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TURNING RETRACTIONS OF AN ALGEBRA INTO AN ALGEBRA

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Abstract. One can turn the set of retractions of a lattice $\langle L, \leq \rangle$ into a poset $R_f(\mathbf{L})$ by letting $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in L$. In 1982 H. Crapo raised the following two problems: (1) Is it true that $R_f(\mathbf{L})$ is a lattice for any lattice $\mathbf{L}$? (2) Is it true that $R_f(\mathbf{L})$ is a complete lattice if $\mathbf{L}$ is a complete lattice?

In 1990 and 1991 B. Li published two papers dealing with the above two questions. He showed that $R_f(\mathbf{L})$ is not necessarily a lattice and that $\mathbf{L}$ is a complete lattice if and only if $R_f(\mathbf{L})$ is a complete lattice.

Motivated by the idea of extending the structure from the base set to the set of all retractions, we introduce the notion of R-algebra as follows. Let $R_f(\mathbf{A})$ denote the set of all retractions of an algebra $\mathbf{A}$. We say that $\mathbf{A}$ is an R-algebra if the set $R_f(\mathbf{A})$ is closed with respect to operations of $\mathbf{A}$ applied pointwise. We give some necessary and some sufficient conditions for $\mathbf{A}$ to be an R-algebra. We show that the property of being an R-algebra carries over to retracts of the algebra. In a set of examples we show that almost no classical algebra is an R-algebra. In particular, a lattice $\mathbf{L}$ is an R-algebra iff $|L| \leq 2$.

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1. Introduction

One can turn the set of retractions of a lattice $\langle L, \leq \rangle$ into a poset by letting $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in L$. In 1982 H. Crapo raised the following two problems [1]: (1) Is it true that for any lattice $\mathbf{L}$, the set of retractions of a lattice partially ordered as above is again a lattice? (2) Is it true that the set of retractions of a lattice is a complete lattice if the original lattice is a complete lattice?

In 1990 and 1991 B. Li published two papers [2, 3] dealing with the above two questions. He showed that the set of retractions of a lattice is not necessarily a lattice, and that $\mathbf{L}$ is a complete lattice if and only if the set of retractions of $\mathbf{L}$ is a complete lattice.

Motivated by the idea of extending the structure from the base set to the set of all retractions, we introduce the notion of R-algebra as follows. Let

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$\mathbf{A} = \langle A, \mathcal{F} \rangle$ be an algebra. By a *retraction* of $\mathbf{A}$ we mean any idempotent endomorphism of $\mathbf{A}$. Let $R_f(\mathbf{A})$ denote the set of all retractions of $\mathbf{A}$. We say that $\mathbf{A}$ is an *R-algebra* if $R_f(\mathbf{A})$ is a subuniverse of $\mathbf{A}^A$. By $\mathbf{R}_f(\mathbf{A})$ we denote the corresponding algebra on the set of retractions. Lattices, groups etc. that are R-algebras shall be referred to as R-lattices, R-groups and so on.

In this paper we give some necessary and some sufficient conditions for $\mathbf{A}$ to be an R-algebra. We show that the property of being an R-algebra carries over to retracts of the algebra. In a set of examples we show that almost no classical algebra is an R-algebra. In particular, a lattice $\mathbf{L}$ is an R-algebra iff $|L| \leq 2$, while a semilattice is an R-algebra iff it is a zero-semilattice.

Let Inv and Pol be the standard clone-theoretic operators. For an algebra $\mathbf{A}$ let $\operatorname{Clo}\mathbf{A}$ denote the clone of all term operations of $\mathbf{A}$ and $\operatorname{Clo}^{(n)}\mathbf{A}$ the set of all n-ary term operations of $\mathbf{A}$. For an operation $f : A^n \to A$ let $f^\bullet = \{\langle x_1, \ldots, x_n, f(x_1, \ldots, x_n)\rangle : \langle x_1, \ldots, x_n\rangle \in A^n\}$ denote the *graph of* f; for a set of operations F let $F^\bullet = \{f^\bullet : f \in F\}$.

Proposition 1. *Let $\langle x_\alpha : \alpha < \lambda\rangle$ be a well ordering of A with $\lambda = |A|$. For $f \in A^A$ let $\mathbf{r}_f = \langle f(x_\alpha) : \alpha < \lambda\rangle$ and for $S \subseteq A^A$ put $\mathbf{r}_S = \{\mathbf{r}_f : f \in S\}$.*

Now let $\mathbf{A} = \langle A, \mathcal{F}\rangle$ be an algebra and let $S = \{f \in A^A : f^2 = f\} \cap \operatorname{Pol}(\operatorname{Clo}\mathbf{A})^\bullet$. Then $\mathbf{A}$ is an R-algebra if and only if $\mathbf{r}_S \in \operatorname{Inv}\operatorname{Clo}\mathbf{A}$.

