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# ON THE IMPLEMENTATION OF SET–VALUED NON–BOOLEAN FUNCTIONS

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**Abstract.** In the set of functions $F : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$ the subset of Boolean functions is not complete. We study the ways of partitioning the definition domain $\mathcal{P}^n(\mathbf{r})$ of a set–valued function $F$ into equivalence classes with respect to equivalence relations generated by $F$ so that on these classes a Boolean function $f$ equal to $F$ exists.

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## 1. Introduction

Let $\mathbf{r} = \{0, 1, ..., r-1\},\ r \geq 1$, and let $\mathcal{P}(\mathbf{r})$ be the set of subsets of $\mathbf{r}$. Then $(\mathcal{P}(\mathbf{r}), \emptyset, \mathbf{r}, \cup, \cap, ^-)$ is a Boolean algebra. There are $2^{r2^{rn}}$ set-valued functions $F : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$, and only $2^{r2^n}$ of them are Boolean. (More on set-valued functions and their applications can be found in [1], [2], [6].)

Let $\oplus$ denote the symmetric difference over $\mathcal{P}(\mathbf{r})$. It is well known that a function $F : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$ is Boolean if and only if it can be represented in the form

$$F(X_1, \ldots, X_n) = A_0 \oplus \sum_{m=1}^{n} \sum_{i_1,\ldots,i_m}^{1,2,\ldots,n} A_{i_1,\ldots,i_m} X_{i_1} \ldots X_{i_m}$$

for all $X_1, \ldots, X_n \in \mathcal{P}(\mathbf{r})$, where $A_0$ and $A_{i_1,\ldots,i_m}$ are constants of $\mathcal{P}(\mathbf{r})$, and the sum is extended over all $\binom{n}{m}$ subsets $\{i_1, \ldots, i_m\}$ of $m$ distinct indices from the set $\{1, \ldots, n\}$. The coefficients $A_0$ and $A_{i_1,\ldots,i_m}$ are uniquely determined by $F$.

The following property of Boolean functions, given in [4], is the generalization of the results of McKinsey and Scognamiglio.

**Theorem 1.1.** *If $f : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$ is a Boolean function then*

$$f(X_1, \ldots, X_n) \oplus f(Y_1, \ldots, Y_n) \subseteq \bigcup_{i=1}^{n} (X_i \oplus Y_i)$$

*for all $X = (X_1, \ldots, X_n)$, $Y = (Y_1, \ldots, Y_n) \in \mathcal{P}^n(\mathbf{r})$.*

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The next theorem describes the partitions of $\mathcal{P}^n(\mathbf{r})$ into classes on which it is possible to approximate the given function $F$ by a Boolean function $f$.

**Theorem 1.2.** *Let $F : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$, and let $\sim$ be an equivalence relation on $\mathcal{P}^n(\mathbf{r})$ such that for $(X_1, \ldots, X_n) \sim (Y_1, \ldots, Y_n)$,*

$$F(X_1, \ldots, X_n) \oplus F(Y_1, \ldots, Y_n) \subseteq \bigcup_{i=1}^{n} (X_i \oplus Y_i)$$

*holds. Then for every element $X = (X_1, \ldots, X_n) \in \mathcal{P}^n(\mathbf{r})$ there exists a Boolean function $f_X : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$ which coincides with $F$ on $[X]$, where $[X]$ is the equivalence class of $X$. This Boolean function is given by*

$$f_X(U_1, \ldots, U_n) = \bigcup_{Y \in [X]} F(Y_1, \ldots, Y_n) \bigcap_{i=1}^{n} (Y_i \oplus U_i \oplus \mathbf{r})$$

*for every $U = (U_1, \ldots, U_n) \in \mathcal{P}^n(\mathbf{r})$.*

## 2. The Equivalence Relations Generated by a Set–Valued Function

**Definition 2.1.** *Let $X = (X_1, \ldots, X_n)$, $Y = (Y_1, \ldots, Y_n) \in \mathcal{P}^n(\mathbf{r})$. We say that $(X_1, \ldots, X_n) \sim_1 (Y_1, \ldots, Y_n)$ if*

$$F(X_1, \ldots, X_n) \oplus F(W_1, \ldots, W_n) \subseteq \bigcup_{i=1}^{n} (X_i \oplus W_i)$$

