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ON THE PERIODIC SOLUTION OF A CLASS OF DIFFERENCE EQUATIONS

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Abstract. Solutions of first-order difference equations are investigated with respect to their periodicities. A complete answer to one open problem of Kulenović and Ladas is given.

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1. Introduction

In the recent book by Kulenović and Ladas [2] a large number of open problems and conjectures, concerning the dynamics of rational difference equations were given.

We are focusing our attention on the open problem 3.4.3: Assume that the difference equation is of the form

$$x_{n+1} = f(x_n) \tag{1}$$

where

$$f \in C[(0,\infty),(0,\infty)]. \tag{2}$$

Obtain necessary and sufficient condition on f so that every positive solution of (1) is periodic with period $k,\ k \geq 2$.

The sequence

$$x_0,\ f(x_0),\dots,f^k(x_0),\dots,$$

where $f^{n+1} = f(f^n);\ n \in N$, is called the orbit of x_0. We are interested to describe the dynamics of f, the behaviour of points under iteration of f.

We have periodic orbit or cycle of period k if $f^k(x_0) = x_0$. f is k-periodic if $f^k(x) = x$ for all x belonging to the domain D of f. It is obvious that k-periodicity implies $k\ell$-periodicity for $\ell \in N$, such a $k\ell$-periodicity for $\ell = 2, 3, \dots$, we will call trivial.

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2. Results

Let us give some characterizations of the f.

Lemma 1. *If $k \geq 2$ then k-periodic function f which satisfies (1) and (2) is monotonously decreasing.*

Proof. Suppose, on the contrary, that for different points x_0, y_0 we have $f(x_0) = f(y_0)$. After $(k-1)$ iterations we get $x_0 = f^k(x_0) = f^k(y_0) = y_0$, a contradiction.

Suppose that f is increasing and f is not 1-periodic ($f(x) \not\equiv x$). If for some x_0 is $f(x_0) \neq x_0$, then the orbit of x_0 is monotonically increasing if $f(x_0) > x_0$ or monotonically decreasing if $f(x_0) < x_0$. In any case f is not periodic.

According to (2) and observation above $f(x)$ is monotonically decreasing such that $\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim\limits_{x\to\infty} f(x) = 0$. If f is 2-periodic then $f(x) \equiv f^{-1}(x)$ and such a function is uniquely determined if we know it on the interval $(0, x_T)$ where $f(x_T) = x_T$. □

Lemma 2. *There are no k-periodic functions if k is a prime number greater than 2.*

Proof. It is an immediate consequence of the fact that

$$(f^n(x_0) - x_0)(f^{n+1}(x_0) - x_0) < 0$$

for $x_0 \neq x_T$. □

Now, we can prove the following theorem.

Theorem 1. *There are only trivial $2n$-periodic functions which satisfy (1) and (2).*

Proof. Suppose that $f(x)$ is not a trivial $2n$-periodic function i.e. $f(x) \not\equiv f^{-1}(x)$. Then there exists an interval $I = (a, b)$ such that $x_T \leq a < b \leq \infty$, $f(a) = f^{-1}(a)$, $f(b) = f^{-1}(b)$ (if $b < \infty$) and $f(x) \neq f^{-1}(x)$ for $x \in I$. Then for $x_0 \in I$ we get

$$b = f^2(b) > f^2(x_0 > x_0 > f^2(a) = a \quad \text{if} \quad f(x_0) < f^{-1}(x_0)$$

or

$$b = f^2(b) > x_0 > f^2(x_0) > f^2(a) = a \quad \text{if} \quad f(x_0) > f^{-1}(x_0).$$

It completes the proof. □

At the same time we see that for $x_0 \in I$ solutions of (1) converge to a prime period-two solutions:

$a,\ f(a), a, \ldots$ if $f(x) > f^{-1}(x)$ on I and $a > x_T$,

$b,\ f(b), b, \ldots$ if $f(x) < f^{-1}(x)$ on I and $b < \infty$.

References

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