Let $\mathbf{A}$ and $\mathbf{A}'$ be term equivalent algebras on the same carrier set A. Then $\mathbf{A}$ is an R-algebra if and only if $\mathbf{A}'$ is an R-algebra. If both $\mathbf{A}$ and $\mathbf{A}'$ are R-algebras then $R_f(\mathbf{A}) = R_f(\mathbf{A}')$, and moreover $\mathbf{R}_f(\mathbf{A})$ and $\mathbf{R}_f(\mathbf{A}')$ are term equivalents.

Proof. For the first part of the proposition, note that S is exactly the set of retractions of $\mathbf{A}$ and that $\mathbf{r}_S \in \operatorname{Inv}\operatorname{Clo}\mathbf{A}$ means that S is closed with respect to term operations on $\mathbf{A}$ applied pointwise. The second part of the proposition now follows immediately. □

Proposition 2. *If $\mathbf{A} = \langle A, \mathcal{F}\rangle$ is an R-algebra, then $\operatorname{Clo}^{(1)}\mathbf{A} \subseteq R_f(\mathbf{A})$.*

Proof. Take any $g \in \operatorname{Clo}^{(1)}\mathbf{A}$. The proof proceeds by induction on the complexity of the unary term giving rise to g. Let $g = f(x, \ldots, x)$ for some $f \in \mathcal{F}$. Since $\mathrm{id} \in R_f(\mathbf{A})$ and since $\mathbf{R}_f(\mathbf{A})$ is an algebra, $f(\mathrm{id}, \ldots, \mathrm{id}) \in R_f(\mathbf{A})$. But, $f(\mathrm{id}, \ldots, \mathrm{id})(x) = f(x, \ldots, x) = g(x)$. So, $g \in R_f(\mathbf{A})$. If $g = f(t_1, \ldots, t_n)$ for some $f \in \mathcal{F}$ and some unary terms t_i, induction hypothesis and the same argument apply. □

Proposition 3. *Let $\mathbf{A} = \langle A, \mathcal{F}\rangle$ be an algebra such that for all $f, f_1, f_2 \in \mathcal{F}$ the following two identities hold on $\mathbf{A}$:*

(i) $f(f(x_{11}, x_{12}, \ldots, x_{1n}), f(x_{21}, x_{22}, \ldots, x_{2n}), \ldots, f(x_{n1}, x_{n2}, \ldots, x_{nn})) = f(x_{11}, x_{22}, \ldots, x_{nn})$,

(*ii*) $f_1(f_2(x_{11},\ldots,x_{1n}),\ldots,f_2(x_{m1},\ldots,x_{mn})) = f_2(f_1(x_{11},\ldots,x_{m1}),\ldots,f_1(x_{1n},\ldots,x_{mn}))$.

Then $\mathbf{A}$ *is an R-algebra. In particular, every rectangular algebra is an R-algebra.*

Proof. It suffices to show that $R_f(\mathbf{A})$ is closed with respect to operations in $\mathcal{F}$. Let $f \in \mathcal{F}$ and $\varphi_1,\ldots,\varphi_n \in R_f(\mathbf{A})$ be arbitrary and let $\psi = f(\varphi_1,\ldots,\varphi_n)$. We shall prove that ψ is a retraction of $\mathbf{A}$.

To prove that ψ is a homomorphism of $\mathbf{A}$, let $f_1 \in \mathcal{F}$ be arbitrary.

$$\begin{aligned}
\psi(f_1(x_1,\ldots,x_n)) = \\
&= f(\varphi_1,\ldots,\varphi_n)(f_1(x_1,\ldots,x_m)) \\
&= f(\varphi_1(f_1(x_1,\ldots,x_m)),\ldots,\varphi_n(f_1(x_1,\ldots,x_m))) \\
&\text{[because } \varphi_j\text{'s are homomorphisms of } \mathbf{A}] \\
&= f(f_1(\varphi_1(x_1),\ldots,\varphi_1(x_m)),\ldots,f_1(\varphi_n(x_1),\ldots,\varphi_n(x_m))) \\
&\text{[because of } (ii)] \\
&= f_1(f(\varphi_1(x_1),\ldots,\varphi_n(x_1)),\ldots,f(\varphi_1(x_m),\ldots,\varphi_n(x_m))) \\
&= f_1(\psi(x_1),\ldots,\psi(x_m)).
\end{aligned}$$