*is equivalent to*

$$F(Y_1, \ldots, Y_n) \oplus F(W_1, \ldots, W_n) \subseteq \bigcup_{i=1}^{n} (Y_i \oplus W_i)$$

*for every $W = (W_1, \ldots, W_n) \in \mathcal{P}^n(\mathbf{r})$.*

**Theorem 2.1.** *Relation $\sim_1$ is an equivalence relation on $\mathcal{P}^n(\mathbf{r})$, and on the equivalence classes of $\sim_1$ it is possible to approximate the function $F$ by a Boolean function.*

*Proof.* Relation $\sim_1$ is obviously reflexive, symmetric and transitive.

For $X = (X_1, \ldots, X_n) \in \mathcal{P}^n(\mathbf{r})$ we introduce the collection of sets $Q_F(X) = \{(W_1, \ldots, W_n) \in \mathcal{P}^n(\mathbf{r}) | F(X_1, \ldots, X_n) \oplus F(W_1, \ldots, W_n) \subseteq \bigcup_{i=1}^{n} (X_i \oplus W_i)\}$. Then $X \sim_1 Y$ if and only if $Q_F(X) = Q_F(Y)$. Since $X \in Q_F(X)$, we have that for $X \sim_1 Y$, $F(X_1, \ldots, X_n) \oplus F(Y_1, \ldots, Y_n) \subseteq \bigcup_{i=1}^{n} (X_i \oplus Y_i)$ holds, i.e., it is possible to approximate $F$ by Boolean functions on the equivalence classes of the relation $\sim_1$. □

Next we give some properties of the relation $\sim_1$ for some values of $n$ and $r$.

**Case** $r = 1$

In this case the set $\mathcal{P}(\mathbf{r})$ is isomorphic to the two-element Boolean algebra $\mathbf{B_2}$, $\mathcal{P}^n(\mathbf{r})$ is isomorphic to $\mathbf{B_2^n}$ so that every function $F : \mathcal{P}^n(\mathbf{1}) \to \mathcal{P}(\mathbf{1})$ is Boolean and has one equivalence class.

**Case** $n = 1$, $r = 2$

This case is studied in [3], and the following results are obtained:

| $k$ – number of classes | number of functions with $k$ classes |
|---|---|
| 1 | 16 |
| 2 | 16 |
| 3 | 128 |
| 4 | 96 |

In [5] we obtained the following results for the relation $\sim_1$:

**Case** $n = 1$, $r = 3$

| $k$ – number of classes | number of functions with $k$ classes |
|---|---|
| 1 | 64 |
| 2 | 1024 |
| 3 | 5504 |
| 4 | 34880 |
| 5 | 165888 |
| 6 | 779520 |
| 7 | 3386880 |
| 8 | 12403456 |

**Case** $n = 2$, $r = 2$

**Theorem 2.2.** *There is no function* $F : \mathcal{P}^2(\mathbf{2}) \to \mathcal{P}(\mathbf{2})$ *such that the relation* $\sim_1$ *has four classes.*

**Case** $n \geq 2$, $r \geq 2$

**Theorem 2.3.** *There is no function* $F : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$, $n \geq 2$ *such that the relation* $\sim_1$ *has two classes.*

Another partition of $\mathcal{P}^n(\mathbf{r})$, independent of the values of the function $F : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$, can be obtained in the following way:

**Definition 2.2.** *Let* $F : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$ *and let* $X = (X_1, \ldots, X_n)$, $Y = (Y_1, \ldots, Y_n) \in \mathcal{P}^n(\mathbf{r})$. *We say that* $(X_1, \ldots, X_n) \sim_2 (Y_1, \ldots, Y_n)$ *if and only if* $Y_i = X_i$ *or* $Y_i = \overline{X_i}$, $i = 1, \ldots, n$.

