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LAGUERRE-LIKE METHODS WITH CORRECTIONS FOR THE INCLUSION OF POLYNOMIAL ZEROS

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Abstract. Iterative methods of Laguerre's type for the simultaneous inclusion of all zeros of a polynomial are proposed. Using Newton's and Halley's corrections, the order of convergence of the basic method is increased from 4 to 5 and 6, respectively. Further improvements are achieved by the Gauss-Seidel approach. Using the concept of the R-order of convergence of mutually dependent sequences, we present the convergence analysis of total-step and single-step methods. The suggested algorithms possess a great computational efficiency since the increase of the convergence rate is attained without additional calculations. The case of multiple zeros is also studied. Two numerical examples are given to demonstrate the convergence properties of the proposed methods.

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1. Introduction

This paper is devoted to the construction of inclusion methods with very high computational efficiency for the simultaneous inclusion of polynomial zeros and presents the continuation of a research exposed recently in [8]. We recall that iterative methods for the simultaneous determination of polynomial zeros, realized in interval arithmetic, produce resulting real or complex intervals (disks or rectangles) containing the wanted zeros. In this manner the information about upper error bounds of approximations to the zeros are provided (see the books [1], [10], [14] for more details).

The presentation of the paper is organized as follows. The basic properties of circular complex arithmetic, necessary for the development and convergence analysis of the presented inclusion methods, are given in the introduction. The basic Laguerre-like total-step method of the fourth order, recently proposed in [8], is presented in short in Section 2. The main goal of our study is to achieve remarkably faster convergence with only few additional numerical operations, which significantly increases the computational efficiency. For this purpose, the modified total-step methods with the increased convergence speed is developed in Section 3 using Newton's and Halley's correction. The convergence analysis

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of these improved methods is given in Section 4. Some important tasks, as the construction of Laguerre-like methods with corrections in single-step mode and modified variants for the inclusion of multiple zeros, are studied in Section 5. Numerical results obtained by the considered methods are given in Section 6.

The construction of inclusion methods in circular complex arithmetic and their convergence analysis require the basic properties of circular complex arithmetic. A circular closed region (disk) $Z := \{z : |z-c| \le r\}$ with the center $c :=$ mid Z and radius $r :=$ rad Z we will denote by parametric notation $Z := \{c; r\}$. If $Z_k := \{c_k; r_k\}\,(k = 1, 2)$, then

$$\begin{aligned} Z_1 \pm Z_2 &= \{c_1 \pm c_2; r_1 + r_2\}, \\ Z_1 \cdot Z_2 &= \{c_1c_2; |c_1|r_2 + |c_2|r_1 + r_1r_2\}. \end{aligned}$$

The addition and subtraction of disks are exact operations.

The inversion of a non-zero disk Z is defined by the Möbius transformation,

$$Z^{-1} = \{c; r\}^{-1} = \frac{\{\bar{c}; r\}}{|c|^2 - r^2} \quad (|c| > r,\ \text{i.e. } 0 \notin Z). \tag{1}$$

The inversion Z^{-1} is also an exact operation, that is, $Z^{-1} = \{z^{-1} : z \in Z\}$.

Beside the exact inversion Z^{-1} of a disk Z, the so-called *centered inversion* Z^{I_c} defined by

$$Z^{I_c} = \{c; r\}^{I_c} := \left\{\frac{1}{c}; \frac{r}{|c|(|c| - r)}\right\} \supseteq Z^{-1} \quad (0 \notin Z) \tag{2}$$

is often used. Sometimes, we will use the symbol INV to denote both inversions, that is INV $\in \{()^{-1}, ()^{I_c}\}$.

Having in mind (1) and (2) the division is defined by

$$Z_1 : Z_2 = Z_1 \cdot \mathrm{INV} Z_2 \quad (0 \notin Z_2,\ \mathrm{INV} \in \{()^{-1}, ()^{I_c}\}.$$

The square root of a disk $\{c; r\}$ in the centered form, where $c = |c|e^{i\theta}$ and $|c| > r$, is defined as the union of two disjoint disks (see [3]):

$$\{c; r\}^{1/2} := \left\{\sqrt{|c|}e^{i\theta/2}; R\right\} \bigcup \left\{-\sqrt{|c|}e^{i\theta/2}; R\right\}, \tag{3}$$

where $R = \dfrac{r}{\sqrt{|c|} + \sqrt{|c| - r}}$.

In this paper we will use the following obvious properties:

$$z \in \{c; r\} \iff |z - c| \le r, \tag{4}$$

$$\{c_1; r_1\} \cap \{c_2; r_2\} = \emptyset \iff |c_1 - c_2| > r_1 + r_2, \tag{5}$$

$$|\mathrm{mid}\ Z| - \mathrm{rad}\ Z \le |z| \le |\mathrm{mid}\ Z| + \mathrm{rad}\ Z \quad (z \in Z). \tag{6}$$

More details about circular arithmetic can be found in the books [1, Ch. 5] and [14, Ch. 2]. Throughout this paper disks in the complex plane will be denoted by capital letters.

2. Total–step method without corrections

Let $P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ be a monic polynomial with simple zeros $\zeta_1, \ldots, \zeta_n$ and let $\mathcal{I}_n := \{1, \ldots, n\}$ be the index set. For the point $z = z_i$ $(i \in \mathcal{I}_n)$ let us introduce

$$\Sigma_{k,i} = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{(z_i-\zeta_j)^k} \quad (k=1,2), \quad q_i^* = nT_{2,i} - \frac{n}{n-1}T_{1,i}^2,$$

$$\delta_{1,i} = \frac{P'(z_i)}{P(z_i)}, \quad \delta_{2,i} = \frac{P'(z_i)^2 - P(z_i)P''(z_i)}{P(z_i)^2}, \quad \varepsilon_i = z_i - \zeta_i.$$

The following identity

$$n\delta_{2,i} - \delta_{1,i}^2 - q_i^* = \frac{1}{n-1}\Big(\frac{n}{\varepsilon_i} - \delta_{1,i}\Big)^2 \tag{7}$$

was proved in [8]. From (7) we obtain the fixed point relation

$$\zeta_i = z_i - \frac{n}{\delta_{1,i} \pm \sqrt{(n-1)(n\delta_{2,i} - \delta_{1,i}^2 - q_i^*)}} \quad (i \in \mathcal{I}_n), \tag{8}$$

which is the base for the construction of inclusion methods of Laguerre's type.

To simplify the notation, let us introduce the following vectors of disks

$$\begin{aligned}
\boldsymbol{Z}^{(m)} &= \big(Z_1^{(m)}, \ldots, Z_n^{(m)}\big) \text{ (inclusion disks)},\\
\boldsymbol{Z}_N^{(m)} &= \big(Z_{N,1}^{(m)}, \ldots, Z_{N,n}^{(m)}\big),\ Z_{N,i}^{(m)} = Z_i^{(m)} - N\big(z_i^{(m)}\big) \text{ (Newton's disks)},\\
\boldsymbol{Z}_H^{(m)} &= \big(Z_{H,1}^{(m)}, \ldots, Z_{H,n}^{(m)}\big),\ Z_{H,i}^{(m)} = Z_i^{(m)} - H\big(z_i^{(m)}\big) \text{ (Halley's disks)},
\end{aligned}$$

where $m = 0, 1, 2\ldots$ is the iteration index and

$$\begin{aligned}
N(z) &= \frac{P(z)}{P'(z)} \text{ (Newton's correction)},\\
H(z) &= \left[\frac{P'(z)}{P(z)} - \frac{P''(z)}{2P'(z)}\right]^{-1} \text{ (Halley's correction)}.
\end{aligned}$$

For brevity, we will write sometimes $z_i, r_i, \hat{z}_i, \hat{r}_i, Z_i, \hat{Z}_i, Z_{N,i}, Z_{H,i}$ instead of $z_i^{(m)}, r_i^{(m)}$, $z_i^{(m+1)}$, $r_i^{(m+1)}$, $Z_i^{(m)}, Z_i^{(m+1)}, Z_{N,i}^{(m)}, Z_{H,i}^{(m)}$. In what follows we will write $w_1 \sim w_2$ or $w_1 = O_M(w_2)$ (the same order of magnitude) for two complex numbers w_1 and w_2 that satisfy $|w_1| = O(|w_2|)$.

