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ON THE ${}_vM_m(s;a,z)$ FUNCTION

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Abstract. In this paper we study the special cases $\{{}_vM_m(s;m+n,z)\}_{n=2}^{\infty}$ and $\{{}_vM_m(s;a,n)\}_{n=1}^{\infty}$, where ${}_vM_m(s;a,z)$ is the function defined in [8]. Also, we give connections between ${}_vM_m(s;a,z)$ function and figured numbers, Stirling numbers of the first kind and Riemann Zeta function.

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1. Introduction

In 1971, Kurepa (see [4, 5]) defined so-called the left factorial $!n$ by:

$$!0 = 0, \quad !n = \sum_{k=0}^{n-1} k! \quad (n \in \mathbb{N})$$

and extended it to the complex half-plane $\mathrm{Re}\,(z) > 0$ as

$$!z = \int_0^{+\infty} \frac{t^z - 1}{t - 1}\, e^{-t}\, dt.$$

Such function can be also extended analytically to the whole complex plane by $!z =!(z+1) - \Gamma(z+1)$, where $\Gamma(z)$ is the gamma function defined by

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t}\, dt \quad (\mathrm{Re}\,(z) > 0).$$

Recently, Milovanović [6] defined and studied a sequence of the factorial functions $\{M_m(z)\}_{m=-1}^{+\infty}$ where $M_{-1}(z) = \Gamma(z)$ and $M_0(z) = !z$. Namely,

$$M_m(z) = \int_0^{+\infty} \frac{t^{z+m} - Q_m(t,z)}{(t-1)^{m+1}}\, e^{-t}\, dt \quad (\mathrm{Re}\,(z) > -(m+1)), \tag{1.1}$$

where the polynomials $Q_m(t;z)$, $m = -1, 0, 1, 2, \ldots$, are given by

$$Q_{-1}(t,z) = 0, \quad Q_m(t,z) = \sum_{k=0}^{m} \binom{m+z}{k} (t-1)^k.$$

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Since

$$M_m(z) = M_m(z+1) - M_{m-1}(z+1) \quad (m \in \mathbb{N}_0), \tag{1.2}$$

similar to the gamma function, the functions $z \mapsto M_m(z)$, for each $m \in \mathbb{N}_0$, can be extended analytically to the whole complex plane, starting from the corresponding analytic extension of the gamma function. Suppose that we have analytic extensions for all functions $z \mapsto M_\nu(z)$, $\nu < m$. Let the function $z \mapsto M_m(z)$ be defined by (1.1) for z in the half-plane $\mathrm{Re}\,(z) > -(m+1)$. Using successively (1.2), we define at first $M_m(z)$ for z in the strip $-(m+2) < \mathrm{Re}\,(z) < -(m+1)$, then for z such that $-(m+3) < \mathrm{Re}\,(z) < -(m+2)$, etc. In this way we obtain the function $M_m(z)$ in the whole complex plane.

In [8] the generalization is given of Milovanović's factorial function:

Definition 1.1 *For* $m = -1, 0, 1, 2, ...$ *and* $\mathrm{Re}\,(z) > v - m - 2$ *the function* ${}_vM_m(s; a, z)$ *defined by*

$${}_vM_m(s; a, z) = \sum_{k=1}^{v} (-1)^{k-1} \binom{z+m+1-k}{m+1} \mathcal{L}[s;\, {}_2F_1(a, k-z, m+2; 1-t)],$$

where v *is a positive integer,* s, a, z *are complex variables.*

The hypergeometric function ${}_2F_1(a, b, c; x)$ is defined by the series

$${}_2F_1(a, b, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \qquad (|x| < 1),$$

and has the integral representation

$${}_2F_1(a, b, c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a} dt,$$

in the x plane cut along the real axis from 1 to ∞, if $\mathrm{Re}\,(c) > \mathrm{Re}\,(b) > 0$. The symbols $(z)_n$ and $\mathcal{L}[s; F(t)]$ represent the Pochhammer symbol

$$(z)_0 = 1, \quad (z)_n = z(z+1)...(z+n-1) = \frac{\Gamma(z+n)}{\Gamma(z)},$$

and Laplace transform

$$\mathcal{L}[s; F(t)] = \int_0^{\infty} e^{-st} F(t) dt.$$

The binomial coefficient function defined via

$$\binom{z}{m} = \frac{\Gamma(z+1)}{\Gamma(m+1)\Gamma(z-m+1)}\,.$$

These function are of interest because their special cases include:

$$ {}_1M_m(1,1,z) = M_m(z) \qquad {}_1M_{-1}(1;1,z) = \Gamma(z) $$

$$ {}_1M_0(1;1,z) = !z \qquad {}_nM_{-1}(1;1,n+1) = A_n, $$

where A_n denotes the alternating factorial numbers

$$ A_n = \sum_{k=1}^{n} (-1)^{n-k} k!. $$

For the complex half-plane $\mathrm{Re}\,(z) > 0$, the function A_z is defined by (see [8])

$$ A_z \stackrel{\text{def}}{=} \int_0^\infty \frac{x^{z+1} - (-1)^z x}{x+1}\, e^{-x} dx. $$

However, apart from $n!$, $!n$ and A_n twenty-five more well-known integer sequences in [10] are special cases of the function ${}_vM_m(s;a,z)$.