To complete the proof, let us show that ψ is idempotent:

$$\begin{aligned}
\psi(\psi(x)) = f(\varphi_1,\ldots,\varphi_n)(\psi(x)) = \\
&= f(\varphi_1(\psi(x)),\ldots,\varphi_n(\psi(x))) \\
&= f(\varphi_1(f(\varphi_1(x),\ldots,\varphi_n(x))),\ldots\varphi_n(f(\varphi_1(x),\ldots,\varphi_n(x)))) \\
&\text{[because } \varphi_j\text{'s are homomorphisms of } \mathbf{A}] \\
&= f(f(\varphi_1\varphi_1(x),\ldots,\varphi_1\varphi_n(x)),\ldots,f(\varphi_n\varphi_1(x),\ldots,\varphi_n\varphi_n(x))) \\
&\text{[because of } (i)] \\
&= f(\varphi_1\varphi_1(x),\ldots,\varphi_n\varphi_n(x)) \\
&\text{[because } \varphi_j\text{'s are idempotent]} \\
&= f(\varphi_1(x),\ldots,\varphi_n(x)) = \psi(x).
\end{aligned}$$

□

Proposition 4. *Let* $\mathbf{A}$ *be an R-algebra and let* $\mathbf{R}$ *be a retract of* $\mathbf{A}$*. Then* $\mathbf{R}$ *is an R-algebra.*

Proof. Let $\varphi : A \to R$ be the corresponding retraction and let

$$\begin{aligned}
R_f(\mathbf{A}, R) &:= \{\psi \in R_f(\mathbf{A}) : \psi(A) \subseteq R\} \\
R_f(\mathbf{A}, R)|_R &:= \{\psi|_R : \psi \in R_f(\mathbf{A}, R)\} \\
R_f(\mathbf{R}) \circ \varphi &:= \{\psi \circ \varphi : \psi \in R_f(\mathbf{R})\}.
\end{aligned}$$

Clearly, $R_f(\mathbf{A}, R) \leq \mathbf{A}^A$ and $R_f(\mathbf{A}, R)|_R \subseteq R_f(\mathbf{R})$. Also, $R_f(\mathbf{R}) \circ \varphi \subseteq R_f(\mathbf{A}, R)$. To see this, it suffices to show that $\psi \circ \varphi$ is idempotent for all

$\psi \in R_f(\mathbf{R})$. Take any $\psi \in R_f(\mathbf{R})$ and $a \in A$. Then $\psi \circ \varphi(a) \in R$, whence $\psi \circ \varphi(a) = \varphi(x)$ for some $x \in A$. Now $\varphi \circ \psi \circ \varphi(a) = \varphi \circ \varphi(x) = \varphi(x) = \psi \circ \varphi(a)$ and thus $\psi \circ \varphi \circ \psi \circ \varphi(a) = \psi \circ \varphi(a)$.

To prove that $\mathbf{R}$ is an R-algebra, let $f \in \mathcal{F}$ and $\psi_1, \ldots, \psi_n \in R_f(\mathbf{R})$ be arbitrary. Then $\psi_1 \circ \varphi, \ldots, \psi_n \circ \varphi \in R_f(\mathbf{R}) \circ \varphi \subseteq R_f(\mathbf{A}, R)$ implying $f(\psi_1 \circ \varphi, \ldots, \psi_n \circ \varphi) \in R_f(\mathbf{A}, R)$ as well. From this we get $f(\psi_1 \circ \varphi, \ldots, \psi_n \circ \varphi)|_R \in R_f(\mathbf{A}, R)|_R \subseteq R_f(\mathbf{R})$. Since φ is a retraction, we have that $\varphi|_R = \mathrm{id}_R$, whence $f(\psi_1 \circ \varphi, \ldots, \psi_n \circ \varphi)|_R = f(\psi_1, \ldots, \psi_n)$. So, $f(\psi_1, \ldots, \psi_n) \in R_f(\mathbf{R})$. □

Lemma 5. *If* $\mathbf{A}$ *is an idempotent R-algebra then* $\mathbf{A}$ *can be embedded into* $\mathbf{R}_f(\mathbf{A})$.

Proof. Let c_a be the constant mapping $c_a(x) = a$ and let $\mathrm{Const}(A) = \{c_a : a \in A\}$. Since $\mathbf{A}$ is an idempotent algebra, $\mathrm{Const}(A) \subseteq R_f(\mathbf{A})$ and $\Phi : A \to R_f(\mathbf{A})$ defined by $\Phi(a) = c_a$ is an embedding of $\mathbf{A}$ into $\mathbf{R}_f(\mathbf{A})$. □

For a class $\mathcal{K}$ of R-algebras let $R_f(\mathcal{K}) = \{\mathbf{R}_f(\mathbf{A}) : \mathbf{A} \in \mathcal{K}\}$ (modulo abuse of set notation). Let $S(\mathcal{K})$ denote the class of all isomorphic copies of subalgebras of algebras from $\mathcal{K}$ and $V(\mathcal{K})$ the variety generated by $\mathcal{K}$.