Let us define the disk

$$S_{k,i}(\boldsymbol{X}, \boldsymbol{W}) := \sum_{j=1}^{i-1}\Big(\mathrm{INV}_1(z_i - X_j)\Big)^k + \sum_{j=i+1}^{n}\Big(\mathrm{INV}_1(z_i - W_j)\Big)^k, \tag{9}$$

for $k = 1, 2$, where $\boldsymbol{X} = (X_1, ..., X_n)$ and $\boldsymbol{W} = (W_1, ..., W_n)$ are vectors whose components are disks and $\mathrm{INV}_1 \in \{()^{-1}, ()^{I_c}\}$, and define the disk

$$Q_i(\boldsymbol{X}, \boldsymbol{W}) = nS_{2,i}(\boldsymbol{X}, \boldsymbol{W}) - \frac{n}{n-1} S_{1,i}^2(\boldsymbol{X}, \boldsymbol{W}).$$

Then, using (9) and the definition of q_i^*, according to the inclusion isotonicity we have $q_i^* \in Q_i(\boldsymbol{X}, \boldsymbol{W})$.

Let $Z_1^{(0)}, ..., Z_n^{(0)}$ be initial disjoint disks containing the zeros $\zeta_1, ..., \zeta_n$, that is, $\zeta_i \in Z_i^{(0)}$ for $i \in \mathcal{I}_n$. Taking inclusion disks $Z_1^{(m)}, \ldots, Z_n^{(m)}$ instead of these zeros in (8), we define the disk

$$A_i^{(m)} = \delta_{1,i}^{(m)} + \Big[(n-1)\big(n\delta_{2,i}^{(m)} - \big(\delta_{1,i}^{(m)}\big)^2 - Q_i\big(\boldsymbol{Z}^{(m)}, \boldsymbol{Z}^{(m)}\big)\Big]_*^{1/2}$$

and state the following *total-step* method for the simultaneous inclusion of all zeros of P,

$$Z_i^{(m+1)} = z_i^{(m)} - n\,\mathrm{INV}_2(A_i^{(m)}) \quad (i \in \mathcal{I}_n), \tag{10}$$

where $z_i^{(m)} = \operatorname{mid}\ Z_i^{(m)}$, $\mathrm{INV}_2 \in \{()^{-1}, ()^{I_c}\}$. In the realization of the iterative formula (10) we first apply the inversion INV_1 to the sums (9), and then the inversion INV_2 in the final step. The interval Laguerre-like method (10) was recently stated in [8].

According to (3), the square root of a disk in (10) produces two disks; the symbol $*$ indicates that one of the two disks has to be chosen. That disk will be called a "*proper*" disk. From (7) and the inclusion $q_i^* \in Q_i$ we conclude that the proper disk is one which contains $n/\varepsilon_i - \delta_{1,i}$. Taking into account (3), we have

$$\Big((n-1)\big(n\delta_{2,i} - \delta_{1,i}^2 - Q_i\big)\Big)^{1/2} = G_{1,i} \cup G_{2,i}, \quad \operatorname{mid}\ G_{k,i} = g_{k,i}, \quad g_{1,i} = -g_{2,i}$$

for $i \in \mathcal{I}_n$, $k = 1, 2$. The criterion for the choice of a proper disk is considered in [3] (see also [9]) and reads:

If the disks $Z_1, \ldots, Z_n$ are reasonably small, then we have to choose that disk (between $G_{1,i}$ and $G_{2,i}$), whose center minimizes $|P'(z_i)/P(z_i) - g_{k,i}|$ $(k = 1, 2)$.

The iterative method (10) with $\mathrm{INV}_1, \mathrm{INV}_2 = ()^{-1}$ or $()^{I_c}$ has the order of convergence equal to *four* (see [8]). The convergence of this method can be accelerated using already calculated disks in the current iteration (Gauss-Seidel approach). In this manner we obtain the *single-step* method

$$Z_i^{(m+1)} = z_i^{(m)} - n\,\mathrm{INV}_2(B_i^{(m)}) \quad (i \in \mathcal{I}_n), \tag{11}$$

where

$$B_i^{(m)} = \delta_{1,i}^{(m)} + \Big[(n-1)\big(n\delta_{2,i}^{(m)} - \big(\delta_{1,i}^{(m)}\big)^2 - Q_i\big(\boldsymbol{Z}^{(m+1)}, \boldsymbol{Z}^{(m)}\big)\Big]_*^{1/2}.$$

The R-order of convergence of the single-step method (11) is at least $3+x_n$, where $x_n>1$ is the unique positive root of the equation $x^n-x-3=0$ (see [10]).

3. Laguerre-like methods with corrections

Let us introduce the abbreviations

$$r^{(m)}=\max_{1\le i\le n} r_i^{(m)},\quad \rho^{(m)}=\min_{\substack{1\le i,j\le n\\ i\ne j}}\left\{\left|z_i^{(m)}-z_j^{(m)}\right|-r_j^{(m)}\right\},$$
$$\varepsilon_i^{(m)}=z_i^{(m)}-\zeta_i,\quad \left|\epsilon^{(m)}\right|=\max_{1\le i\le n}\left|\varepsilon_i^{(m)}\right|$$

for $i\in\mathcal{I}_n$, $m=0,1,\ldots$. Further increase of the convergence speed of the iterative methods (10) and (11) can be achieved using Newton's or Halley's correction in the similar way as in [2], [11] and [12]. In this construction we assume that initial inclusion disks $Z_1^{(0)},\ldots,Z_n^{(0)}$, containing the zeros $\zeta_1,\ldots,\zeta_n$, have been chosen in such a way that each disk $Z_i^{(0)}-N\big(\mathrm{mid}\ \big(Z_i^{(0)}\big)\big)$ or $Z_i^{(0)}-H\big(\mathrm{mid}\ \big(Z_i^{(0)}\big)\big)$ also contains the zero ζ_i $(i\in\mathcal{I}_n)$. This point is the subject of the following assertion where, for simplicity, the iteration indices are omitted.

Lemma 1. *Let $Z_1,\ldots,Z_n$ be inclusion disks for the zeros $\zeta_1,\ldots,\zeta_n$, $\zeta_i\in Z_i$, and let $z_i=\mathrm{mid}\ Z_i$, $r_i=\mathrm{rad}\ Z_i$. If the inclusion disks $Z_1,\ldots,Z_n$ are chosen so that the inequality*

$$\rho>3(n-1)r \tag{12}$$

is satisfied, then for $i\in\mathcal{I}_n$ we have the implications:

(*i*) $\zeta_i\in Z_i\quad\Rightarrow\quad \zeta_i\in Z_{N,i}:=Z_i-N(z_i)$;

(*ii*) $\zeta_i\in Z_i\quad\Rightarrow\quad \zeta_i\in Z_{H,i}:=Z_i-H(z_i)$.

This lemma can be proved in a similar way as in [13] so that we omit the proof.

Starting from the fixed-point relation (8) we can construct the total-step Laguerre-like inclusion methods with Newton's and Halley's corrections. We will study the convergence rate of these methods simultaneously, using a uniform approach. For this purpose we indicate these methods with the additional superscript indices $\lambda=1$ (for Newton's correction) and $\lambda=2$ (for Halley's correction). Consequently, we denote the corresponding vectors of disk approximations as follows:

$$\begin{aligned}\boldsymbol{Z}^{(1)} &= \big(Z_1^{(1)},\ldots,Z_n^{(1)}\big)=\big(Z_{N,1},\ldots,Z_{N,n}\big)\\ \boldsymbol{Z}^{(2)} &= \big(Z_1^{(2)},\ldots,Z_n^{(2)}\big)=\big(Z_{H,1},\ldots,Z_{H,n}\big).\end{aligned}$$

Both corrections $N(z_i)$ and $H(z_i)$ will be also denoted by $C^{(1)}(z_i)$ and $C^{(2)}(z_i)$, respectively. For simplicity, we will omit the iteration index for all quantities at the m-th iteration, while the quantities at the $(m+1)$-st iteration will be denoted with the additional symbol ˆ ("hat"). Now we can write both methods in the unique form as

$$\hat{Z}_i = z_i - n\mathrm{INV}_2\Big(\delta_{1,i} + \Big[(n-1)\big(n\delta_{2,i} - \delta_{1,i}^2 - Q_i\big(\boldsymbol{Z}^{(\lambda)}, \boldsymbol{Z}^{(\lambda)}\big)\Big]_*^{1/2}\Big) \tag{13}$$

for $i \in \mathcal{I}_n$ and $\lambda = 1, 2$. Since we can apply two types of inversions in the calculation of the sums (9), by combining the inversions $()^{-1}$ and $()^{I_c}$ in (13) we are in the possibility to construct four inclusion methods.

4. Convergence of the improved methods

Before considering convergence properties of the simultaneous interval method (13) and initial conditions for its convergence, we will give some necessary estimates.

