_M}} \mu_{E_i \cap E}\left(\frac{j}{x_i}\right).$$

We denote by $\mathcal{E}(\mathcal{S}, \Gamma_\mu)$ the $\Gamma_\mu$-integrable step functions set.

**Theorem 1.** (*i*) *Relatively to the operation of addition $(f+g)(s) = f(s) + g(s)$, $\mathcal{E}(\mathcal{S}, \Gamma_\mu)$ is a subsemigroup of $[0,\infty]^S$.*

(*ii*) *For $E \in \mathcal{S}$, the map $f \to \int_E f d\mu$ of $\mathcal{E}(\mathcal{S}, \Gamma_\mu)$ in $D_+$ is additive in the sense that*

$$\int_E (f+g)\, d\mu = \tau_M\left(\int_E f d\mu, \int_E g d\mu\right), f, g \in \mathcal{E}(\mathcal{S}, \Gamma_\mu).$$

(*iii*) *For $f \in \mathcal{E}(\mathcal{S}, \Gamma_\mu)$, the map $E \xrightarrow{\theta} \theta(E)$, that is denoting $\theta_E = \theta(E) = \int_E f d\mu$, we have*

$$\theta_{\bigcup\limits_{i=1}^{n} E_i} = \underset{i=1}{\overset{n}{\tau_M}} (\theta_{E_i}),\ E_i \cap E_j = \emptyset,\ i \neq j,$$

*and $\theta_\emptyset = \varepsilon_0$.*

(*iv*) *For $f \in \mathcal{E}(\mathcal{S}, \Gamma_\mu)$,* $\lim\limits_{E \overset{\Gamma_\mu}{\underset{E \in \mathcal{S}}{\to}} \emptyset} \theta_E = \lim\limits_{E \overset{\Gamma_\mu}{\underset{E \in \mathcal{S}}{\to}} \emptyset} \int_E f d\mu = \varepsilon_0.$

*Proof.* (i) Suppose that $f$ and $g$ are $\Gamma_\mu$-integrable step functions having the form $f = \sum\limits_{i=1}^{n} x_i \chi_{A_i}$ , $g = \sum\limits_{l=1}^{m} y_l \chi_{B_l}$. Then the values $z_1, z_2, \ldots, z_p$ of $f+g$ are found among the elements $x_i + y_l$, $1 \le i \le n$, $1 \le l \le m$ and $f+g = \sum\limits_{k=1}^{p} z_k \chi_{E_k}$, where $E_k$ is the union of all the sets $A_i \cap B_l$ for which $x_i + y_l = z_k$. If $z_k = \infty$ and $x_i + y_l = z_k$, then $E_k$ is $\Gamma_\mu$-neglijable. Thus $f+g$ is $\Gamma_\mu$-integrable.

If $P_k$ is the set of all pairs $(i,l)$ with $x_i + y_l = z_k$, then $A_i \cap B_l$ is a void if $(i,l)$ is in none of the sets $P_k$, $k = 1, 2, \ldots, p$ and hence

$$\begin{aligned}
\int_E (f+g)\, d\mu &= \underset{k=1}{\overset{p}{\tau_M}} \mu_{E \cap E_k}\left(\frac{j}{z_k}\right) \\
&= \underset{k=1}{\overset{n}{\tau_M}} \left[\underset{(\text{ i,l}) \in P_k}{\tau_M} \mu_{E \cap A_i \cap B_l}\right]\left(\frac{j}{z_k}\right) \\
&= \underset{i=1}{\overset{n}{\tau_M}} [\underset{l=1}{\overset{m}{\tau_M}} \mu_{E \cap A_i \cap B_l}]\left(\frac{j}{x_i + y_l}\right)
\end{aligned}$$

$$= \tau_M \left[ \underset{i=1}{\overset{n}{\tau_M}} \left[ \underset{l=1}{\overset{m}{\tau_M}} \mu_{E\cap A_i \cap B_l} \right] \left( \frac{j}{x_i} \right), \underset{i=1}{\overset{n}{\tau_M}} \left[ \underset{l=1}{\overset{m}{\tau_M}} \mu_{E\cap A_i \cap B_l} \right] \left( \frac{j}{y_l} \right) \right]$$

$$= \tau_M \left[ \underset{i=1}{\overset{n}{\tau_M}} \mu_{E\cap A_i} \left( \frac{j}{x_i} \right), \underset{l=1}{\overset{m}{\tau_M}} \mu_{E\cap B_l} \left( \frac{j}{y_l} \right) \right] = \tau_M \left[ \int_E f d\mu \int_E g d\mu \right].$$