Proposition 6. *Let* $\mathcal{K}$ *be a class of idempotent R-algebras of the same type. Then* $V(\mathcal{K}) = V(R_f(\mathcal{K}))$.

Proof. Since $\mathbf{R}_f(\mathbf{A}) \leq \mathbf{A}^A$ for any algebra $\mathbf{A}$, we have $R_f(\mathcal{K}) \subseteq V(\mathcal{K})$ and thus $V(R_f(\mathcal{K})) \subseteq V(\mathcal{K})$. For the other inclusion take any $\mathbf{A} \in \mathcal{K}$. According to Lemma 5 algebra $\mathbf{A}$ embeds into $\mathbf{R}_f(\mathbf{A})$, whence $\mathcal{K} \subseteq S(R_f(\mathcal{K}))$. Thus $V(\mathcal{K}) \subseteq V(R_f(\mathcal{K}))$. □

2. Examples

Unary algebras. Let $\mathbf{A}$ be a unary algebra. According to Proposition 2, if $\mathbf{A}$ is an R-algebra, each fundamental operation of $\mathbf{A}$ is a retraction of $\mathbf{A}$. The converse is also obvious. Thus we have that a unary algebra $\mathbf{A} = \langle A, \mathcal{F} \rangle$ is an R-algebra if and only if $\mathcal{F} \subseteq R_f(\mathbf{A})$.

Some semigroups. Let $\mathbf{S} = \langle S, \cdot \rangle$ be a semigroup such that $\mathbf{S} \models xyz = xz$. One easily verifies that $\mathbf{S}$ satisfies both conditions listed in Proposition 3. Therefore, $\mathbf{S}$ is an R-semigroup.

Bounded complemented algebras. We say that an algebra $\mathbf{A} = \langle A, \mathcal{F} \rangle$ is *bounded complemented* if there are constants $0, 1 \in \mathcal{F}$ and a unary operation $\overline{} \in \mathcal{F}$ such that $\overline{0} = 1$, $\overline{1} = 0$, and $|A| = 1$ if and only if $0 = 1$.

A bounded complemented algebra $\mathbf{A}$ *is an R-algebra if and only if it is* $|A| = 1$.

Proof. $\Leftarrow$: obvious.

$\Rightarrow$: Let $\mathbf{A}$ be a bounded complemented R-algebra. According to Proposition 2, $\overline{}$ is a retraction of $\mathbf{A}$, whence $\overline{\overline{x}} = \overline{x}$ for each $x \in A$. Therefore $0 = \overline{1} = \overline{\overline{1}} = 1$, implying that $|A| = 1$. □

As a corollary, we have the following. Let $\mathbf{L} = \langle L, \wedge, \vee, \overline{}, 0, 1\rangle$ be a complemented lattice. $\mathbf{L}$ is an R-algebra if and only if $L = \{0\}$. In particular, a boolean algebra $\mathbf{B}$ is an R-algebra if and only if $B = \{0\}$.

Groups. Let $\mathbf{C}_n$ denote the n-element cyclic group and let $\mathbf{E}$ denote the trivial one element group.

A group is an R-group if and only if it isomorphic either to $\mathbf{E}$ *or to* $\mathbf{C}_2$.

Proof. $\Leftarrow$: obvious.

$\Rightarrow$: Let us first show that $\mathbf{C}_2 \times \mathbf{C}_2$ is not an R-group.

Consider $\varphi_1, \varphi_2 : C_2 \times C_2 \to C_2 \times C_2$ defined by $\varphi_1(\langle x, y\rangle) = \langle x + y, 0\rangle$ and $\varphi_2(\langle x, y\rangle) = \langle 0, x + y\rangle$. One easily verifies that φ_1 and φ_2 are retractions of $\mathbf{C}_2 \times \mathbf{C}_2$. On the other hand, $\varphi_1 + \varphi_2$ is not since $(\varphi_1 + \varphi_2) \circ (\varphi_1 + \varphi_2)(\langle 1, 0\rangle) = \langle 0, 0\rangle \neq \langle 1, 1\rangle = (\varphi_1 + \varphi_2)(\langle 1, 0\rangle)$.