It is easy to show that

$$z_i - Z_j + C^{(\lambda)}(z_j) = \{z_i - \zeta_j + \xi_j^{(\lambda)}\varepsilon_j^{\lambda+1}; r_j\},$$

where

$$\xi_j^{(1)} = -\frac{\Sigma_{1,j}}{1 + \varepsilon_j\Sigma_{1,j}} \quad \text{and} \quad \xi_j^{(2)} = -\frac{\Sigma_{1,j}^2 + \Sigma_{2,j}}{2 + 2\varepsilon_j\Sigma_{1,j} + \varepsilon_j^2(\Sigma_{1,j}^2 + \Sigma_{2,j})}.$$

For brevity, let us set for $\lambda = 1, 2$:

$$\begin{aligned}
h_{ij}^{(\lambda)} &= \mathrm{mid}\ \big(z_i - Z_j + C^{(\lambda)}(z_j)\big) = z_i - \zeta_j + \xi_j^{(\lambda)}\varepsilon_j^{\lambda+1}, \quad w_{ij}^{(\lambda)} = \frac{1}{h_{ij}^{(\lambda)}},\\
d_{ij}^{(\lambda)} &= \frac{r_j}{|h_{ij}^{(\lambda)}|\big(|h_{ij}^{(\lambda)}| - r_j\big)}, \quad s_{k,i}^{(\lambda)} = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{\big(z_i - z_j + C_j^{(\lambda)}\big)^k} \quad (k = 1, 2),\\
q_i^{(\lambda)} &= ns_{2,i}^{(\lambda)} - \frac{n}{n-1}\big(s_{1,i}^{(\lambda)}\big)^2, \quad f_i^{(\lambda)} = n\delta_{2,i} - \delta_{1,i}^2 - q_i^{(\lambda)},\\
v_i^{(\lambda)} &= \frac{(n-1)\big(q_i^* - q_i^{(\lambda)}\big)}{(n/\varepsilon_i - \delta_{1,i})^2}, \quad \eta = \frac{25}{2}n(n-1)\frac{r|\varepsilon|}{\rho^3}, \quad \gamma = \frac{41n(n-1)}{5\rho^3}.
\end{aligned}$$

Lemma 2. *Let the inequality* (12) *hold. Then*

(*i*) $d_{ij}^{(\lambda)} < \dfrac{5r}{3\rho^2}$;

(*ii*) $|w_{ij}^{(\lambda)}| < \dfrac{12}{11\rho}$;

(*iii*) $\left|f_i^{(\lambda)}\right| - \gamma r > \dfrac{1}{\varepsilon_i^2}\Big(n - \dfrac{5}{2}\Big) > 0.$

(*iv*) $\sqrt{(n-1)\{f_i^{(\lambda)};\gamma r\}} \subset \Big\{\sqrt{(n-1)f_i^{(\lambda)}};\eta\Big\};$

(*v*) $\sqrt{1+v_i^{(\lambda)}} \in \Big\{1;\dfrac{|\varepsilon_i|}{5\rho}\Big\};$

The proofs of the assertions (i)–(v) are similar with those given in [13] and will be omitted to save a space.

Let IM be an iterative numerical method which generates k sequences $\{z_1^{(m)}\},\ldots,\{z_k^{(m)}\}$ for the approximation of the solutions $z_1^*,\ldots,z_k^*$. To estimate the order of convergence of the iterative method IM we usually introduce the error-sequences

$$\varepsilon_i^{(m)} = ||z_i^{(m)} - z_i^*|| \quad (i = 1,\ldots,k).$$

The convergence analysis of inclusion methods with corrections needs the following assertion, which is a special case of Theorem 3 given in [5]:

Theorem 1. *Given the error-recursion*

$$\varepsilon_i^{(m+1)} \le \alpha_i \prod_{j=1}^{k} \big(\varepsilon_j^{(m)}\big)^{t_{ij}}, \quad (i \in \mathcal{I}_k;\ m = 0,1,2,\ldots), \tag{14}$$

where $t_{ij} \ge 0,\ \alpha_i > 0,\ 1 \le i,j \le k$. *Denote the matrix of exponents appearing in* (14) *with* T_k, *that is* $T_k = [t_{ij}]_{k\times k}$. *If the non-negative matrix* T_k *has the spectral radius* $\rho(T_k) > 1$ *and a corresponding eigenvector* $\boldsymbol{x}_\rho > 0$, *then the* R*-order of all sequences* $\{\varepsilon_i^{(m)}\}$ $(i \in \mathcal{I}_k)$ *is at least* $\rho(T_k)$.

In the sequel the matrix $T_k = [t_{ij}]$ will be called the *R-matrix* because of its connection with the R-order of convergence.

Let $O_R(IM)$ denote the R-order of convergence of an iteration method IM. For the total-step methods (13) we can state

Theorem 2. *Assume that initial disks* $Z_1^{(0)},\ldots,Z_n^{(0)}$ *are chosen so that* $\zeta_i \in Z_i^{(0)}$ $(i \in \mathcal{I}_n)$ *and the inequality*

$$\rho^{(0)} > 3(n-1)r^{(0)} \tag{15}$$

holds. Then the inclusion methods (13) *are convergent and the following is true for each* $i \in \mathcal{I}_n$ *and* $m = 1,2,\ldots$:

1° $\rho^{(m)} > 3(n-1)r^{(m)};$

2° $\zeta_i \in Z_i^{(m)}$ *for each* $i \in \mathcal{I}_n$ *and* $m = 1, 2, \ldots$;

3° *the lower bound of the R-order of convergence of the interval methods* (13) *is*

$$O_R(13) \geq \begin{cases} \lambda + 4 \quad (\lambda = 1, 2), & \textit{if} \quad \mathrm{INV}_1 = ()^{I_c}, \\ 2 + \sqrt{7} \cong 4.646, & \textit{if} \quad \mathrm{INV}_1 = ()^{-1}. \end{cases}$$

Proof. Let us note that the condition (15) provides that initial disks $Z_1^{(0)}, \ldots, Z_n^{(0)}$ be disjoint. Indeed, for arbitrary pair $i, j \in \mathcal{I}_n$ $(i \neq j)$ we have

$$|z_i^{(0)} - z_j^{(0)}| > \rho^{(0)} > 3(n-1)r^{(0)} > 2r^{(0)} \geq r_i^{(0)} + r_j^{(0)},$$

which means that $Z_i^{(0)} \cap Z_j^{(0)} = \emptyset$ (according to (5)).

The assertions of Theorem 2 will be proved by mathematical induction. In the sequel we will often use the inequality (12) in the form

$$\frac{r}{\rho} < \frac{1}{3(n-1)} \leq \frac{1}{6}, \tag{16}$$

often without explicit citation.

First, let $m = 0$ and let us take into consideration the initial condition (15). Then, according to Lemma 1, we immediately obtain the implications

$$\zeta_i \in Z_i \Rightarrow \zeta_i \in Z_i^{(\lambda)} := Z_i - C^{(\lambda)}(z_i) \quad (i \in \mathcal{I}_n;\ \lambda = 1, 2).$$

We should also prove that the inclusion disks $Z_1^{(\lambda)}, \ldots, Z_n^{(\lambda)}$ $(\lambda = 1, 2)$ are also disjoint. It is not difficult to estimate

$$|N(z_i)| < 2r, \quad |H(z_i)| < 2r,$$

so that we have

$$\begin{aligned} |\mathrm{mid}\ Z_i^{(\lambda)} - \mathrm{mid}\ Z_j^{(\lambda)}| &= |z_i - C^{(\lambda)}(z_i) - z_j + C^{(\lambda)}(z_j)| \\ &\geq |z_i - z_j| - |C^{(\lambda)}(z_i)| - |C^{(\lambda)}(z_j)| \\ &> \rho - 4r > 3(n-1)r - 4r \geq r_i + r_j. \end{aligned}$$

Thus, $Z_i^{(\lambda)} \cap Z_j^{(\lambda)} = \emptyset$ $(i \neq j)$ because of (5). The above facts are necessary for the inclusion method (13) to be well defined.

As mentioned above, we can combine two types of inversions in the iterative formulas (13). In what follows super(sub)script indices "e" and "c" will be used to mark the type of the used inversion in (13).