(ii) Suppose that $f \in \mathcal{E}(\mathcal{S}, \Gamma_\mu)$, having the form $f = \sum_{l=1}^{m} x_l \chi_{A_l}$, $A_l \in \mathcal{S}$ and let $E = \cup E_i$, be, $E_i \cap E_j = \emptyset$, $i \neq j$, $E_i \in \mathcal{S}$. If

$$\theta_E = \theta(E) = \int_E f d\mu = \underset{l=1}{\overset{m}{\tau_M}} \mu_{E\cap A_l} \left( \frac{j}{x_l} \right)$$

then we have

$$\mu_{E\cap A_l} \left( \frac{j}{x_l} \right) = \mu_{\left(\overset{n}{\underset{i=1}{\cup}} E_i\right)\cap A_l} \left( \frac{j}{x_l} \right) = \mu_{\overset{n}{\underset{i=1}{\cup}} (E_i \cap A_l)} \left( \frac{j}{x_l} \right) = \underset{i=1}{\overset{n}{\tau_M}} \mu_{E_i\cap A_l} \left( \frac{j}{x_l} \right).$$

It comes out that

$$\theta_E = \theta(E) = \underset{l=1}{\overset{m}{\tau_M}} \left[ \underset{i=1}{\overset{n}{\tau_M}} \mu_{E_i \cap A_l} \left( \frac{j}{x_l} \right) \right] = \underset{i=1}{\overset{n}{\tau_M}} \left[ \underset{l=1}{\overset{m}{\tau_M}} \mu_{E_i \cap A_l} \left( \frac{j}{x_l} \right) \right] =$$

$$= \underset{i=1}{\overset{n}{\tau_M}} \left[ \int_{E_i} f d\mu \right] = \underset{i=1}{\overset{n}{\tau_M}} (\theta_{E_i}).$$

The remaining conclusions of the theorem follow from Definitions 1.1 and 3.1 and by hypothesis $C_1$) and $C_2$). □

The extension of the integral from the step functions to the arbitrary functions from $[0,\infty]^S$ is based on the following result:

**Lemma 1.** *Let* $\{f_\alpha\}$ *be a generalized sequence from the space* $\mathcal{E}(\mathcal{S}, \Gamma_\mu)$, *which is Cauchy in* $[0,\infty]^S (\Gamma_\mu)$. $\{\int_E f_\alpha d\mu\}$ *in order to be a Cauchy sequence in* $D_+$ *uniform with respect to* $E \in \mathcal{S}$ *it is necessary and sufficient that:*

(*a*) *For any neighborhood* $\mathcal{N}$ *of* $\varepsilon_0$ *in* $(D_+, \tau_M, d_L)$, *there exists an index* $\alpha_0$, *a finite set* $K \subseteq I$ *and* $v \in \mathcal{V}$ *so that:*

$$\alpha \geq \alpha_0 \text{ and } \gamma_E^{i*} \in v,\ i \in K, \text{ imply } \int_E f_\alpha d\mu \in \mathcal{N}.$$

(*b*) *For any neighborhood* $\mathcal{N}$ *of* $\varepsilon_0$ *in* $(D_+, \tau_M, d_L)$, *there exists an index* $\alpha_0$ *and* $F \in \mathcal{S}$, *with* $F$ *a* $\Gamma_\mu$*-finite, so that*

$$\int_E f d\mu \in \mathcal{N} \text{ if } \alpha \geq \alpha_0 \text{ and } E \in \mathcal{S},\ E \subset S - F.$$

*Proof.* To prove the first assertion, we notice that for any neighborhood $\mathcal{N}$ of $\varepsilon_0$, there exists a symmetric entourage $\mathcal{W}$ of the uniform structure from $(D_+, \tau_M, d_L)$ so that $\mathcal{W}^2(\varepsilon_0) \subseteq \mathcal{N}$.

Let $\alpha_0$ be such that if $\alpha \geq \alpha_0$, we have $\left(\int_E f_\alpha d\mu, \int_E f_\alpha d\mu\right) \in \mathcal{W}$ for any $E \in \mathcal{S}$.