Now, let $\mathbf{G} = \langle G, +, -, 0\rangle$ be an R-group and suppose that $\mathbf{G}$ is isomorphic neither to $\mathbf{E}$ nor to $\mathbf{C}_2$. According to Proposition 2, "$-$" is a retraction of $\mathbf{G}$, and that is possible if and only if $-x = x$ for all $x \in G$. Therefore, $\mathbf{G}$ is a 2-elementary abelian group and is isomorphic to a direct sum of certain number of $\mathbf{C}_2$'s. Since $\mathbf{G}$ is isomorphic neither to $\mathbf{E}$ nor to $\mathbf{C}_2$, $\mathbf{G}$ is a direct sum of at least two $\mathbf{C}_2$'s. Without loss of generality we can assume that elements of $\mathbf{G}$ are 01-sequences, the length of each being at least two. Consider the mapping $\varphi : G \to G$ given by

$$\varphi(\langle x_1, x_2, x_3, x_4, \ldots\rangle) = \langle x_1, x_2, 0, 0, \ldots\rangle.$$

φ is a retraction of $\mathbf{G}$ onto its subalgebra isomorphic to $\mathbf{C}_2 \times \mathbf{C}_2$. According to Proposition 4, $\mathbf{G}$ is not an R-group. □

Modules. *Let* ${}_{\mathbf{P}}\mathbf{A}$ *be a* $\mathbf{P}$*-module for some ring* $\mathbf{P}$. ${}_{\mathbf{P}}\mathbf{A}$ *is an R-algebra if and only if* $|A| = 1$ *or* $\mathbf{A} \cong \mathbf{C}_2$ *and there is an ideal* I *of* $\mathbf{P}$ *such that* $\mathbf{P}/I \cong \mathbf{GF}(2)$.

Proof. $\Leftarrow$: obvious.

$\Rightarrow$: Let $\mathbf{P}$ be a ring. As in the case of groups we show that ${}_{\mathbf{P}}(\mathbf{C}_2 \times \mathbf{C}_2)$ is not an R-algebra.

Now, let $\mathbf{A} = \langle A, +, -, 0\rangle$ be a $\mathbf{P}$-module that is an R-algebra and $|A| > 1$. As in the case of groups we show that $\mathbf{A} \cong \mathbf{C}_2$. For the sake of simplicity, let $\mathbf{A} = \mathbf{C}_2$. Let $I = \{p \in P : p \cdot 1 = 0\}$. Clearly, I is an ideal of $\mathbf{P}$, so let us show that $\mathbf{P}/I \cong \mathbf{GF}(2)$. Take any $r, s \in P \setminus I$. Then $r \cdot 1 = s \cdot 1 = 1$, whence $s - r \in I$ and thus $s + I \subseteq r + I$. The other inclusion follows analogously. □

In particular, we have the following

A vector space $\mathbf{V}$ *is an R-vector space if and only either* $V = \{0\}$ *or* $\mathbf{V}$ *is isomorphic to* $\mathbf{C}_2$ *over* $\mathbf{GF}(2)$.

Rings with unity. *Let* $\mathbf{P} = \langle P, +, -, 0, \cdot, 1\rangle$ *be a ring with unity.* $\mathbf{P}$ *is an R-ring if and only if* $|P| = 1$.

Proof. $\Leftarrow$: obvious.

$\Rightarrow$: Let $\mathbf{P} = \langle R, +, -, 0, \cdot\rangle$ be an R-ring. Then $\mathbf{P} \models x = -x$, $xy = yx$, $x^4 \approx x^2$. The first identity follows from the fact that "$-$" is a retraction of $\mathbf{P}$, whence $-(-x) = -x$. As for the last two identities, note that $\varphi(x) = x^2$ being a unary term operation of $\mathbf{P}$ is also a retraction of $\mathbf{P}$, whence $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(\varphi(x)) = \varphi(x)$, for all $x, y \in P$.

Let $|P| \geq 2$ and $P' = \{x^2 : x \in P\}$. Since $\varphi : P \to P'$ given by $\varphi(x) = x^2$ is a retraction of $\mathbf{P}$, $\mathbf{P'}$ is a retract of $\mathbf{P}$. Note that $0, 1 \in P'$, whence $|P'| \geq 2$. Let us show that $\mathbf{P'}$ is a boolean ring. Since $\mathbf{P}$ is a commutative ring with unity, so is $\mathbf{P'}$. For each $y \in P'$ we have that $y^2 = y$ since $y^2 = (x^2)^2 = x^4 = x^2 = y$. Therefore, $\mathbf{P'}$ is a boolean ring with at least two elements. Boolean rings are term equivalent to boolean algebras so from $|P'| \geq 2$ it follows that $\mathbf{P'}$ is not an R-ring. Proposition 4 ensures that $\mathbf{P}$ is not an R-ring. □

3. Lattices and semilattices

In this paragraph we characterise R-lattices and R-semilattices. We show that R-lattices have at most two elements, while R-semilattices coincide with zero-semilattices.