1) *The case* $\mathrm{INV}_1 = ()^{I_c}$

Let us consider first the case $\mathrm{INV}_1, \mathrm{INV}_2 = ()^{I_c}$. Applying the centered inversion (2) and using circular arithmetic operations, we get

$$\begin{aligned} S_{1,i}\big(\boldsymbol{Z}^{(\lambda)}, \boldsymbol{Z}^{(\lambda)}\big) &= \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{z_i - Z_j + C^{(\lambda)}(z_j)} = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{\{h_{ij}^{(\lambda)}; r_j\}} \\ &= \sum_{\substack{j=1\\ j\neq i}}^{n} \big\{u_{ij}^{(\lambda)}; d_{ij}^{(\lambda)}\big\} \subset \Big\{s_{1,i}^{(\lambda)}; \frac{5(n-1)r}{3\rho^2}\Big\}, \end{aligned}$$

wherefrom

$$\begin{aligned} S_{1,i}^2\big(\boldsymbol{Z}^{(\lambda)}, \boldsymbol{Z}^{(\lambda)}\big) &\subset \left\{\big(s_{1,i}^{(\lambda)}\big)^2; 2\big|s_{1,i}^{(\lambda)}\big|\frac{5(n-1)r}{3\rho^2} + \Big(\frac{5(n-1)r}{3\rho^2}\Big)^2\right\} \\ &\subset \Big\{\big(s_{1,i}^{(\lambda)}\big)^2; \gamma_1 r\Big\}, \qquad \gamma_1 = \frac{41(n-1)^2}{10\rho^3}. \end{aligned}$$

Applying (i) and (ii) of Lemma 2, we get

$$\begin{aligned} S_{2,i}\big(\boldsymbol{Z}^{(\lambda)}, \boldsymbol{Z}^{(\lambda)}\big) &= \sum_{\substack{j=1\\ j\neq i}}^{n} \Big(\frac{1}{z_i - Z_j + C^{(\lambda)}(z_j)}\Big)^2 = \sum_{\substack{j=1\\ j\neq i}}^{n} \big\{u_{ij}^{(\lambda)}; d_{ij}^{(\lambda)}\big\}^2 \\ &\subset \Big\{s_{2,i}^{(\lambda)}; \gamma_2 r\Big\}, \qquad \gamma_2 = \frac{41(n-1)}{10\rho^3}. \end{aligned}$$

Using the above inclusions of the sums $S_{1,i}^2$ and $S_{2,i}$, we obtain

$$\begin{aligned} Q_i\big(\boldsymbol{Z}^{(\lambda)}, \boldsymbol{Z}^{(\lambda)}\big) &= nS_{2,i}\big(\boldsymbol{Z}^{(\lambda)}, \boldsymbol{Z}^{(\lambda)}\big) - \frac{n}{n-1}S_{1,i}^2\big(\boldsymbol{Z}^{(\lambda)}, \boldsymbol{Z}^{(\lambda)}\big) \\ &\subset \left\{ns_{2,i}^{(\lambda)} - \frac{n}{n-1}\big(s_{1,i}^{(\lambda)}\big)^2; \gamma r\right\} = \{q_i^{(\lambda)}; \gamma r\}. \end{aligned}$$

Since $f_i^{(\lambda)} = n\delta_{2,i} - \delta_{1,i}^2 - q_i^{(\lambda)}$, according to the assertion (iii) of Lemma 2 we conclude that $0 \notin \{f_i^{(\lambda)}; \gamma r\}$, so that we can calculate square root of a disk

$$(n-1)\Big(n\delta_{2,i} - \delta_{1,i}^2 - Q_i\big(\boldsymbol{Z}^{(\lambda)}, \boldsymbol{Z}^{(\lambda)}\big)\Big) = (n-1)\big\{f_i^{(\lambda)}; \gamma r\big\}.$$

Further, putting $u_i^{(\lambda)} = \delta_{1,i} + \big[(n-1)f_i^{(\lambda)}\big]_*^{1/2}$, and using the assertion (iv) of Lemma 2, from the iterative formula (13) we obtain

$$\hat{Z}_i \subset z_i - n\mathrm{INV}_2\Big(\big\{u_i^{(\lambda)}; \eta\big\}\Big). \tag{17}$$

Using the identity (7) we find

$$\begin{aligned} f_i^{(\lambda)} &= n\delta_{2,i} - \delta_{1,i}^2 - q_i^* + q_i^* - q_i^{(\lambda)} = \frac{1}{n-1}(n/\varepsilon_i - \delta_{1,i})^2 + q_i^* - q_i^{(\lambda)} \\ &= \frac{1}{n-1}(n/\varepsilon_i - \delta_{1,i})^2\big(1 + v_i^{(\lambda)}\big). \end{aligned}$$

According to this and (v) of Lemma 2 we obtain

$$\begin{aligned} u_i^{(\lambda)} &= \delta_{1,i} + \big[(n-1)f_i^{(\lambda)}\big]_*^{1/2} = \delta_{1,i} + (n/\varepsilon_i - \delta_{1,i})\sqrt{1+v_i^{(\lambda)}} \\ &\in \delta_{1,i} + (n/\varepsilon_i - \delta_{1,i})\Big\{1; \frac{|\varepsilon_i|}{5\rho}\Big\} \\ &= \Big\{n/\varepsilon_i; \frac{|n-\varepsilon_i\delta_{1,i}|}{5\rho}\Big\} =: U_i. \end{aligned}$$

Here we have taken into account the criterion for the selection of the proper value of the square root which yields $\sqrt{(n/\varepsilon_i - \delta_{1,i})^2} = +(n/\varepsilon_i - \delta_{1,i})$. Using (6) and the inequality

$$|n-\varepsilon_i\delta_{1,i}| \le n-1+|\varepsilon_i|\sum_{\substack{j=1\\ j\ne i}}^{n}\frac{1}{|z_i-\zeta_j|} \le (n-1)\Big(1+\frac{r}{\rho}\Big) < \frac{7(n-1)}{6},$$

we find

$$\begin{aligned} \big|u_i^{(\lambda)}\big| &> |\text{mid } U_i| - \text{rad } U_i = \frac{n}{|\varepsilon_i|} - \frac{|n-\varepsilon_i\delta_{1,i}|}{5\rho} > \frac{n}{r} - \frac{7(n-1)}{6}\cdot\frac{1}{5\rho} \\ &= \frac{1}{r}\Big(n - \frac{7(n-1)r}{30\rho}\Big) > \frac{1}{r}\Big(n-\frac{7}{90}\Big). \end{aligned} \tag{18}$$

By (12) and (18) we obtain

$$\begin{aligned} \big|u_i^{(\lambda)}\big| - \eta &> \frac{90n-7}{90r} - \frac{25}{2}n(n-1)\frac{r^2}{\rho^3} \\ &> \frac{1}{r}\left(n - \frac{7}{90} - \frac{25}{2}n(n-1)\Big(\frac{1}{3(n-1)}\Big)^3\right) \\ &= \frac{1}{r}\Big(n-\frac{7}{90}-\frac{25n}{54(n-1)^2}\Big) > \frac{1}{r}\Big(n-\frac{11}{20}\Big) > 0. \end{aligned}$$

According to the last inequality we conclude that $0 \notin \{u_i^{(\lambda)}; \eta\}$ so that the iterative processes (13) are well defined and $\hat{Z}_i$ is a closed disk. Then from (17) we obtain

$$\hat{Z}_i \subset \hat{D}_i^{(c)} := z_i - n\Big\{\frac{1}{u_i^{(\lambda)}}; \frac{\eta}{|u_i^{(\lambda)}|(|u_i^{(\lambda)}|-\eta)}\Big\}.$$

Hence

$$\begin{aligned} \hat{r}_i &= \text{rad } \hat{Z}_i < \frac{n\eta}{|u_i^{(\lambda)}|(|u_i^{(\lambda)}|-\eta)} = \frac{\dfrac{25n^2(n-1)|\varepsilon_i|^3 r}{2\rho^3}}{\Big(n-\dfrac{7}{90}\Big)\cdot\Big(n-\dfrac{11}{20}\Big)} \\ &< \frac{16(n-1)|\varepsilon_i|^3 r}{\rho^3}, \end{aligned}$$

because of

$$\frac{25n^2}{2\Big(n-\dfrac{7}{90}\Big)\Big(n-\dfrac{11}{20}\Big)} < 16, \quad \text{for all } n \geq 3.$$

