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ON A NONEQUDISTANT DIFFERENCE SCHEME OF CHAWLA TYPE

Dragoslav Herceg[1], **Djordje Herceg**[1]

Abstract. We present a fourth-order finite difference method for general second-order nonlinear boundary value problem $-y''+f(x,y,y')=0$ subject to two-point boundary conditions. We use nonequidistant discretization mesh and each discretization of the differential equation at an interior mesh point is based on just three evaluations of f. The present paper extends the results given in Chawla (1978) to the case of nonequidistant mesh. Numerical examples are considered to demonstrate computationally the fourth order of the method.

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1. Introduction

We consider the general second-order nonlinear differential equation

$$-y''+f(x,y,y')=0, \quad x\in I=[0,1], \tag{1.1}$$

subject to homogeneous boundary conditions

$$y(0)=y(1)=0. \tag{1.2}$$

If we have

$$-u''+f(x,u,u')=0, \quad x\in I=[0,1],$$

$$u(0)=A, \quad u(1)=B,$$

we can use the usual procedure to homogenize the boundary conditions. If u is a solution of this problem and if we put $y=u-\sigma(x)$, where $\sigma(x)=Bx+A(1-x)$, then y is a solution of the boundary value problem of the form (1.1), (1.2) with the function $f^*(x,y,y')=f(x,y+\sigma(x),y'+\sigma'(x))$ instead of the function $f(x,y)$.

[1]Department of Mathematics and Informatics, Faculty of Science, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia and Montenegro, hercegd@im.ns.ac.yu, herceg@im.ns.ac.yu

Our aim in this paper is to construct the fourth order difference scheme on a nonequidistant discretization mesh in such a way that discretization of the differential equation at each interior mesh point is based on just three evaluations of f. The idea was developed in the papers [2], [3].

For simplicity, we shall assume that

$$f \in C^\infty (I \times \mathbb{R} \times \mathbb{R})$$

and

$$\frac{\partial f}{\partial y} > 0, \quad \left| \frac{\partial f}{\partial z} \right| \le W,$$

for some positive constant W.

These conditions assure us of the existence of the unique solution of the boundary value problem, [6], [1].

In this paper we describe a fourth-order finite difference method for the boundary value problem (1.1), (1.2), based on Hermitian approximation of (1.1) on a suitable nonequidistant mesh. For some classes of singularly perturbed boundary value problem appropriate meshes one can find in [5] and [8]. In section 2 we describe the finite difference method. In the case where the differential equation is linear, the method leads to tridiagonal linear system. In section 3 we obtain local truncation errors. In the fourth section we consider numerical examples to illustrate the method. Our results show the fourth order convergence if we use the appropriate discretization mesh.

2. Difference scheme

Let us now introduce the notation. Let n be a positive integer and x_k the mesh points of the mesh I_h' such that

$$0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1$$

and let

$$h_k = x_k - x_{k-1}, \quad k = 1, 2, \ldots, n.$$

In the following, we consider obtaining three-point finite difference approximations for the differential equation at a fixed point $x_k \in I_h = I_h' \setminus \{0, 1\}$. For simplicity, we define for a fixed k

$$h = x_k - x_{k-1}, \quad H = x_{k+1} - x_k.$$

So, we have $x_{k-1} = x_k - h$ and $x_{k+1} = x_k + H$.

Let w^h be a mesh function. Mesh functions will be defined with the $\mathbb{R}^{n+1}$ column vectors

$$w^h = [w_0, w_1, \ldots, w_n]^\top$$

(for simplicity, the superscript h is omitted in the components). In particular,

$$y^h = \left[y\left(x_0\right), y\left(x_1\right), \ldots, y\left(x_n\right)\right]^\top .$$

In order to form a discretization of the problem (1.1) we approximate the differential equation of (1.1) by a difference formula of Hermite type in $x_k \in I_h$. The coefficients in this formula are not constant, i.e. they depend on x_{k-1}, x_k and x_{k+1} for all $k = 1, 2, \ldots, n-1$. One can obtain these coefficients in a similar way as on an equidistant mesh. Let the values of the exact solution $y\left(x\right)$ at mesh points x_k be denoted by y_k; similarly $y_k' = y\left(x_k\right)$, $y_k'' = f\left(x_k, y_k, y_k'\right)$, etc.