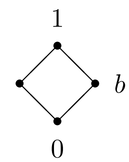
Let us recall that c_a denotes the constant mapping $c_a(x) = a$ and that $\mathrm{Const}(A)$ denotes the set of all the constant mappings $A \to A$.

Lattices. A sublattice $\mathbf{I}$ of a lattice $\mathbf{L}$ is said to be an *ideal* of $\mathbf{L}$ if $i \in I$ and $x \leq i$ imply $x \in I$. An ideal $\mathbf{I}$ is *prime* if $x \vee y \in I$ implies $x \in I$ or $y \in I$. A sublattice $\mathbf{F}$ of $\mathbf{L}$ is said to be a *filter* of $\mathbf{L}$ if $f \in F$ and $x \geq f$ imply $x \in F$. A filter $\mathbf{F}$ is *prime* if $x \wedge y \in F$ implies $x \in F$ or $y \in F$. If $\mathbf{I}$ is a prime ideal of $\mathbf{L}$, then $L \setminus I$ is a prime filter of $\mathbf{L}$, and vice versa, if $\mathbf{F}$ is a prime filter of $\mathbf{L}$, then $L \setminus F$ is a prime ideal of $\mathbf{L}$. Let $(a]$ denote the ideal of all the lattice elements below a: $(a] = \{x \in L : x \leq a\}$.

Lemma 7.

(*a*) *Let* $\mathbf{L} = \langle L, \wedge, \vee\rangle$ *be a chain.* $\mathbf{L}$ *is an R-lattice if and only if* $|L| \leq 2$.

(*b*) *The following lattice is not an R-lattice:*

$\mathbf{M}_2 =$ 

Proof. (a) $\Leftarrow$: obvious.

$\Rightarrow$: Let $|L| \geq 3$ and choose $0, 1, 2 \in L$ such that $0 < 1 < 2$. Consider $\varphi : L \to L$ given by:

$$\varphi(x) = \begin{cases} 2, & x \geq 2 \\ 0, & x < 2. \end{cases}$$

Obviously $\varphi, c_1 \in R_f(\mathbf{L})$. On the other hand, $\varphi \wedge c_1 : 2 \mapsto 1 \mapsto 0$, whence $\varphi \wedge c_1 \notin R_f(\mathbf{L})$. Thus, $\mathbf{L}$ is not an R-lattice.

(b) Consider $\varphi, \psi : L \to L$ given by:

$$\varphi = \begin{pmatrix} 0 & a & b & 1 \\ 0 & a & 0 & a \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} 0 & a & b & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

It is a routine to check that $\varphi, \psi \in R_f(\mathbf{L})$. On the other hand, $\varphi \wedge \psi : 1 \mapsto a \mapsto 0$, whence $\varphi \wedge \psi \notin R_f(\mathbf{L})$. Thus, $\mathbf{L}$ is not an R-lattice. □

Lemma 8. *If* $\mathbf{L}$ *is an R-lattice, then* $\mathbf{L}$ *is a distributive lattice.*

Proof. Let $\mathbf{L} = \langle L, \wedge, \vee \rangle$ be an R-lattice. Let us recall that $\{\mathrm{id}\} \cup \mathrm{Const}(L) \subseteq R_f(\mathbf{L})$. Consider the following mappings: $\varphi_a(x) = a \wedge x$ and $\psi_a(x) = a \vee x$. Since $\mathbf{L}$ is an R-algebra, we have $\varphi_a = c_a \wedge \mathrm{id} \in R_f(\mathbf{L})$ and $\psi_a = c_a \vee \mathrm{id} \in R_f(\mathbf{L})$ for each $a \in L$. Therefore, φ_a and ψ_a are homomorphisms of $\mathbf{L}$, i.e.:

$$\varphi_x(y \vee z) = \varphi_x(y) \vee \varphi_x(z) \quad \text{and} \quad \psi_x(y \wedge z) = \psi_x(y) \wedge \psi_x(z),$$

or, equivalently,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \text{and} \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \quad \square$$

Theorem 9. *Let* $\mathbf{L}$ *be a lattice.* $\mathbf{L}$ *is an R-lattice if and only if* $|L| \leq 2$.

Proof. $\Leftarrow$: obvious.