From the above relation we conclude that

$$\hat{r} = O\big(\epsilon^3 r\big) \tag{19}$$

and also, by (12),

$$\hat{r} < \frac{r}{6}. \tag{20}$$

Since $\xi_j^{(\lambda)} = O_M(1)$, we have

$$\Sigma_{1,i} - s_{1,i}^{(\lambda)} = \sum_{\substack{j=1\\ j\neq i}}^{n}\Big(\frac{1}{z_i-\zeta_j} - w_{ij}^{(\lambda)}\Big) = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\xi_j^{(\lambda)}\varepsilon_j^{\lambda+1}}{(z_i-\zeta_j)h_{ij}^{(\lambda)}} = O_M\big(\epsilon^{\lambda+1}\big),$$

wherefrom

$$\Sigma_{1,i}^2 - \big(s_{1,i}^{(\lambda)}\big)^2 = \Big(\Sigma_{1,i} - s_{1,i}^{(\lambda)}\Big)\Big(\Sigma_{1,i} + s_{1,i}^{(\lambda)}\Big) = O_M\big(\epsilon^{\lambda+1}\big)$$

and

$$\begin{aligned} \Sigma_{2,i} - s_{2,i}^{(\lambda)} &= \sum_{\substack{j=1\\ j\neq i}}^{n}\Bigg(\frac{1}{(z_i-\zeta_j)^2} - \big(w_{ij}^{(\lambda)}\big)^2\Bigg) \\ &= \sum_{\substack{j=1\\ j\neq i}}^{n}\Bigg(\frac{1}{z_i-\zeta_j} - w_{ij}^{(\lambda)}\Bigg)\Bigg(\frac{1}{z_i-\zeta_j} + w_{ij}^{(\lambda)}\Bigg) = O_M\big(\epsilon^{\lambda+1}\big). \end{aligned}$$

Furthermore, using the relation

$$q_i^* - q_i^{(\lambda)} = n\Big(\Sigma_{2,i} - s_{2,i}^{(\lambda)}\Big) - \frac{n}{n-1}\Big(\Sigma_{1,i}^2 - \big(s_{1,i}^{(\lambda)}\big)^2\Big),$$

we conclude that

$$q_i^* - q_i^{(\lambda)} = O_M\big(\epsilon^{\lambda+1}\big)$$

and

$$v_i = O_M\big(\epsilon^{\lambda+3}\big).$$

Therefore, the quantity $v_i^{(\lambda)}$ is very small so that we can use the approximation

$$\Big[1+v_i^{(\lambda)}\Big]_*^{1/2} \cong 1 + \frac{v_i^{(\lambda)}}{2}.$$

According to this we have

$$\begin{aligned} u_i^{(\lambda)} &= \delta_{1,i} + \Big(\frac{n}{\varepsilon_i} - \delta_{1,i}\Big)\sqrt{1+v_i^{(\lambda)}} = \delta_{1,i} + \Big(\frac{n}{\varepsilon_i} - \delta_{1,i}\Big)\sqrt{1+O\big(\epsilon^{\lambda+3}\big)} \\ &= \frac{n}{\varepsilon_i} + O_M\big(\epsilon^{\lambda+2}\big). \end{aligned}$$

For the center $\hat{z}_i$ of $\hat{Z}_i$ we obtain from (13) and (17)

$$\hat{z}_i = \text{mid }\ \hat{Z}_i = z_i - \frac{n}{u_i^{(\lambda)}},$$

whence

$$(21)\quad |\hat{\varepsilon}_i| = |\hat{z}_i - \zeta_i| \cong \Big|\varepsilon_i - \frac{n\varepsilon_i}{n + O_M\big(\epsilon^{\lambda+3}\big)}\Big| = \Big|\frac{O_M\big(\epsilon^{\lambda+4}\big)}{n + O_M\big(\epsilon^{\lambda+3}\big)}\Big| = O\big(\epsilon^{\lambda+4}\big)$$

since $n + O_M\big(\epsilon^{\lambda+3}\big)$ is bounded.

Using a geometric construction and the fact that the disks $Z_i^{(m)}$ and $Z_i^{(m+1)}$ must have at least one joint point (the zero ζ_i), the following relation can be derived (see [4])

$$\rho^{(m+1)} \ge \rho^{(m)} - r^{(m)} - 3r^{(m+1)}.$$

Using (4.7) and the last inequality (for $m = 0$), we find

$$\rho^{(1)} \ge \rho^{(0)} - r^{(0)} - 3r^{(1)} > 3(n-1)r^{(0)} - r^{(0)} - \frac{r^{(0)}}{2} > 6r^{(1)}\Big(3(n-1) - 1 - \frac{1}{2}\Big),$$

wherefrom it follows

$$(22)\qquad \rho^{(1)} > 4(n-1)r^{(1)}.$$

This is the condition (14) for the index $m = 1$, which means that all assertions of Lemmas 1 and 2 are valid for $m = 1$. Especially, the inequality (20) of the form $r^{(1)} < r^{(0)}/6$ points to the contraction of the new circular approximations $Z_1^{(1)}, \dots, Z_n^{(1)}$.

Using the definition of ρ and (28), for arbitrary pair of indices $i, j \in \mathcal{I}_n$ $(i \ne j)$ we have

$$|z_i^{(1)} - z_j^{(1)}| \ge \rho^{(1)} > 3(n-1)r^{(1)} > 2r^{(1)} \ge r_i^{(1)} + r_j^{(1)}.$$

Therefore, in regard to (5), the disks $Z_1^{(1)}, \dots, Z_n^{(1)}$ produced by (13) are disjoint.

Repeating the above procedure and the argumentation for an arbitrary index $m \ge 0$ we can derive all above relations for the index $m+1$. Since these relations have already been proved for $m = 0$, by mathematical induction it follows that, if the condition (12) holds, they are valid for all $m \ge 1$. In particular, we have

$$(23)\qquad \rho^{(m)} > 3(n-1)r^{(m)}$$

(the assertion 1°) and

$$r^{(m+1)} < \frac{r^{(m)}}{6}. \tag{24}$$

With regard to the inequality (24) we conclude that the sequence $\{r^{(m)}\}$ tends to 0; consequently, the inclusion methods (13) are *convergent*. Furthermore, taking into account that (23) holds, the assertions of Lemmas 1 and 2 are valid for arbitrary m, which means that the Laguerre-like methods (13) are well defined at each iterative step.

Suppose that $\zeta_i \in Z_i^{(m)}$ for each $i \in \mathcal{I}_n$. Then from (8) and (13) it follows that $\zeta_i \in Z_i^{(m+1)}$ (according to the inclusion isotonicity). Since $\zeta_i \in Z_i^{(0)}$ (the assumption of the theorem), by mathematical induction we prove that $\zeta_i \in Z_i^{(m)}$ for each $i \in \mathcal{I}_n$ and $m = 0, 1, \ldots$ (the assertion 2°).

Finally, we will determine the lower bound for the R-order of convergence of the methods (13) (the assertion 3°). The sequences $\{z_i^{(m)}\}$ and $\{r_i^{(m)}\}$ of the centers and radii of the disks $Z_i^{(m)}$ produced by the algorithms (13) are mutually dependent. For simplicity, we adopt $1 > \epsilon^{(0)} = r^{(0)} > 0$, which means that we deal with the "worst case" model. We note that such assumption is usual in this type of analysis and has no influence on the final result of the limit process which we apply in order to obtain the lower bound for the R-order of convergence. By virtue of (19) and (21) we notice that these sequences behave as follows

$$\epsilon^{(m+1)} \sim \left(\epsilon^{(m)}\right)^{\lambda+4}, \quad r^{(m+1)} \sim \left(\epsilon^{(m)}\right)^{3} r^{(m)}.$$

From these relations we form the R-matrix $T_2^{(c)} = \begin{bmatrix} \lambda+4 & 0 \\ 3 & 1 \end{bmatrix}$. Its spectral radius is $\rho(T_2^{(c)}) = \lambda+4$ and the corresponding eigenvector $\boldsymbol{x}_\rho^{(c)} = \big((\lambda+3)/3, 1\big) > 0$. Hence, according to Theorem 1, we get

$$O_R((13)_c) \geq \rho\,(T_2^{(c)}) = \lambda + 4 \quad (\lambda = 1, 2).$$

It remains to discuss the case when the exact inversion (1) is applied in the final step, that is, $\mathrm{INV}_2 = (\,)^{-1}$. Starting from the inclusion (17) we obtain

$$\hat{Z}_i \subset \hat{D}_i^{(e)} := z_i - n\left\{ \frac{\bar{u}_i^{(\lambda)}}{\big|u_i^{(\lambda)}\big|^2 - \eta^2}; \frac{\eta}{\big|u_i^{(\lambda)}\big|^2 - \eta^2} \right\} \tag{25}$$

and

$$\hat{r}_i = \mathrm{rad}\ \hat{Z}_i < \frac{n\eta}{\big|u_i^{(\lambda)}\big|^2 - \eta^2} < \frac{14(n-1)|\varepsilon_i|^3 r}{\rho^3}. \tag{26}$$