**Theorem 2.4.** *Relation* $\sim_2$ *is an equivalence relation on* $\mathcal{P}^n(\mathbf{r})$ *which induces the partition of* $\mathcal{P}^n(\mathbf{r})$ *into equivalence classes such that on these classes the function* $F$ *can be approximated by a Boolean function.*

*Proof.* Relation $\sim_2$ is obviously reflexive, symmetric and transitive.

For any two distinct elements $(X_1, \ldots, X_n), (Y_1, \ldots, Y_n)$ from the same equivalence class there exists a $j \in \{1, \ldots, n\}$ such that $Y_j = \overline{X_j}$, so that

$$F(X_1, \ldots, X_n) \oplus F(Y_1, \ldots, Y_n) \subseteq \bigcup_{i=1}^{n} (X_i \oplus Y_i) =$$
$$= (X_1 \oplus Y_1) \cup \ldots \cup (X_j \oplus Y_j) \cup \ldots \cup (X_n \oplus Y_n) = \mathbf{r},$$

and also

$$F(X_1, \ldots, X_n) \oplus F(X_1, \ldots, X_n) = \emptyset \subseteq \bigcup_{i=1}^{n} (X_i \oplus X_i),$$

i.e., the function $F$ can be approximated by a Boolean function on the equivalence classes generated by the relation $\sim_2$.

The set $\mathcal{P}^n(\mathbf{r})$, which has $2^{rn}$ elements, is divided by $\sim_2$ into $2^{(r-1)n}$ equivalence classes, each containing $2^n$ elements of the form

$$\{(X_1, X_2, \ldots, X_n), (X_1, X_2, \ldots, \overline{X_n}), \ldots (\overline{X_1}, X_2, \ldots, X_n) \ldots (\overline{X_1}, \overline{X_2}, \ldots, \overline{X_n})\},$$

so that every non–Boolean function $F : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$ can be approximated by $2^{(r-1)n}$ Boolean functions. □

Function $F : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$ induces another equivalence relation on $\mathcal{P}^n(\mathbf{r})$ in the following way:

**Definition 2.3.** *Let* $X = (X_1, \ldots, X_n)$ *and* $Y = (Y_1, \ldots, Y_n)$ *be two elements from* $\mathcal{P}^n(\mathbf{r})$ *and* $F : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$. *We say that* $X \sim_3 Y$ *if and only if* $F(X) = F(Y)$.

**Theorem 2.5.** *Relation* $\sim_3$ *is an equivalence relation on* $\mathcal{P}^n(\mathbf{r})$. *On each equivalence class of* $\sim_3$, *the function* $F$ *can be approximated by a Boolean function.*

*Proof.* Relation $\sim_3$ is obviously reflexive, symmetric and transitive.

Let $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ be two elements from the same equivalence class. Then

$$F(X_1, \ldots, X_n) \oplus F(Y_1, \ldots, Y_n) = \emptyset \subseteq \bigcup_{i=1}^{n} (X_i \oplus Y_i),$$

and on the equivalence classes of $\sim_3$ function $F$ can be approximated by (constant) Boolean functions. □

**Theorem 2.6.** *The number $N_{k,r,n}$ of functions $F : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$ with $k$ equivalence classes with respect to the relation $\sim_3$ is given by*

$$N_{k,r,n} = \binom{2^r}{k}(-1)^k \sum_{j=1}^{k}(-1)^j \binom{k}{j} j^{2^{rn}} = \binom{2^r}{k}(-1)^k((1-e^t)^k)^{(2^{rn})}|_{t=0}.$$

*Proof.* The set–valued function $F : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$ is completely determined by its table of values, i.e., by the array of $2^{rn}$ elements from $\mathcal{P}(\mathbf{r})$. We can choose $k$ values out of $2^r$ values in $\binom{2^r}{k}$ ways, and the number of ways we can arrange those $k$ values in $2^{rn}$ places is given by

$$k^{2^{rn}} - \binom{k}{k-1}(k-1)^{2^{rn}} + \ldots + (-1)^{k-1}\binom{k}{1}1^{2^{rn}} =$$
$$= \sum_{i=0}^{k-1}(-1)^i \binom{k}{k-i}(k-i)^{2^{rn}} = \sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j} j^{2^{rn}}.$$

(From the $k^{2^{rn}}$ variations of $k$ elements of order $2^{rn}$ we subtract those that have less than $k$ different elements.)