From Theorem 3.1 $(iv)$, it results that there exists $v \in \mathcal{V}$, $K =$ finite $\subseteq I$ so that we have $\int_E f_{\alpha_0} d\mu \in \mathcal{W}(\varepsilon_0)$ if $\gamma_E^{i*} \in v, i \in K$. Therefore $\int_E f_\alpha d\mu \in \mathcal{N}$ if $\alpha \geq \alpha_0$ and $\gamma_E^{i*} \in v, i \in K$, that is the condition (a) holds. The condition (b) is obtained by taking $F = \{s \in S; f_{\alpha_0}(s) \neq 0\}$. We have $F \in \mathcal{S}$, $\Gamma_\mu$-finite and $\int_E f_{\alpha_0} d\mu = 0$ for any $E \in \mathcal{S}$, with $E \subset S - F$.

To prove the converse, let $W$ be a symmetric entourage for $(D_+, \tau_M, d_L)$ and let $\alpha_0$, a finite set $K \subseteq I, v \in \mathcal{V}$ and $F$ be chosen depending on the neighborhood $\mathcal{W}(\varepsilon_0)$ and satisfying the conditions $a)$ and $b)$.

For $F$ and $\mathcal{W}$, let the entourage $\sigma \in \Sigma$ be chosen according to axiom $C_1$. We write: $F_{\alpha\alpha'} = \{s \in S; (f_\alpha(s), f_{\alpha'}(s)) \notin \sigma\}$, $F_{\alpha\alpha'} \in \mathcal{S}$.

Since $\{f_\alpha\}$ is Cauchy in $[0,\infty]^S(\Gamma_\mu)$, there exists $\alpha \geq \alpha_0$ so that $\gamma_{F_{\alpha\alpha'}}^{i*} \in v$, $i \in K$, for $\alpha, \alpha' \geq \alpha_1$. For $E \in \mathcal{S}$, in the semigroup $D_+ \times D_+$ we can write

$$\left(\int_E f_\alpha d\mu, \int_E f_{\alpha'} d\mu\right) =$$

$$= \tau_M \left\{ \tau_M \left[ \left( \int_{E \cap F_{\alpha\alpha'}} f_\alpha d\mu, \int_{E \cap F_{\alpha\alpha'}} f_{\alpha'} d\mu \right), \left( \int_{E-(F_{\alpha\alpha'} \cap F)} f_\alpha d\mu, \int_{E-(F_{\alpha\alpha'} \cap F)} f_{\alpha'} d\mu \right) \right], \left( \int_{E-(F_{\alpha\alpha'} \cap F)} f_\alpha d\mu, \int_{E-(F_{\alpha\alpha'} \cap F)} \int f_{\alpha'} d\mu \right) \right\}$$

$$\in \tau_M \{\tau_M [(\mathcal{W}(\varepsilon_0) \times \mathcal{W}(\varepsilon_0), \mathcal{W}(\varepsilon_0)), \mathcal{W}(\varepsilon_0)], \mathcal{W}\}$$

$$\subseteq \tau_M \left[\tau_M\left(\mathcal{W}^2, \mathcal{W}^2\right), \mathcal{W}^2\right] \subseteq \mathcal{W} \quad \text{for} \quad \alpha, \alpha' \geq \alpha_1.$$

Note: $(i)\, \tau_M(\mathcal{U}, \mathcal{N}) = \{\tau_M(F,G); F \in \mathcal{U}, G \in \mathcal{N}\}, \mathcal{U}, \mathcal{N} \subset D_+ \times D_+$.

$(ii)$ In the semigroup $D_+ \times D_+$ for $(F_1, G_1)$,$(F_2, G_2) \in D_+ \times D_+$ we note $\tau_M[(F_1, G_1), (F_2, G_2)] = (\tau_M(F_1, F_2), \tau_M(G_1, G_2))$. □

**Corollary 1.** *Let $\{f_\alpha\}$ and $\{g_\beta\}$ be two generalized sequences from the space $\mathcal{E}(\mathcal{S}, \Gamma_\mu)$ convergent in $[0,\infty]^S(\Gamma_\mu)$ to the same function. If $\left\{\int_E f_\alpha d\mu\right\}$ and $\left\{\int_E g_\beta d\mu\right\}$ are generalized Cauchy sequences in $(D_+, \tau_M, d_L)$ uniformly in $E \in \mathcal{S}$, then for any entourage $\mathcal{W}$ from $D_+$ there exist $\alpha_0$ and $\beta_0$ so that if $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$, it results in $\left(\int_E f_\alpha d\mu, \int_E g_\beta d\mu\right) \in \mathcal{W}$, uniformly in $E \in \mathcal{S}$.*