From (25) we find

$$\begin{aligned}\hat{z}_i &= \operatorname{mid}\ \hat{Z}_i \cong \operatorname{mid}\ \hat{D}_i^{(e)} = z_i - \frac{n\bar{u}_i^{(\lambda)}}{\big|u_i^{(\lambda)}\big|^2 - \eta^2} \\ &= z_i - \frac{n}{u_i^{(\lambda)}\Big(1 - \eta^2/\big|u_i^{(\lambda)}\big|^2\Big)}. \end{aligned} \tag{27}$$

According to the estimations derived above, we have $\eta = O(r\epsilon)$ and $u_i^{(\lambda)} = O_M(1/\epsilon)$, which gives $\eta^2/\big|u_i^{(\lambda)}\big|^2 = O(r^2\epsilon^4)$. Using the development into geometric series, from (27) we obtain

$$\hat{z}_i \cong z_i - \frac{n}{u_i^{(\lambda)}}\left(1 + \frac{\eta^2}{\big|u_i^{(\lambda)}\big|^2} + \cdots\right) = z_i - \frac{n}{u_i^{(\lambda)}} + O_M\big(r^2\epsilon^5\big),$$

wherefrom

$$\begin{aligned}\hat{\varepsilon}_i &\cong \varepsilon_i - \frac{n}{u_i^{(\lambda)}} + O_M\big(r^2\epsilon^5\big) = \varepsilon_i^3 O_M\big(\epsilon^{\lambda+1}\big) + O_M\big(r^2\epsilon^5\big) \\ &= \varepsilon_i^3 O_M\big(\alpha\epsilon^{\lambda+1} + \beta r^2\epsilon^2\big),\end{aligned}$$

where α and β are some complex quantities such that $|\alpha| = O(1)$ and $|\beta| = O(1)$. Hence

$$\hat{\varepsilon}_i = \varepsilon_i^3 O_M\big(\epsilon^{\lambda+1}\big) \quad (\lambda = 1, 2), \tag{28}$$

and we see that the relations (26) and (28) coincide with (19) and (21). Consequently, the lower bound of the R-order of convergence of the inclusion methods (13) when $\mathrm{INV}_1 = ()^{I_c}$, $\mathrm{INV}_2 = ()^{-1}$ is the same as in the case when $\mathrm{INV}_1, \mathrm{INV}_2 = ()^{I_c}$.

2) *The case* $\mathrm{INV}_1 = ()^{-1}$

Having in mind that $\xi_j^{(\lambda)} = O_M(1)$, in this case we get

$$\begin{aligned}\Sigma_{1,i} - s_{1,i}^{(\lambda)} &= \sum_{\substack{j=1 \\ j\neq i}}^{n}\left(\frac{1}{z_i - \zeta_j} - \frac{\bar{h}_{ij}^{(\lambda)}}{\big|h_{ij}^{(\lambda)}\big|^2 - r_j^2}\right) \\ &= \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\xi_j^{(\lambda)}\bar{h}_{ij}^{(\lambda)}\varepsilon_j^2 - r_j^2}{(z_i - \zeta_j)\big(\big|h_{ij}^{(\lambda)}\big|^2 - r_j^2\big)} = O_M\big(\alpha\epsilon^2 + \beta r^2\big).\end{aligned}$$

According to this we get

$$\Sigma_{1,i}^2 - \big(s_{1,i}^{(\lambda)}\big)^2 = O_M\big(\alpha'\epsilon^2 + \beta' r^2\big)$$

and

$$\Sigma_{2,i} - s_{2,i}^{(\lambda)} = O_M\big(\alpha''\epsilon^2 + \beta'' r^2\big).$$

Then

$$q_i^* - q_i^{(\lambda)} = O_M\big(\alpha'''\epsilon^2 + \beta''' r^2\big) \quad \text{and} \quad v_i^{(\lambda)} = \varepsilon^2 O_M\big(\alpha^*\epsilon^2 + \beta^* r^2\big).$$

The sequences $\{\epsilon^{(m)}\}$ and $\{r^{(m)}\}$ behave as follows

$$\epsilon^{(m+1)} \sim \big(\epsilon^{(m)}\big)^3\big(r^{(m)}\big)^2, \quad r^{(m+1)} \sim \big(\epsilon^{(m)}\big)^3 r^{(m)}.$$

The R-matrix is $T_2^{(e)} = \begin{bmatrix} 3 & 2 \\ 3 & 1 \end{bmatrix}$ with the spectral radius $\rho\big(T_2^{(e)}\big) = 2 + \sqrt{7}$ and the corresponding eigenvector $\boldsymbol{x}_\rho^{(e)} = \big((1+\sqrt{5})/2, 1\big) > 0$. Hence, according to Theorem 1, we obtain

$$O_R((13)_e) \geq \rho\,(T_2^{(e)}) = 2 + \sqrt{7} \approx 4.646. \qquad \square$$

5. Some modifications

In this Section we will comment some results presented in the previous sections which are concerned with the possibility of increasing the convergence rate of the considered methods. Also, we will construct the corresponding methods with corrections in single-step mode and methods for the inclusion of multiple zeros.

Improvements of convergence rate

As was shown in [15], the Laguerre-like simultaneous methods in ordinary complex arithmetic of the form

$$\hat{z}_i = z_i - \frac{n}{\delta_1(z_i) + a_i} \quad (i \in \mathcal{I}_n), \tag{29}$$

where

$$a_i = \Big[(n-1)(n\delta_2(z_i) - \delta_1^2(z_i) - n s_{2,i} + \frac{n}{n-1}(s_{1,i})^2\Big]_*^{1/2}$$

and

$$s_{\lambda,i} = \sum_{j\neq i}\big(z_i - z_j\big)^{-\lambda}\ (\lambda = 1, 2),$$

have the order of convergence equal to 4, 5 and 6 if $C_j = 0,\ C_j = N(z_j)$ (Newton's correction), and $C_j = H(z_j)$ (Halley's correction), respectively. However, according to the results of Theorem 2, we can infer that the increase of convergence order from 4 to 5 (by Newton's correction) and from 4 to 6 (by Halley's correction) is feasible only if we apply the centered inversion (2) to the sum

(9). The explanation lies in the fact that the application of the centered inversion enables faster convergence of the midpoints of disks produced by (13), which behave as the approximations defined by the iterative methods (29) for $C_j = N(z_j)$ and $C_j = H(z_j)$. Consequently, the accelerated convergence of the midpoints of disks forces faster convergence of radii, which can be observed from the proof of Theorem 2 where we manipulate with the coupling of the convergence of the midpoints and the radii. However, the following natural question now arises:

Which kinds of corrections increase the convergence rate if the exact inversion INV_1 *is applied to the sums* (9)?

In order to answer this question, let us consider a correction $C_j^{(\lambda)}$ of the order $\lambda+1$, which is involved in an iterative method $\hat{z}_i = z_i - C_j^{(\lambda)}$ of the order $\lambda + 1 \geq 2$ (that is, $\hat{\varepsilon}_i = O_M(\epsilon^{\lambda+1})$). If the exact inversion is applied to the sum (9), then the correction of order $\lambda + 1 \geq 2$ yields

$$\hat{\epsilon} = \epsilon^3 O_M(\alpha\epsilon^{\lambda+1} + \beta r^2).$$

Taking into account that $\epsilon < r$ but preserving $\epsilon = O(r)$, from the above relation we observe that for $\lambda > 1$ the term r^2 has *dominant* influence in reference to the first term in the parenthesis. Indeed, if $\lambda > 1$ then only

$$\alpha\epsilon^{\lambda+1} + \beta r^2 = r^2(\beta + \alpha\epsilon^{\lambda-1}) = O(r^2),$$

which means that any correction of the order higher than *two* (e.g., Halley's correction $H(z_j)$ of the order 3) *cannot* provide the increase of the convergence rate of the Lagguere-like inclusion algorithms (13).

Single-step methods with corrections

The proposed inclusion methods with corrections (13) can be further accelerated by using the already calculated disk in the current iteration (the Gauss-Seidel procedure or single-step mode). Starting from (11) we can construct the following single-step inclusion methods with correction

$$(30)\ \hat{Z}_i = \hat{z}_i - n\mathrm{INV}_2\left(\delta_{1,i} + \Big[(n-1)\big(n\delta_{2,i} - \delta_{1,i}^2 - Q_i(\widehat{\boldsymbol{Z}}, \boldsymbol{Z}^{(\lambda)})\big)\Big]_*^{1/2}\right),$$

for $i \in \mathcal{I}_n$ and $\lambda = 1, 2$. In this case it is very difficult to find the R-order of convergence of the considered methods (30) for a specific value of the degree n (see [11]). From this reason we will restrict our attention to the estimation of the bounds of the R-order taking the limit cases $n = 2$ and very large n. Since the R-order now depends on the number of zeros n (= the polynomial degree since all the zeros are simple), we will use the notation $O_R(IM, n)$ for the R-order.