On the other hand, for

$$g(t) = (1-e^t)^k = \sum_{j=0}^{k}(-1)^j \binom{k}{j} e^{jt}$$

we have

$$g^{(2^{rn})}(t) = \frac{d^{2^{rn}}}{dt^{2^{rn}}} g(t) = \sum_{j=1}^{k}(-1)^j \binom{k}{j} j^{2^{rn}} e^{jt}$$

and

$$g^{(2^{rn})}(0) = \sum_{j=1}^{k}(-1)^j \binom{k}{j} j^{2^{rn}}.$$

□

For the relation $\sim_3$ we have the following results:

**Case** $n = 1$**,** $r = 2$

| $k$ – number of classes | number of functions with $k$ classes |
|---|---|
| 1 | 4 |
| 2 | 84 |
| 3 | 144 |
| 4 | 24 |

**Case** $n = 1$**,** $r = 3$

| $k$ − number of classes | number of functions with $k$ classes |
|---|---|
| 1 | 8 |
| 2 | 7112 |
| 3 | 324576 |
| 4 | 2857680 |
| 5 | 7056000 |
| 6 | 5362560 |
| 7 | 1128960 |
| 8 | 40320 |

It can be shown that for $r > 1$ there exist functions such that the relation $\sim_3$ partitions the set $\mathcal{P}^n(\mathbf{r})$ into more classes than the relation $\sim_1$.

**Theorem 2.7.** *For every $r > 1$ and $n \in N$ there exists a function $F : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$ such that the relation $\sim_1$ partitions the set $\mathcal{P}^n$ in three classes, while the relation $\sim_3$ partitions it in $2^r$ classes.*

*Proof.* Let the function $F : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r}),\ r > 1$ be given by

$$F(X_1, X_2, \ldots, X_n) = \begin{cases} \bigcap\limits_{i=1}^{n} X_i & \text{for } (X_1, X_2, \ldots, X_n) \neq (\emptyset, \emptyset, \ldots, \emptyset) \\ \mathbf{r} & \text{for } (X_1, X_2, \ldots, X_n) = (\emptyset, \emptyset, \ldots, \emptyset) \end{cases}$$

First, we show that this function is not Boolean. Let us suppose on the contrary that it is. Then there exist $A_0,\ A_{i_1,\ldots,i_m} \in P(\mathbf{r})$ such that

$$F(X_1, X_2, \ldots, X_n) = A_0 \oplus \sum_{m=1}^{n} \sum_{i_1,\ldots,i_m}^{1,2,\ldots,n} A_{i_1,\ldots,i_m} X_{i_1} \ldots X_{i_m}.$$

We have

$$F(\emptyset, \emptyset, \ldots, \emptyset) = A_0 = \mathbf{r},\ \text{so } A_0 = \mathbf{r};$$

$$F(\mathbf{r}, \emptyset, \ldots, \emptyset) = A_0 \oplus A_1 = \mathbf{r} \oplus A_1 = \emptyset,$$

so

$$A_1 = \mathbf{r} \text{ and similarly } A_2 = \ldots = A_n = \mathbf{r};$$

$$F(\mathbf{r}, \mathbf{r}, \emptyset, \ldots, \emptyset) = A_0 \oplus A_1 \oplus A_2 \oplus A_{1,2} = \mathbf{r} \oplus \mathbf{r} \oplus \mathbf{r} \oplus A_{1,2} = \mathbf{r} \oplus A_{1,2} = \emptyset,$$

so

$$A_{1,2} = \mathbf{r} \text{ and similarly } A_{1,3} = \ldots = A_{n-1,n} = \mathbf{r};$$