For $k \in \{1, 2, \ldots, n-1\}$ Let

$$\bar{y}_k' = \frac{1}{H+h} y_{k+1} - \frac{1}{H+h} y_{k-1},$$

$$\bar{y}_{k+1}' = \frac{h+2H}{H\left(H+h\right)} y_{k+1} - \frac{h+H}{Hh} y_k + \frac{H}{h\left(H+h\right)} y_{k-1},$$

$$\bar{y}_{k-1}' = -\frac{h}{H\left(H+h\right)} y_{k+1} + \frac{h+H}{Hh} y_k - \frac{H+2h}{h\left(H+h\right)} y_{k-1},$$

$$\bar{f}_{k\pm1} = f\left(x_{k\pm1}, y_{k\pm1}, \bar{y}_{k\pm1}'\right),$$

$$\bar{f}_k = f\left(x_k, y_k, \bar{y}_k' + K\right),$$

$$K = \alpha \bar{f}_{k-1} + \beta \bar{f}_{k+1} = \alpha f\left(x_{k-1}, y_{k-1}, \bar{y}_{k-1}'\right) + \beta f\left(x_{k+1}, y_{k+1}, \bar{y}_{k+1}'\right),$$

$$\alpha = \frac{h^2 + 4hH - 4H^2}{10(h+H)}, \quad \beta = \frac{H^2 + 4hH - 4h^2}{10(h+H)}.$$

Then, at each x_k, the differential equation (1.1) is discretized by

$$\begin{aligned} 0 \;=\; & a_1\left(k\right) y_{k-1} + a_0\left(k\right) y_k + a_2\left(k\right) y_{k+1} + b_1\left(k\right) f\left(x_{k-1}, y_{k-1}, \bar{y}_{k-1}'\right) \\ & + b_0\left(k\right) f\left(x_k, y_k, \bar{y}_k' + K\right) + b_2\left(k\right) f\left(x_{k+1}, y_{k+1}, \bar{y}_{k+1}'\right) + \tau_k, \end{aligned}$$

where $\tau_k = \mathcal{O}\left(\max\left\{h, H\right\}^4\right)$.

Let

$$T^h w_k = a_1\left(k\right) w_{k-1} + a_0\left(k\right) w_k + a_2\left(k\right) w_{k+1} + b_1\left(k\right) w_{k-1}'' + b_0\left(k\right) w_k'' + b_2\left(k\right) w_{k+1}'',$$

where $w_k = w\left(x_k\right)$ and $w_k'' = w''\left(x_k\right)$ for a function $w\left(x\right)$. The coefficients $a_j\left(k\right)$ and $b_j\left(k\right)$, $j = 0, 1, 2$ one can obtain as in [5] and [8] from the system

$$T^h x_k^m = 0, \quad m = 0, 1, 2, 3,$$

$$b_0\left(k\right) = \frac{5}{6},$$

$$b_1(k) + b_0(k) + b_2(k) = 1.$$

So, we have for $k = 1, 2, \ldots n-1$

$$a_1(k) = \frac{-2}{h_k(h_k + h_{k+1})}, \quad a_0(k) = \frac{2}{h_k h_{k+1}}, \quad a_2(k) = \frac{-2}{h_{k+1}(h_k + h_{k+1})},$$

$$b_1(k) = \frac{2h_k - h_{k+1}}{6(h_k + h_{k+1})}, \quad b_0(k) = \frac{5}{6}, \quad b_2(k) = \frac{2h_{k+1} - h_k}{6(h_k + h_{k+1})}.$$

For a fixed k, with $h = x_k - x_{k-1},\ H = x_{k+1} - x_k$, we obtain

$$a_1(k) = \frac{-2}{h(h+H)}, \quad a_0(k) = \frac{2}{hH}, \quad a_2(k) = \frac{-2}{H(h+H)},$$

$$b_1(k) = \frac{2h - H}{6(h+H)}, \quad b_0(k) = \frac{5}{6}, \quad b_2(k) = \frac{2H - h}{6(h+H)}.$$

In equidistant case, i.e. if $h = H$, we have the same results as in [2] and [3] :

$$\bar{y}'_k = \frac{y_{k+1} - y_{k-1}}{2h},$$

$$\bar{y}'_{k+1} = \frac{3y_{k+1} - 4y_k + y_{k-1}}{2h},$$

$$\bar{y}'_{k-1} = \frac{-y_{k+1} + 4y_k - 3y_{k-1}}{2h},$$

$$\alpha = \frac{h}{20}, \quad \beta = -\frac{h}{20}.$$

$$a_1(k) = -h^{-2}, \quad a_0(k) = 2h^{-2}, \quad a_2(k) = -h^{-2}$$

$$b_1(k) = \frac{1}{12}, \quad b_0(k) = \frac{5}{6}, \quad b_2(k) = \frac{1}{12}.$$

Let us now describe the method. Let

$$\Phi_0(y_0) = 0, \quad \Phi_n(y_n) = 0$$

and for $k = 1, 2, \ldots, n-1$

$$\Phi_k(y_{k-1}, y_k, y_{k+1}) = b_1(k)\bar{f}_{k-1} + b_0(k)\bar{f}_k + b_2(k)\bar{f}_{k+1}.$$