*Proof.* Given a symmetric entourage $\mathcal{W}$, from $(D_+, \tau_M, d_L)$ so that

$$\tau_M \left[\tau_M \left(\mathcal{W}_1^2, \mathcal{W}_1^2\right), \mathcal{W}_1^2\right] \subseteq \mathcal{W},$$

we choose an entourage $\sigma \in \Sigma$ according to $W_1$ in conformity with the axiom $C_1$. We denote $F_{\alpha\beta} = \{s \in S; (f_\alpha(s), g_\beta(s)) \notin \sigma\}$. From the previous lemma it results that there exist $\alpha_0, \beta_0, v \in \mathcal{V}, K = \text{finite} \subseteq I$ so that if $F \in \mathcal{S}, F, \Gamma_\mu$-finite and $\alpha \geq \alpha_0, \beta \geq \beta_0, \gamma_E^{i*} \in v, i \in K, E \subset S - F, E \in \mathcal{S}$ imply $\int_E f_\alpha d\mu \in \mathcal{W}_1(\varepsilon_0)$, and $\int_E g_\beta d\mu \in \mathcal{W}_1(\varepsilon_0)$.

By hypothesis there exist $\alpha_1 \geq \alpha_0$ and $\beta_1 \geq \beta_0$ so that for $\alpha > \alpha_1, \beta > \beta_1$ we have $\gamma^{i*}(F_{\alpha\beta}) \in v, i \in K$. Expressing the pair $\left(\int_E f_\alpha d\mu, \int_E g_\beta d\mu\right)$ in the same way as in the proof of the second statement from Lemma 3.1, the result is obtained. □

**Definition 5.** *The function $f \in [0,\infty]^S$ is called $\Gamma_\mu$-integrable if there exists a generalized sequence $\{f_\alpha\}$ from $\mathcal{E}(\mathcal{S},\Gamma_\mu)$ so that $f_\alpha \xrightarrow{\Gamma_\mu} f$ and $\left\{\int_E f_\alpha d\mu\right\}$ is a generalized Cauchy sequence in $(D_+, \tau_M, d_L)$, uniformly in $E \in \mathcal{S}$. Then the $\Gamma_\mu$-integral is an element from $\widehat{D}_+$, the completed space of $D_+$, defined by*

$$\int_E f d\mu = \lim_\alpha \int_E f_\alpha d\mu.$$

From Corollary 3.1, it results that the above $\Gamma_\mu$-integral is properly defined. We denote by $\mathcal{L}(\mathcal{S},\Gamma_\mu)$ the set of $\Gamma_\mu$-integrable functions from $[0,\infty]^S$. It is obvious that $\mathcal{E}(\mathcal{S},\Gamma_\mu) \subseteq \mathcal{L}(\mathcal{S},\Gamma_\mu)$ and the $\Gamma_\mu$-integral restricted to $\mathcal{E}(\mathcal{S},\Gamma_\mu)$ coincides with the $\Gamma_\mu$-integral from definition.

**Theorem 2.** $(i)$ *Relatively to the operation of addition, the set $\mathcal{L}(\mathcal{S},\Gamma_\mu)$ is a subsemigroup of $[0,\infty]^S$.*

$(ii)$ *For $E \in \mathcal{S}$, the mapping $f \to \int_E f d\mu$ of $\mathcal{L}(\mathcal{S},\Gamma_\mu)$ in $\widehat{D}_+$ is additive*

$$\left(i.e.: \int_E (f+g)\, d\mu = \tau_M \left(\int_E f d\mu, \int_E g d\mu\right), f, g \in \mathcal{L}(\mathcal{S},\Gamma_\mu)\right).$$

$(iii)$ *For $f \in \mathcal{L}(\mathcal{S},\Gamma_\mu)$ the mapping $E \to \theta(E) = \theta_E = \int_E f d\mu$ is additive $\left(i.e.: \theta_{\cup_{i=1}^n E_i} = \underset{i=1}{\overset{n}{\tau_M}} (\theta_{E_i}), E_i \cap E_j = \emptyset, i \neq j, \text{ and } \theta_\emptyset = \varepsilon_0\right)$.*

$(iv)$ *For $f \in \mathcal{L}(\mathcal{S},\Gamma_\mu)$ we have $\lim\limits_{\substack{E \overset{\Gamma_\mu}{\to} \emptyset \\ E \in \mathcal{S}}} \theta_E = \varepsilon_0$.*

The proof follows from Corollary 3.1 and Definition 3.2.

**Remark 3.** *If the family of submeasures is defined by the above probabilistic $\mathcal{V}$-structure we obtain: the integrability with probabilistic $\beta$-structure, the integrability with probabilistic $H$-submeasure, the integrability with probabilistic $f$-submeasure and the integrability with $\mathcal{V}$-Šerstnev submeasure, respectively.*

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