$\Rightarrow$: Let $\mathbf{L}$ be an R-lattice. According to Lemma 8, $\mathbf{L}$ is a distributive lattice. We shall show that $\mathbf{L}$ must be a chain. Suppose to the contrary that $\mathbf{L}$ is not a chain and let a and b be two incomparable elements in $\mathbf{L}$. Let $\mathbf{I}_a$ be the prime ideal of $\mathbf{L}$ such that $(a] \subseteq I_a \not\ni b$ and let $\mathbf{I}_b$ be the prime ideal of $\mathbf{L}$ such that $(b] \subseteq I_b \not\ni a$. Obviously, $I_a \not\subseteq I_b$ and $I_b \not\subseteq I_a$.

Let $F_a := L \setminus I_a$ and $F_b := L \setminus I_b$. $\mathbf{F}_a$ and $\mathbf{F}_b$ are prime filters of $\mathbf{L}$. Furthermore, let $0 := a \wedge b$ and $1 := a \vee b$. We have that $0 \in I_a \cap I_b$, $1 \in F_a \cap F_b$.

Consider a mapping $\varphi : L \to L$ defined by:

$$\varphi(x) = \begin{cases} 0, & x \in I_a \cap I_b \\ a, & x \in I_a \cap F_b \\ b, & x \in I_b \cap F_a \\ 1, & x \in F_a \cap F_b. \end{cases}$$

It is easy to verify that φ is a retraction of $\mathbf{L}$ onto $\mathbf{M}_2$. Hence, $\mathbf{M}_2$ is a retract of $\mathbf{L}$, which implies that $\mathbf{L}$ is not an R-lattice (Lemma 7(b), Proposition 4). Therefore, $\mathbf{L}$ is a chain. According to Lemma 7(a), $|L| \leq 2$. $\square$

Semilattices. A subsemilattice $\mathbf{I}$ of a semilattice $\mathbf{S} = \langle S, \cdot \rangle$ is said to be an *ideal* of $\mathbf{S}$ if $i \in I$ and $x \leq i$ imply $x \in I$. An ideal $\mathbf{I}$ is *prime* if $xy \in I$ implies $x \in I$ or $y \in I$. A subsemilattice $\mathbf{F}$ of $\mathbf{S}$ is said to be a *filter* of $\mathbf{S}$ if $f \in F$ and $x \geq f$ imply $x \in F$. If $\mathbf{I}$ is a prime ideal of $\mathbf{S}$, then $S \setminus I$ is a filter of $\mathbf{S}$, and vice versa, if $\mathbf{F}$ is a filter of $\mathbf{S}$, then $S \setminus F$ is a prime ideal of $\mathbf{S}$. Let $[a)$ denote the filter of all the semilattice elements above a: $[a) = \{x \in L : x \geq a\}$.

The proof of the following lemma is analogous to the proof of Lemma 7(a):

Lemma 10. *Let* $\mathbf{S} = \langle S, \cdot \rangle$ *be a chain.* $\mathbf{S}$ *is an R-semilattice if and only if* $|S| \leq 2$.

Lemma 11. *Let* $\mathbf{S}$ *be an R-semilattice. Let* $\mathbf{I}_1 \neq \mathbf{I}_2$ *be distinct prime ideals of* $\mathbf{S}$ *and let* $\emptyset \neq I_1 \subset I_2$*. Then* $\mathbf{I}_2 = \mathbf{S}$.

Proof. Suppose to the contrary that $\mathbf{I}_2 \neq \mathbf{S}$. Let $F_2 := S \setminus I_2$ be the corresponding filter of $\mathbf{S}$. It is obvious that $I_1 \cap F_2 = \emptyset$ and $I_1 \cup F_2 \neq S$. Choose arbitrary $1 \in F_2$ and $q \in S \setminus (I_1 \cup F_2)$. Set $p := 1 \cdot q$. One easily verifies that $p \in S \setminus (I_1 \cup F_2)$. Choose arbitrary $i \in I_1$ and set $0 := p \cdot i$. Obviously, $0 \in I_1$.