$$\ldots$$

$$F(\mathbf{r}, \mathbf{r}, \ldots, \mathbf{r}, \emptyset) = \underbrace{\mathbf{r} \oplus \mathbf{r} \oplus \ldots \oplus \mathbf{r}}_{2^{n-1}-1} \oplus A_{1,2,\ldots,n-1} = \mathbf{r} \oplus A_{1,2,\ldots,n-1} = \emptyset,$$

so

$$A_{1,2,\ldots,n-1} = \mathbf{r} \text{ and similarly } A_{1,2,\ldots,n-2,n} = \ldots = A_{2,3,\ldots n} = \mathbf{r};$$

$$F(\mathbf{r},\mathbf{r},\ldots,\mathbf{r}) = \underbrace{\mathbf{r}\oplus\mathbf{r}\oplus\ldots\oplus\mathbf{r}}_{2^n-1}\oplus A_{1,2,\ldots,n} = \mathbf{r}\oplus A_{1,2,\ldots,n} = \mathbf{r},\ A_{1,2,\ldots,n} = \emptyset.$$

As $r > 1$, $\{0\} \neq \mathbf{r}$ and

$$F(\{0\},\emptyset,\ldots,\emptyset) = A_0 \oplus A_1\{0\} = \mathbf{r}\oplus\mathbf{r}\{0\} = \mathbf{r}\oplus\{0\} = \overline{\{0\}}.$$

On the other hand, by definition of the function $F$, we have

$$F(\{0\},\emptyset,\ldots,\emptyset) = \emptyset,$$

a contradiction, so $F$ is not Boolean.

Function $F$ obviously takes all the values from $\mathcal{P}(\mathbf{r})$, that is $im(F) = \mathcal{P}(\mathbf{r})$, so the relation $\sim_3$ partitions the set $\mathcal{P}^n(\mathbf{r})$ into $2^r$ equivalence classes.

Relation $\sim_2$ partitions the set $\mathcal{P}^n(\mathbf{r})$ into $2^{(r-1)n}$ equivalence classes.

We shall show that the relation $\sim_1$ partitions the set $\mathcal{P}^n(\mathbf{r})$ into three equivalence classes by determining the collection $Q_F$ for every element from $\mathcal{P}^n(\mathbf{r})$.

Since, for every $(X_1,X_2,\ldots,X_n) \in \mathcal{P}^n(\mathbf{r})$, $(X_1,X_2,\ldots,X_n) \neq (\emptyset,\emptyset,\ldots,\emptyset)$, we have

$$\begin{aligned} &F(\emptyset,\emptyset,\ldots,\emptyset)\oplus F(X_1,X_2,\ldots,X_n) = \\ &\quad = \mathbf{r}\oplus X_1X_2\ldots X_n = \overline{X_1X_2\ldots X_n} = \overline{X_1}\cup\overline{X_2}\cup\ldots\cup\overline{X_n}, \end{aligned}$$

and

$$\bigcup_{i=1}^{n}(\emptyset\oplus X_i) = \bigcup_{i=1}^{n} X_i,$$

then

$$F(\emptyset,\emptyset,\ldots,\emptyset)\oplus F(X_1,X_2,\ldots,X_n) \subseteq \bigcup_{i=1}^{n} X_i$$

if and only if $\bigcup\limits_{i=1}^{n} X_i = \mathbf{r}$. (If $\bigcup\limits_{i=1}^{n} X_i = \mathbf{r}$, then surely

$$F(\emptyset,\emptyset,\ldots,\emptyset)\oplus F(X_1,X_2,\ldots,X_n) \subseteq \bigcup_{i=1}^{n} X_i\,.$$

If, on the other hand, there is a $k \in \{0,1,\ldots,r-1\}$ such that $k \notin \bigcup\limits_{i=1}^{n} X_i$, then $k \in \overline{\bigcup\limits_{i=1}^{n} X_i} = \bigcap\limits_{i=1}^{n}\overline{X_i} \subseteq \bigcup\limits_{i=1}^{n}\overline{X_i}$, that is $F(\emptyset,\emptyset,\ldots,\emptyset)\oplus F(X_1,X_2,\ldots,X_n) \not\subseteq \bigcup\limits_{i=1}^{n} X_i$.)