2. The statement results and proofs

Investigation of the function ${}_vM_m(s;a,z)$ by using the two statements could be translated to the investigation of the function ${}_1M_m(s;a,z)$ as follows: applying the relation ${}_2M_m(s;a,z) = {}_1M_m(s;a,z) - {}_1M_m(s;a,z-1)$, induction on v we have

$$ {}_vM_m(s;a,z) = \sum_{k=1}^{v} (-1)^{k-1}\, {}_1M_m(s;a,z-k+1). \tag{2.3} $$

Using the relation

$$ {}_2F_1(a, 1-z, m+2; 1-t) = {}_2F_1(1-z, a, m+2; 1-t) \qquad (|1-t| < 1) $$

we have

$$ \binom{a+m}{m+1} {}_1M_m(s;a,z) = \binom{z+m}{m+1} {}_1M_m(s;1-z,a). $$

Hence

$$ {}_1M_m(s;a,z) = \frac{(z)_{m+1}}{(a)_{m+1}}\, {}_1M_m(s;1-z,a). $$

The relation (2.3) yields

$$ \sum_{k=1}^{v} (-1)^{k-1}\, {}_1M_m(s;a,z-k+1) = \sum_{k=1}^{v} (-1)^{k-1} \frac{(z-k+1)_{m+1}}{(a)_{m+1}}\, {}_1M_m(s;k-z,a) $$

i.e.,

$$ {}_vM_m(s;a,z) = \sum_{k=1}^{v} (-1)^{k-1} \frac{(z-k+1)_{m+1}}{(a)_{m+1}}\, {}_1M_m(s;k-z,a)\,. \tag{2.4} $$

2.1 The special case $\{{}_vM_m(s;m+n,z)\}_{n=2}^{\infty}$

For the function ${}_vM_m(s;a,z)$, the following special cases hold (see [8]):

$$ {}_1M_m(1;-n,z)=\frac{n!\,z(z+1)...(z+m)}{(n+m+1)!}\sum_{k=0}^{\infty}\binom{n+m+1}{k+m+1}\binom{z-1}{k}(-1)^kD_k, $$

$$ {}_1M_m(1/n;m+2,z)=\frac{n^z\Gamma(z+m+1)}{(m+1)!}, $$

where D_k denotes the *derangement numbers* (sequence A000166 in [10]). We now introduce a generalization of the last relation:

Theorem 2.1. *For every natural number $n>1$ and $\operatorname{Re}(s)>0,\ \operatorname{Re}(z)>0$ we have*

$$ {}_vM_m(s;m+n,z)= $$

$$ =\sum_{k=1}^{v}\frac{(-1)^{k-1}s^{k-z-1}}{(m+n-1)!}\sum_{i=0}^{n-2}\binom{n-2}{i}(-s)^i\Gamma(z+m+n-k-i)\,. $$

Proof. Let the function $\beta_i^d(m,n)$ be defined by

$$ \beta_i^d(m,n)=\begin{cases}0, & \text{if } i+d\le n-2,\\ m+n-1, & \text{if } i+d>n-2.\end{cases} $$

Since

$$ {}_2F_1(m+n,1-z,m+2;1-t)=\sum_{j=0}^{\infty}\frac{(m+n)_j(1-z)_j}{(m+2)_j}\frac{(1-t)^j}{j!}\qquad(|1-t|<1) $$

using relations (see [1])

$$ \begin{aligned} a(1-z)\,{}_2F_1(a+1,b,c;z) &= (2a-c-az+bz)\,{}_2F_1(a,b,c;z)+\\ &+ (c-a)\,{}_2F_1(a-1,b,c;z) \end{aligned} $$

and

$$ (1-z)_n=\frac{(-1)^n\Gamma(z)}{\Gamma(z-n)} $$

we have

$$ {}_2F_1(m+n,1-z,m+2;1-t)= $$

$$ =\frac{t^{z-n+1}(m+1)!}{(m+n-1)!}\sum_{i=0}^{n-2}\binom{n-2}{i}(-1)^{n-i}t^i\prod_{d=1}^{n-2}(z-d+\beta_i^d(m,n)). $$