Consider the mapping $\varphi : S \to S$ defined by:

$$\varphi(x) = \begin{cases} 0, & x \in I_1 \\ p, & x \in S \setminus (I_1 \cup F_2) \\ 1, & x \in F_2. \end{cases}$$

φ is a retraction of $\mathbf{S}$ onto the three element chain $0 < p < 1$, which implies that $\mathbf{S}$ is not an R-semilattice (Lemma 10, Proposition 4). Contradiction. $\square$

Lemma 12. *If a semilattice has a subsemilattice isomorphic to a three-element chain, then the semilattice is not an R-semilattice.*

Proof. Let $a < b < c$ be a three-element chain in $\mathbf{S}$. Let $I_b = S \setminus [b)$ and $I_c = S \setminus [c)$. $\mathbf{I}_b$ and $\mathbf{I}_c$ are distinct prime ideals and $a \in I_b \subset I_c$. According to Lemma 11, $\mathbf{I}_c = \mathbf{S}$. But, $c \notin I_c$. Contradiction. $\square$

A semilattice $\mathbf{S} = \langle S, \cdot \rangle$ is called a *zero-semilattice* if $(\exists 0 \in S)(\forall x, y \in S)(x \neq y \Rightarrow xy = 0)$.

Theorem 13. *Let* **S** *be a semilattice.* **S** *is an R-semilattice if and only if* **S** *is a zero-semilattice.*

Proof. $\Rightarrow$: Let $\mathbf{S} = \langle S, \cdot \rangle$ be an R-semilattice. If **S** is a chain, then $|S| \leq 2$ (Lemma 10) and every such chain is trivially a zero-semilattice.

Let **S** be a semilattice that is not a chain. Let a and b be arbitrary incomparable elements in **S** and put $0 := ab$. Lemma 12 implies that **S** does not have a three-element chain.

Note that 0 is the least element in **S** (if $c < 0$ then $c < 0 < a$ is a three-element chain; if c and 0 are incomparable elements, then $c \cdot 0 < 0 < a$ is a three element chain). Using this fact, it is easy to prove that $x \neq y \Rightarrow xy = 0$. If $x = 0$ or $y = 0$, then $xy = 0$ since 0 is the least element in **S**. Suppose that $x \neq 0$, $y \neq 0$ and $xy \neq 0$. If $x < y$ then $0 < x < y$ is a three-element chain. If, on the other hand, x and y are incomparable, then $0 < xy < x$ is a three-element chain. Therefore, if $x \neq y$ then $xy = 0$.

$\Leftarrow$: Let 0 be the zero of **S**. For $X \subseteq S$, let $\varphi_X : S \to S$ denote the following mapping:

$$\varphi_X(x) = \begin{cases} 0, & x \notin X \\ x, & x \in X. \end{cases}$$

If $0 \in X$, then φ_X is a retraction of **S**. We shall prove that $R_f(\mathbf{S}) = \mathrm{Const}(\mathbf{S}) \cup \{\varphi_X : 0 \in X \subseteq S\}$.

$\supseteq$: obvious.

$\subseteq$: Let $\psi : S \to S$ be a retraction of **S**.

Case 1: $\psi(0) \neq 0$. Let $\psi(0) = a \neq 0$. We shall prove that $\psi = c_a$. Let x be an arbitrary element of S. If $x = a$ then $\psi(x) = \psi(a) = \psi(\psi(0)) = \psi(0) = a$. Suppose therefore that $x \neq a$. Since $xa = 0$, we have $\psi(x)\psi(a) = \psi(xa) = \psi(0) = a$. It is easy to see that $\psi(a) = a$: $\psi(a) = \psi(\psi(0)) = \psi(0) = a$. Thus, $\psi(x) \cdot a = a$ whence $\psi(x) = a$. Thus, $\psi = c_a$.

Case 2: $\psi(0) = 0$. First, we shall prove that for each $x \in S$, $\psi(x) \in \{0, x\}$. Let x be arbitrary element of S and suppose that $\psi(x) = y \notin \{0, x\}$. Obviously, $\psi(y) = y$. Since $x \neq y$, we have $xy = 0$ implying $\psi(0) = \psi(xy) = \psi(x)\psi(y) = yy = y \neq 0$. Contradiction.

Therefore, $\psi(x) \in \{0, x\}$ for each $x \in S$. Let $X = \{x \in S : \psi(x) = x\}$. It is easy to verify that $\psi = \varphi_X$.

Now, when we know that $R_f(\mathbf{S}) = \mathrm{Const}(\mathbf{S}) \cup \{\varphi_X : 0 \in X \subseteq S\}$, in order to complete the proof it suffices to show that $R_f(\mathbf{S})$ is closed with respect to "$\cdot$". This, however, follows easily from the following observations:

$$c_a \cdot c_b = \begin{cases} c_0, & a \neq b \\ c_a, & a = b \end{cases} ; \qquad c_a \cdot \varphi_X = \begin{cases} c_0, & a \notin X \\ \varphi_{\{0,a\}}, & a \in X \end{cases} ; \qquad \varphi_X \cdot \varphi_Y = \varphi_{X \cap Y}$$

(note that $\varphi_{\{0\}} = c_0$). □

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