So $Q_F(\emptyset,\emptyset,\dots,\emptyset) = \{(\emptyset,\emptyset,\dots,\emptyset)\} \cup \{(X_1,X_2,\dots,X_n) \in \mathcal{P}^n(\mathbf{r}) | \bigcup_{i=1}^{n} X_i = \mathbf{r}\}$.

Further, for any two elements $(X_1,X_2,\dots,X_n),\ (Y_1,Y_2,\dots,Y_n)$ from $\mathcal{P}^n(\mathbf{r})$, distinct from $(\emptyset,\emptyset,\dots,\emptyset)$, we have

$$\begin{aligned}
&F(X_1,X_2,\dots,X_n) \oplus F(Y_1,Y_2,\dots,Y_n) = \\
&\quad = X_1X_2\dots X_n \oplus Y_1Y_2\dots Y_n \\
&\quad = \overline{X_1X_2\dots X_n}Y_1Y_2\dots Y_n \cup X_1X_2\dots X_n\overline{Y_1Y_2\dots Y_n} \\
&\quad = (\overline{X_1} \cup \overline{X_2} \cup \dots \cup \overline{X_n})Y_1Y_2\dots Y_n \cup X_1X_2\dots X_n(\overline{Y_1} \cup \overline{Y_2} \cup \dots \cup \overline{Y_n}) \\
&\quad = \overline{X_1}Y_1Y_2\dots Y_n \cup \overline{X_2}Y_1Y_2\dots Y_n \cup \dots \cup \overline{X_n}Y_1Y_2\dots Y_n \\
&\qquad \cup X_1X_2\dots X_n\overline{Y_1} \cup X_1X_2\dots X_n\overline{Y_2} \cup \dots \cup X_1X_2\dots X_n\overline{Y_n} \\
&\quad \subseteq \overline{X_1}Y_1 \cup \overline{X_2}Y_2 \cup \dots \cup \overline{X_n}Y_n \cup X_1\overline{Y_1} \cup X_2\overline{Y_2} \cup \dots \cup X_n\overline{Y_n} \\
&\quad = (X_1 \oplus Y_1) \cup (X_2 \oplus Y_2) \cup \dots \cup (X_n \oplus Y_n)
\end{aligned}$$

and for any $(X_1,X_2,\dots,X_n) \neq (\emptyset,\emptyset,\dots,\emptyset)$,

$$\begin{aligned}
&Q_F(X_1,X_2,\dots,X_n) = \mathbf{r} && \text{if } \bigcup_{i=1}^{n} X_i = \mathbf{r}, \\
&Q_F(X_1,X_2,\dots,X_n) = \mathbf{r} - \{(\emptyset,\emptyset,\dots,\emptyset)\} && \text{if } \bigcup_{i=1}^{n} X_i \neq \mathbf{r}.
\end{aligned}$$

So we have shown that the relation $\sim_1$ partitions the set $\mathcal{P}^n(\mathbf{r})$ in three equivalence classes, one of which contains only the element $(\emptyset,\emptyset,\dots,\emptyset)$, the second contains all the elements $(X_1,X_2,\dots,X_n)$ from $\mathcal{P}^n(\mathbf{r}) - \{(\emptyset,\emptyset,\dots,\emptyset)\}$ for which $\bigcup_{i=1}^{n} X_i = \mathbf{r}$, and the third contains all the elements $(X_1,X_2,\dots,X_n)$ from $\mathcal{P}^n(\mathbf{r}) - \{(\emptyset,\emptyset,\dots,\emptyset)\}$ for which $\bigcup_{i=1}^{n} X_i \neq \mathbf{r}$. □

**Example 2.1.** *Let the function $F : \mathcal{P}^2(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$ be given by*

$$F(X,Y) = \begin{cases} X \cup Y & \textit{if } X = \emptyset \\ X \cap Y & \textit{otherwise} \end{cases}$$