Let D be the tridiagonal matrix

$$D = \begin{bmatrix} 1 & & & & & & \\ a_1(1) & a_0(1) & a_2(1) & & & & \\ & a_1(2) & a_0(2) & a_2(1) & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & a_1(n-1) & a_0(n-1) & a_2(n-1) \\ & & & & & 1 \end{bmatrix}$$

and let $Y = [y_0, y_1, \ldots, y_n]^T$,

$$\Phi(Y) = [\Phi_0, \Phi_1, \ldots, \Phi_n]^T, \qquad \Upsilon = [0, \tau_1, \tau_2, \ldots, \tau_{n-1}, 0]^T.$$

Now, our difference scheme can be written in the following form

$$DY + \Phi(Y) + \Upsilon = 0. \tag{2.1}$$

We form a discrete analogue of the problem (1.1) in the form

$$Dw + \Phi(w) = 0. \tag{2.2}$$

The solution w of (2.2) is an approximation for Y. The system (2.2) is an $(n+1) \times (n+1)$ nonlinear system. In the case when differential equation (1.1) is linear (2.2) is tridiagonal linear system. The nonlinear system (2.2) can be solved by the Newton-Raphson method and tridiagonal linear solver.

3. Local truncation errors

A simple Taylor series argument shows:

Lemma 1 *Let $y \in C^3(I)$, $x_k \in I_h$. Then there exists $\omega_k \in (x_k - h, x_k + H)$ such that*

$$\bar{y}'_k - y'_k = \frac{H-h}{2} y''_k + \Omega_k^3,$$

where

$$\Omega_k^3 = \frac{h^2 - hH + H^2}{6} y^{(3)}(\omega_k)$$

Lemma 2 *Let $y \in C^5(I)$, $x_k \in I_h$. Then there exists $\omega_k \in (x_k - h, x_k + H)$ such that*

$$\bar{y}'_k - y'_k = \frac{H-h}{2} y''_k + \frac{h^2 - hH + H^2}{6} y^{(3)}(x_k) + \frac{(H-h)\left(h^2 + H^2\right)}{24} y_k^{(4)} + \Omega_k^5,$$

where

$$\Omega_k^5 = \frac{h^5 + H^5}{120(h+H)} y^{(5)}(\omega_k).$$

In a manner similar to previous lemmas we can prove the following results.

Lemma 3 *Let $y \in C^3(I)$, $x_k \in I_h$. Then there exist $\tau_k \in (x_k - h, x_k + H)$ and $\tau_{k,0}^+ \in (x_k, x_k + H)$ such that*

$$\bar{y}'_{k+1} - y'_{k+1} = \Upsilon_k^3,$$

where

$$\Upsilon_k^3 = \frac{H^2(h+2H)}{6(h+H)} y^{(3)}\left(\tau_{k,0}^+\right) - \frac{H\left(h^2 + 3hH + 3H^2\right)}{6(h+H)} y^{(3)}(\tau_k).$$

Lemma 4 *Let $y \in C^5(I)$, $x_k \in I_h$. Then there exist $\tau_k \in (x_k - h, x_k + H)$ and $\tau_{k,0}^+ \in (x_k, x_k + H)$ such that*

$$\bar{y}'_{k+1} - y'_{k+1} = -\frac{H(h+H)}{6} y_k^{(3)} - \frac{H(h+H)(2H-h)}{24} y_k^{(4)} + \Upsilon_k^5,$$

where

$$\Upsilon_k^5 = \frac{H^4(h+2H)}{120(h+H)} y^{(5)}\left(\tau_{k,0}^+\right) - \frac{H\left(h^4 + 5hH^3 + 5H^4\right)}{120(h+H)} y^{(5)}(\tau_k).$$

Lemma 5 *Let $y \in C^3(I)$, $x_k \in I_h$. Then there exist $\theta_k \in (x_k - h, x_k + H)$ and $\theta_{k,0}^- \in (x_k - h, x_k)$ such that*

$$\bar{y}'_{k-1} - y'_{k-1} = \Theta_k^3,$$

where

$$\Theta_k^3 = \frac{h^2(2h+H)}{6(h+H)} y^{(3)}\left(\theta_{k,0}^-\right) - \frac{h\left(H^2 + 3hH + 3h^2\right)}{6(h+H)} y^{(3)}(\theta_k).$$

Lemma 6 *Let $y \in C^5(I)$, $x_k \in I_h$. Then there exist $\tau_k \in (x_k - h, x_k + H)$ and $\tau_{k,0}^+ \in (x_k, x_k + H)$ such that*

$$\bar{y}'_{k-1} - y'_{k-1} = \frac{h(h+H)(2h-H)}{24} y_k^{(4)} - \frac{h(h+H)}{6} y_k^{(3)} + \Theta_k^5,$$

where

$$\Theta_k^5 = \frac{h^4(2h+H)}{120(h+H)} y^{(5)}\left(\theta_{k,0}^-\right) - \frac{h\left(H^4 + 5h^3H + 5h^4\right)}{120(h+H)} y^{(5)}(\theta_k).$$