Taking into account the fact that the convergence rate of a considered single-step method becomes almost the same as the one of the corresponding total-step method when the polynomial degree is very large, according to Theorem 2 we have

$$O_R((30,n)) \cong O_R(13) \geq \begin{cases} 2+\sqrt{7} \cong 4.646, & \text{if } \ \mathrm{INV}_1 = ()^{-1}, \\ \lambda + 4 \ (\lambda = 1,2), & \text{if } \ \mathrm{INV}_1 = ()^{I_c}. \end{cases}$$

Let us consider now the single-step method (30) for $n = 2$ and suppose that $1 > |\varepsilon_1^{(0)}| = |\varepsilon_2^{(0)}| = r_1^{(0)} = r_2^{(0)}$ (the "worst case" model). After an extensive and labor calculation we find the following estimates

$$|\hat{\varepsilon}_1| \sim |\varepsilon_1|^3 r_2^2, \quad |\hat{\varepsilon}_2| \sim |\varepsilon_1|^3|\varepsilon_2|^3 r_2^2, \quad \hat{r}_1 \sim |\varepsilon_1|^3 r_2, \quad \hat{r}_2 \sim |\varepsilon_1|^3|\varepsilon_2|^3 r_2,$$

if $\mathrm{INV}_1 = ()^{-1}$ and

$$|\hat{\varepsilon}_1| \sim |\varepsilon_1|^3|\varepsilon_2|^{\lambda+1}, \ |\hat{\varepsilon}_2| \sim |\varepsilon_1|^3|\varepsilon_2|^{\lambda+4}, \ \hat{r}_1 \sim |\varepsilon_1|^3 r_2, \ \hat{r}_2 \sim |\varepsilon_1|^3|\varepsilon_2|^3 r_2,$$

if $\mathrm{INV}_1 = ()^{I_c}$.

The corresponding R-matrices have the form

$$T_4^{(e)} = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 3 & 3 & 0 & 2 \\ 3 & 0 & 0 & 1 \\ 3 & 3 & 0 & 1 \end{bmatrix} \quad \text{if} \quad \mathrm{INV}_1 = ()^{-1}$$

and

$$T_4^{(c)} = \begin{bmatrix} 3 & \lambda+1 & 0 & 0 \\ 3 & \lambda+4 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 3 & 3 & 0 & 1 \end{bmatrix} \quad \text{if} \quad \mathrm{INV}_1 = ()^{I_c}.$$

Their spectral radii and eigenvectors are given by

$$\rho\big(T_4^{(e)}\big) = 6.29654, \quad \boldsymbol{x}_\rho^{(e)} = (1, 1.91, 0.7382, 1.6483) > 0 \quad \text{if} \ \ \mathrm{INV}_1 = ()^{-1},$$

and

$$\begin{aligned} \rho\big(T_4^{(c)}\big) &= \frac{1}{2}\Big(7+\lambda+\sqrt{13+14\lambda+\lambda^2}\Big) = \begin{cases} 6.64575, & \lambda = 1, \\ 7.8541, & \lambda = 2, \end{cases} \\ \boldsymbol{x}_\rho^{(c)} &= \begin{cases} (1, 1.8229, 0.6771, 1.5) > 0, & \lambda = 1, \\ (1, 1.6180, 0.5279, 1.1459) > 0, & \lambda = 2, \end{cases} \end{aligned}$$

if $\mathrm{INV}_1 = ()^{I_c}$.

Let Ω be the range of the lower bounds of the R-order of convergence concerning the single-step methods (30). Taking into account the previous results, we obtain

$$\Omega = (4.646, 6.297) \quad \text{if} \quad \mathrm{INV}_1 = ()^{-1}$$

and

$$\Omega = \begin{cases} (5, 6.646), & \lambda = 1, \\ (6, 7.855), & \lambda = 2, \end{cases} \qquad \text{if} \quad \mathrm{INV}_1 = ()^{I_c}.$$

Since the increased convergence is gained with only a few additional calculations, we infer that the inclusion method (30) has a great computational efficiency.

Algorithms for multiple zeros

The proposed algorithms (13) and (30) with Newton's and Halley's corrections can be easily modified for the application in the case of multiple zeros. Let $\mu_1, ..., \mu_\nu$ $(\mu_1 + \cdots + \mu_\nu = n)$ be the multiplicities of the the zeros $\zeta_1, ..., \zeta_\nu$, $(\nu \leq n)$ of P. We note that efficient procedures for finding the order of multiplicity may be found in [6] and [7].

The total-step and the single-step iterative methods of Laguerre's type for the inclusion of multiple zeros read (omitting the iteration index)

$$\begin{aligned} \widehat{Z}_i &= z_i - n\mathrm{INV}_2\Big(\delta_{1,i} + \Big[\frac{n-\mu_i}{\mu_i}\big(n\delta_{2,i} - \delta_{1,i}^2 - Q_i^*(\boldsymbol{Z}^{(\lambda)}, \boldsymbol{Z}^{(\lambda)})\big)\Big]_*^{1/2}\Big), \\ \widehat{Z}_i &= z_i - n\mathrm{INV}_2\Big(\delta_{1,i} + \Big[\frac{n-\mu_i}{\mu_i}\big(n\delta_{2,i} - \delta_{1,i}^2 - Q_i^*(\widehat{\boldsymbol{Z}}, \boldsymbol{Z}^{(\lambda)})\big)\Big]_*^{1/2}\Big), \end{aligned}$$

where $i \in \mathcal{I}_\nu$ and $\lambda = 0, 1, 2$. In these formulas one should take

$$\begin{aligned} Z_{N,i} &= Z_i - N^*(z_i), \quad N^*(z_i) = \mu_i \frac{P(z_i)}{P'(z_i)}, \\ Z_{H,i} &= Z_i - H^*(z_i), \\ H^*(z_i) &= \frac{P(z_i)}{\Big(\dfrac{1 + 1/\mu_i}{2}\Big)P'(z_i) - \dfrac{P(z_i)P''(z_i)}{2P'(z_i)}}, \\ S_{k,i}^*(\boldsymbol{X}, \boldsymbol{W}) &= \sum_{j=1}^{i-1} \mu_j \big(\mathrm{INV}_1(z_i - X_j)\big)^k + \sum_{j=i+1}^{\nu} \mu_j \big(\mathrm{INV}_1(z_i - W_j)\big)^k, \\ Q_i^* &= nS_{2,i}^* - \frac{n}{n-\mu_i}\big(S_{1,i}^*\big)^2, \end{aligned}$$

for $k = 1, 2$.

6. Numerical results

In Section 4 we have shown that the modified methods (13) with Newton's and Halley's corrections improve the convergence rate of the Laguerre-like inclusion method (10). This acceleration is especially significant when the centered inversion (2) is applied. Since this improvement was attained with only a few additional operations, it is clear that the computational efficiency of the inclusion methods proposed in this paper is increased compared with the basic method (10).

The inclusion methods of Laguerre's type without corrections (10) and with Newton's and Halley's corrections have been tested in solving many polynomial equations. For comparison purpose, we have also tested the interval Weierstrass-like method

$$\hat{Z}_i = z_i - P(z_i) \cdot \Big(\prod_{\substack{j=1 \\ j \neq i}}^{n} (z_i - Z_j) \Big)^{-1} \quad (i \in \mathcal{I}_n), \tag{31}$$

which has quadratic convergence, see [10, Ch. 2], [16]. We note that, sometimes, the following version of (31) of the form

$$\hat{Z}_i = z_i - P(z_i) \cdot \prod_{\substack{j=1 \\ j \neq i}}^{n} (z_i - Z_j)^{-1} \quad (i \in \mathcal{I}_n) \tag{32}$$

is applied since it produces smaller disks. However, its computational cost is greater compared with (31).

Theoretical results concerning the convergence order of the considered Laguerre-like methods mainly well coincide with the convergent behavior of these methods in practice, especially when the number of iterative steps increases. To provide the enclosure of the zeros when the produced disks were very small, we have used multi-precision arithmetic applying the programming package *Mathematica* 5.0. Among many numerical results, we select two examples for demonstration.