*for $r \geq 2$.*

This function is not Boolean. Let us suppose, on the contrary, that it is. Then there exist $A_0, A_1, A_2, A_{12} \in \mathcal{P}^2(\mathbf{r})$ such that $F$ can be written in the form $F(X,Y) = A_0 \oplus A_1X \oplus A_2Y \oplus A_{12}XY$. Then

$$F(\emptyset,\emptyset) = A_0 = \emptyset,$$

$$F(\emptyset,\mathbf{r}) = A_2 = \mathbf{r},$$

$$F(\mathbf{r}, \emptyset) = A_1 = \emptyset,$$

$$F(\mathbf{r}, \mathbf{r}) = \mathbf{r} \oplus A_{12} = \mathbf{r},\ A_{12} = \emptyset,$$

so that $F(X, Y) = Y$. Since $r > 1$, we have $\{0\} \neq \mathbf{r}$, and

$$F(\{0\}, \overline{\{0\}}) = \overline{\{0\}} \neq \emptyset$$

which is a contradiction.

The number of equivalence classes with respect to $\sim_2$ is four, while the number of classes with respect to $\sim_1$ is six, which can be verified using the program given in Appendix. □

## Appendix

```
program n2r2;
type skup = set of 0..15;
var n,n2,k,l,broj,brojac,i,j : integer;
     f : array [0..3,0..3] of 0..3;
     x : array [0..3] of 0..3;
     g : array [0..15] of skup;
     xx : array [0..3,0..3] of 0..15;
     novaklasa : boolean;
     ul,iz : text;
begin
n:=4; n2:=16;
assign (ul,'ulazn2r2.txt');
assign (iz,'izlazn2r2.txt');
reset(ul);
rewrite(iz);
k:=0;
for i:=0 to n-1 do begin
                         x[i]:=i;
for j:=0 to n-1 do begin
                         xx[i,j]:=k;
                         k:=k+1;
                      end;
                      end;
for i:=0 to n-1 do begin
for j:=0 to n-1 do begin
                         read(ul,f[i,j]);
                      end;
                      end;
brojac:=1;
for i:=0 to n-1 do begin
for j:=0 to n-1 do begin
        g[brojac]:=[];
        for k:=0 to n-1 do begin
```

```
        for l:=0 to n-1 do begin
             if (((f[i,j] xor f[k,l]) and ((x[i] xor x[k]) or (x[j] xor x[l])))
             = (f[i,j] xor f[k,l]))
             then begin
                       g[brojac] :=g[brojac] + [xx[k,l]];
                    end;
        end;
        end;
        brojac:=brojac+1;
end;
end;
broj := 1;
for i:=1 to n2-1 do begin
      novaklasa := true;
      for j:=0 to i-1 do begin
           if g[j] = g[i] then novaklasa := false;
      end;
      if novaklasa then broj := broj + 1;
end;
writeln (' ',broj);
close(ul);
close(iz);
end.
```

## References


[1] Ngom, A., Reicher, C., Simovici, D. A., Stojmenović, I., Set-Valued Logic Algebra: A Carrier Computing Foundation. Multi. Val. Logic 2(1997), 183–216.

[2] Reicher, C., Simovici, D. A., Bio–Algebras. Proc. of the 20th Int. Symp. on Multiple–Valued Logic (1990), 48–53.

[3] Reischer, C., Simovici, D. A., On the Implementation of Set–Valued Non–Boolean Switching Functions. Proc. of the 21st Int. Symp. on Multiple–Valued Logic (1991), 166–172.

[4] Rudeanu, S., Boolean Functions and Equations. Amsterdam: North–Holland, 1974.

[5] Tošić, R., Čomić, L., On Set Valued Non-Boolean Functions. Novi Sad J. Math. (2000), 79–88.

[6] Tošić, R., Stojmenović, I., Simovici, D. A., Reischer, C., On Set–Valued Functions and Boolean Collections. Proc. of the 22nd Int. Symp. on Multiple–Valued Logic (1992), 250–254.


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