Hence

$$\mathcal{L}[s;\ {}_2F_1(m+n,1-z,m+2;1-t)] =$$

$$= \int_0^\infty e^{-st}\frac{t^{z-n+1}(m+1)!}{(m+n-1)!}\sum_{i=0}^{n-2}\binom{n-2}{i}(-1)^{n-i}t^i$$

$$\times\prod_{d=1}^{n-2}(z-d+\beta_i^d(m,n))\,dt$$

$$= \frac{(m+1)!}{(m+n-1)!}\sum_{i=0}^{n-2}\binom{n-2}{i}(-1)^{n-i}\prod_{d=1}^{n-2}(z-d+\beta_i^d(m,n))$$

$$\times\int_0^\infty e^{-st}t^{z-n+1+i}\,dt\,.$$

Since, for $\mathrm{Re}\,(s)>0$ and $\mathrm{Re}\,(z)>0$

$$\int_0^\infty e^{-st}t^{z-n+1+i}\,dt = s^{n-z-i-2}\,\Gamma(z-n+i+2)$$

we have

$$\mathcal{L}[s;\ {}_2F_1(m+n,1-z,m+2;1-t)] =$$

$$= \frac{(m+1)!}{(m+n-1)!}\sum_{i=0}^{n-2}\binom{n-2}{i}(-1)^{n-i}s^{n-z-i-2}\,\Gamma(z-n+i+2)$$

$$\times\prod_{d=1}^{n-2}(z-d+\beta_i^d(m,n))$$

$$= \frac{(m+1)!}{(m+n-1)!}\frac{\Gamma(z)}{s^z}\sum_{i=0}^{n-2}\binom{n-2}{i}(-s)^i\frac{\Gamma(z+m+n-1-i)}{\Gamma(z+m+1)},$$

so that

$${}_1M_m(s;m+n,z) =$$

$$= \binom{z+m}{m+1}\frac{(m+1)!}{(m+n-1)!}\frac{\Gamma(z)}{s^z}\sum_{i=0}^{n-2}\binom{n-2}{i}(-s)^i\frac{\Gamma(z+m+n-1-i)}{\Gamma(z+m+1)}$$

$$= \frac{s^{-z}}{(m+n-1)!}\sum_{i=0}^{n-2}\binom{n-2}{i}(-s)^i\Gamma(z+m+n-1-i).$$

The relation (2.3) yields

$$
\begin{aligned}
&{}_vM_m(s;m+n,z) = \\
= \quad & \sum_{k=1}^{v} \frac{(-1)^{k-1} s^{k-z-1}}{(m+n-1)!} \sum_{i=0}^{n-2} \binom{n-2}{i} (-s)^i \Gamma(z+m+n-k-i)\,.
\end{aligned}
$$

2.2 The special case $\{{}_vM_m(s;a,n)\}_{n=1}^{\infty}$

The function $\{{}_vM_m(s;a,n)\}_{n=1}^{\infty}$ can be expressed in terms of the falling factorial polynomials

$$
[x]_0 = 1, \quad [x]_n = x(x-1)...(x-n+1) = \frac{\Gamma(x+1)}{\Gamma(x-n+1)} \qquad (n \in \mathbb{N})
$$

in the form

Theorem 2.2. *For every natural number n and $\mathrm{Re}\,(s) > 0$ we have*

$$
{}_vM_m(s;a,n) = \sum_{k=1}^{v} (-1)^{k-1} \sum_{i=0}^{n-k} \frac{(-1)^i}{i!} \cdot s^{i+k-n-1} \cdot (a)_{n-k-i} \cdot [a-m-2]_i\,.
$$

Proof. For $n \in \mathbb{N}$, using the relation

$$
{}_2F_1(a,1-n,m+2;1-t) = \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} \frac{(a)_i}{(m+2)_i} (1-t)^i
$$

we have

$$
\begin{aligned}
&\mathcal{L}[s;\ {}_2F_1(a,1-n,m+2;1-t)] = \\
= \quad & \int_0^{\infty} e^{-st} \sum_{i=0}^{n-1} \binom{n-1}{i} \frac{(-1)^i\,(a)_i}{(m+2)_i} (1-t)^i \,dt \\
= \quad & \sum_{i=0}^{n-1} \binom{n-1}{i} \frac{(-1)^i\,(a)_i}{(m+2)_i} \int_0^{\infty} e^{-st} (1-t)^i \,dt\,.
\end{aligned}
$$

Since, for $\mathrm{Re}\,(s) > 0$

$$
\int_0^{\infty} e^{-st} (1-t)^i \,dt = (-1)^i s^{-i-1} e^{-s}\,\Gamma(i+1,-s)\,,
$$

where $\Gamma(z,x)$ is the incomplete gamma function defined by

$$
\Gamma(z,x) = \int_x^{+\infty} t^{z-1} e^{-t} \,dt\,,
$$

we have

$$\mathcal{L}[s;\ {}_2F_1(a,1-n,m+2;1-t)] = \sum_{i=0}^{n-1} \binom{n-1}{i} \frac{(a)_i\, s^{-i-1}}{(m+2)_i}\, e^{-s}\, \Gamma(i+1,-s).$$