Example 1 ([8]). To find inclusion disks for the zeros of the polynomial

$$P(z) = z^9 + 3z^8 - 3z^7 - 9z^6 + 3z^5 + 9z^4 + 99z^3 + 297z^2 - 100z - 300,$$

we applied the interval methods (10), (11), (13), (30), (31) and (32). The exact zeros of P are

$$\zeta_1 = -3,\ \zeta_{2,3} = \pm 1,\ \zeta_{4,5} = \pm 2i,\ \zeta_{6,7} = -2 \pm i,\ \zeta_{8,9} = 2 \pm i.$$

The initial disks were selected to be $Z_i^{(0)} = \{z_i^{(0)}; 0.3\}$, with the centers

$$\begin{array}{lll} z_1^{(0)} = -3.1 + 0.2i, & z_2^{(0)} = -1.2 - 0.1i, & z_3^{(0)} = 1.2 + 0.1i, \\ z_4^{(0)} = 0.2 - 2.1i, & z_5^{(0)} = 0.2 + 1.9i, & z_6^{(0)} = -1.8 + 1.1i, \\ z_7^{(0)} = -1.8 - 0.9i, & z_8^{(0)} = 2.1 + 1.1i, & z_9^{(0)} = 1.8 - 0.9i. \end{array}$$

The maximal radii of the inclusion disks produced in the first three iterative steps are given in Table 1, where the notation $A(-q)$ means $A \times 10^{-q}$.

	$r^{(1)}$	$r^{(2)}$	$r^{(3)}$
(2.4)	1.15(−2)	2.08(−10)	1.12(−43)
(2.5)	1.04(−2)	4.30(−11)	3.94(−46)
(3.4), $\lambda = 1$	8.35(−3)	1.19(−11)	3.81(−59)
(5.2), $\lambda = 1$	7.24(−3)	1.55(−12)	1.51(−62)
(3.4), $\lambda = 2$	8.56(−3)	1.65(−13)	7.10(−83)
(5.2), $\lambda = 2$	7.47(−3)	1.56(−14)	1.06(−84)
(6.1)	diverges	—	—
(6.2)	5.85(−1)	3.26(−1)	3.01(−3)

Table 1: The maximal radii of inclusion disks

We note that, at present, the disks produced in the third iteration are pointless from a practical point of view. However, we presented them to emphasize the property of inclusion methods with corrections consisting of the growing accuracy when the number of iterative steps increases. Furthermore, the Weierstrass-like inclusion method (31) is divergent, while its version (32) requires even 7 iterations to produce disks of approximately of the same size ($r^{(7)} = 1.32(-40)$) as the disks obtained by the basic method (10) after the third iteration ($r^{(3)} = 1.12(-43)$).

Example 2 The same interval methods (applied in Example 1) were implemented for the determination of the eigenvalues of Hessenberg's matrix

$$H = \begin{bmatrix} 2+3i & 1 & 0 & 0 & 0 \\ 0 & 4+6i & 1 & 0 & 0 \\ 0 & 0 & 6+9i & 1 & 0 \\ 0 & 0 & 0 & 8+12i & 1 \\ 1 & 0 & 0 & 0 & 10+15i \end{bmatrix},$$

whose characteristic polynomial is

$$\begin{aligned} f(\lambda) &= \lambda^5 - (30+45i)\lambda^4 + (-425+1\,020i)\lambda^3 + (10\,350 - 2\,025i)\lambda^2 \\ &\quad -(32\,606 + 32\,880i)\lambda - 14\,641 + 71\,640i. \end{aligned}$$

We used convenient fact that the eigenvalues of a square matrix belong to the union of the so-called Gerschgorin's disks $\{a_{ii}; R_i\}$ $(i \in \mathcal{I}_n)$, where a_{ii} are the diagonal elements of a matrix and $R_i = \sum_{j \neq i} |a_{ij}|$. For the given matrix H Gerschgorin's disks are

$$\begin{aligned} &Z_1 = \{2+3i; 1\}, \quad Z_2 = \{4+6i; 1\}, \quad Z_3 = \{6+9i; 1\}, \\ &Z_4 = \{8+12i; 1\}, \quad Z_5 = \{10+15i; 1\}. \end{aligned}$$

These disks are mutually disjoint, so that each of them contains one and only one eigenvalue of H. For this reason, we chose these Gerschgorin's disks as initial disks for our inclusion methods.

The entries of the maximal radii of the disks produced in the first two iterations, when both inversions are $()^{I_c}$, are given in Table 2. The behavior of the Weierstrass-like methods (31) and (32) is similar as in Example 1.

	$r^{(1)}$	$r^{(2)}$
(2.4)	2.77(−10)	3.36(−53)
(2.5)	1.32(−10)	2.52(−52)
(3.4), $\lambda = 1$	2.77(−10)	1.26(−61)
(5.2), $\lambda = 1$	1.32(−10)	3.48(−63)
(3.4), $\lambda = 2$	2.77(−10)	8.28(−73)
(5.2), $\lambda = 2$	1.32(−10)	4.11(−73)
(6.1)	diverges	—
(6.2)	2.34(−3)	2.10(−10)

Table 2: The maximal radii of inclusion disks

According to the results of numerous experiments, including those displayed in Tables 1 and 2, we conclude that the proposed Laguerre-like methods with corrections produce very small disks. It is worth noting that, in some particular cases, the proposed methods with corrections give in the first iterative step disks not smaller compared to those produced by the inclusion methods without corrections. The reason is simple: Newton's and Halley's method cannot improve initial (point) approximations if the centers of initial disks are not sufficiently close to the zeros. In later iterations the convergence order of the presented inclusion methods with corrections increases and its value approaches to theoretical one obtained in the presented convergence analysis. Finally, let us note that, beside greater computational efficiency, Laguerre-like inclusion methods demonstrate better convergence properties in practice compared with the Weierstrass-like inclusion methods (31) and (32).

References

[1] Alefeld, G., Herzberger, J., Introduction to Interval Computation, Academic Press, New York, 1983.

[2] Carstensen, C., Petković, M.S., An improvement of Gargantini's simultaneous inclusion method for polynomial roots by Schroeder's correction, Appl. Numer. Math. 25 (1993), 59–67.

[3] Gargantini, I., Parallel Laguerre iterations: Complex case, Numer. Math. 26 (1976), 317–323.

[4] Gargantini, I., Further application of circular arithmetic: Schröder-like algorithms with error bound for finding zeros of polynomials, SIAM J. Numer. Anal. 15 (1978), 497–510.

[5] Herzberger, J., Metzner, L., On the Q-order and R-order of convergence for coupled sequences arising in iterative numerical processes, in: Numerical Methods and Error Bounds (eds G. Alefeld and J. Herzberger), Mathematical Research Vol. 89, Akademie Verlag, Berlin, 1996, pp. 120–131.

[6] Kravanja, P., On Computing Zeros of Analytic Functions and Related Problems in Structured Numerical Linear Algebra, Ph.D. Thesis, Katholieke Universiteit Leuven, Lueven, 1999.

[7] Niu, X. M., Sakurai, T., A Method for finding the zeros of polynomials using a companion matrix, Japan J. Industr. Appl. Math. 20 (2003), 239–256.

[8] Petković, M. S., Laguerre-like inclusion method for polynomial zeros, J. Comput. Appl. Math. 152 (2003), 451–465.

[9] Petković, M. S., On a generalization of the root iterations for polynomial complex zeros, Computing 27 (1981), 37–55.

[10] Petković, M. S., Iterative Methods for Simultaneous Inclusion of Polynomial Zeros, Springer-Verlag, Berlin-Heidelberg-New York, 1989.

[11] Petković, M. S., Halley-like method with corrections for the inclusion of polynomial zeros, Computing 62 (1999), 69–88.

[12] Petković, M. S., Carstensen, C., On some improved inclusion methods for polynomial roots with Weierstrass' correction, Comput. Math. with Appl. 25 (1993), 59–67.

[13] Petković, M. S., Milošević, D. M., Ostrowski-like method with corrections for the inclusion of polynomial zeros, Reliable Computing (to appear).

[14] Petković, M. S., Petković, Lj. D., Complex Interval Arithmetic and Its Applications, Wiley-VCH, Berlin-Weinheim-New York, 1998.

[15] Petković, M. S., Petković, Lj. D., Živković, D., Laguerre-like methods for the simultaneous approximation of polynomial zeros, Computing [Suppl.] 15 (2001), 189–210.

[16] Wang, X., Zheng, S., The quasi-Newton method in parallel circular iteration, J. Comput. Math. 4 (1984), 305–309.

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