Hence

$${}_1M_m(s;a,n) = \binom{n+m}{m+1} \sum_{i=0}^{n-1} \binom{n-1}{i} \frac{(a)_i\, s^{-i-1}}{(m+2)_i}\, e^{-s}\, \Gamma(i+1,-s).$$

Then the following well-known relation

$$\Gamma(i+1,-s) = e^s\, i! \sum_{j=0}^{i} (-1)^j \frac{s^j}{j!}\,, \qquad (\operatorname{Re}(s) > 0)$$

yields

$$\begin{aligned} {}_1M_m(s;a,n) &= \binom{n+m}{m+1} \sum_{i=0}^{n-1} \binom{n-1}{i} \frac{(a)_i\, s^{-i-1} i!}{(m+2)_i} \sum_{j=0}^{i} (-1)^j \frac{s^j}{j!} \\ &= \sum_{i=0}^{n-1} \frac{(n+m)!}{(n-i-1)! \cdot (m+i+1)!}\, s^{-i-1}\, (a)_i \sum_{j=0}^{i} (-1)^j \frac{s^j}{j!} \\ &= \sum_{i=0}^{n-1} \frac{(-1)^i}{i!}\, s^{i-n}\, (a)_{n-i-1}\, [a-m-2]_i\,, \end{aligned}$$

so that

$${}_vM_m(s;a,n) = \sum_{k=1}^{v} (-1)^{k-1} \sum_{i=0}^{n-k} \frac{(-1)^i}{i!} \cdot s^{i+k-n-1} \cdot (a)_{n-k-i} \cdot [a-m-2]_i\,.$$

2.3 Remarks

Remark 2.3 *Let*

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \qquad (\operatorname{Re}(z) > 1)$$

denote the Riemann Zeta-function. Applying Theorem 2.1 for $m = -1$ we have

$$\zeta(z) = \frac{1}{{}_1M_{-1}(1;1,z)} \sum_{n=1}^{\infty} {}_1M_{-1}(n;1,z) \qquad (\operatorname{Re}(z) > 1).$$

Remark 2.4 *According to the previous definition, for $a = 0$ we have*

$$ {}_vM_m(s;0,z) = \frac{1}{s}\sum_{k=1}^{v}(-1)^{k-1}\binom{z+m-k+1}{m+1}. $$

Hence

$$ \begin{aligned} \left\{ z \atop m \right\} &= {}_1M_{m-1}(1;0,z) \\ \sum_{k=0}^{m} s(m,k)\cdot z^k &= {}_1M_{m-1}(1;0,z-m+1)\cdot {}_1M_1(1;1,m+1) \quad (m\in\mathbb{N}), \end{aligned} $$

where $\left\{ z \atop m \right\} = \binom{z+m-1}{m}$ *is the figured number (see [2]) and* $s(n,m)$ *is the Stirling number of the first kind defined via*

$$ x(x-1)...(x-n+1) = \sum_{k=0}^{n} s(n,k)x^k. $$

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References

[1] Abramowitz, M., Stegun, I.A., Handbook of Mathematical Functions, National Bureau of Standards, Washington, 1970.

[2] Butzer, P.L., Hauss, M., Schmidt, M., Factorial functions and Stirling numbers of fractional orders, Results in Matematics, Vol. 16(1989), 147–153.

[3] Carlitz, L., A note on the left factorial function, Math. Balkanica 5 (1975), 37–42.

[4] Kurepa, Đ., On the left factorial function $!n$, Math. Balkanica 1 (1971), 147–153.

[5] Kurepa, Đ., Left factorial function in complex domain, Math. Balkanica 3 (1973), 297–307.

[6] Milovanović, G.V., A sequence of Kurepa's functions, Scientifiv Rewiew No. 19-20 (1996), 137–146.

[7] Milovanović, G.V., Petojević, A., Generalized factorial function, numbers and polynomials and related problems, Math. Balkanica, New Series, Vol. 16, Fasc 1-4, (2002), 113-130.

[8] Petojević, A., The function ${}_vM_m(s;a,z)$ and some well-known sequences, Journal of Integer Sequences, Vol. 5 (2002), Article 02.1.7, 1–16.

[9] Prudnikov, A.P., Brychkov, Yu. A., Marichev, O.I., Integrals and Series. Elementary Functions, Nauka, Moscow, 1981. (in Russian)

[10] Sloane, N.J.A., The On-Linea Encyclopedia of Integer Sequence, published elec. at `http://www.research.att.com/~njas/sequences/`

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