Here and in the following, we have

$$F(x,y,z) = \frac{\partial}{\partial x} f(x,y,z), \quad G(x,y,z) = \frac{\partial}{\partial z} f(x,y,z),$$

$$H(x,y,z) = \frac{\partial^2}{\partial z^2} f(x,y,z), \quad G' = \frac{dG}{dx}, \quad G_m = G(x_m, y(x_m), y'(x_m)).$$

From (2.1) it follows

$$\tau_k = a_1(k) y_{k-1} + a_0(k) y_k + a_2(k) y_{k+1} + b_1(k) \bar{f}_{k-1} + b_0(k) \bar{f}_k + b_2(k) \bar{f}_{k+1}.$$

For simplicity, we shall write a_j, b_j instead of $a_j(k)$ and $b_j(k)$ respectively. So we have

$$\tau_k = a_1 y_{k-1} + a_0 y_k + a_2 y_{k+1} + b_1 \bar{f}_{k-1} + b_0 \bar{f}_k + b_2 \bar{f}_{k+1}.$$

Since

$$\bar{f}_{k\pm1} = f\left(x_{k\pm1}, y_{k\pm1}, \bar{y}'_{k\pm1}\right) = f_{k\pm1} + G_{k\pm1}\left(\bar{y}'_{k\pm1} - y'_{k\pm1}\right) + H^*_{k\pm1}\left(\bar{y}'_{k\pm1} - y'_{k\pm1}\right)^2,$$

$$\bar{f}_k = f\left(x_k, y_k, \bar{y}'_k + K\right) = f_k + G_k\left(\bar{y}'_k - y'_k + K\right) + H^*_k\left(\bar{y}'_k - y'_k + K\right)^2,$$

$$f_{k\pm1} = y''_{k\pm1}, \quad f_k = y''_k,$$

where, for some $\xi_-,\ \xi_+,\ \xi$,

$$H^*_{k-1} = H\left(x_{k-1}, y_{k-1}, \xi_-\right), \quad H^*_{k+1} = H\left(x_{k+1}, y_{k+1}, \xi_+\right), \quad H^*_k = H\left(x_k, y_k, \xi\right),$$

we have

$$\tau_k = R_k + S_k + T_k, \tag{3.1}$$

with

$$R_k = a_1 y_{k-1} + a_0 y_k + a_2 y_{k+1} + b_1 y''_{k-1} + b_0 y''_k + b_2 y''_{k+1},$$

$$\begin{aligned} S_k \quad = \quad & b_1\left(G_{k-1}\left(\bar{y}'_{k-1} - y'_{k-1}\right) + H^*_{k-1}\left(\bar{y}'_{k-1} - y'_{k-1}\right)^2\right) \\ & + b_2\left(G_{k+1}\left(\bar{y}'_{k+1} - y'_{k+1}\right) + H^*_{k+1}\left(\bar{y}'_{k+1} - y'_{k+1}\right)^2\right) \end{aligned}$$

and

$$T_k = b_0\left(G_k\left(\bar{y}'_k - y'_k + K\right) + H^*_k\left(\bar{y}'_k - y'_k + K\right)^2\right).$$

For some $\eta_+,\ \eta_-$ we have

$$G_{k+1} = G_k + HG'_k + \frac{H^2}{2}G''_{\eta+} \tag{3.2}$$

and

$$G_{k-1} = G_k - hG'_k + \frac{h^2}{2}G''_{\eta-}, \tag{3.3}$$

where

$$G''_{\eta+} = G''\left(\eta_+, y\left(\eta_+\right), y'\left(\eta_+\right)\right), \quad G''_{\eta-} = G''\left(\eta_-, y\left(\eta_-\right), y'\left(\eta_-\right)\right).$$

From Lemma 3.2 and Lemma 3.3 it follows

$$\begin{aligned} S_k \quad = \quad & G_k\left(b_1\left(\bar{y}'_{k-1} - y'_{k-1}\right) + b_2\left(\bar{y}'_{k+1} - y'_{k+1}\right)\right) \\ & + G'_k\left(-hb_1\left(\bar{y}'_{k-1} - y'_{k-1}\right) + Hb_2\left(\bar{y}'_{k+1} - y'_{k+1}\right)\right) \\ & + \frac{1}{2}\left(h^2 b_1\left(\bar{y}'_{k-1} - y'_{k-1}\right)G''_{\eta-} + H^2 b_2\left(\bar{y}'_{k+1} - y'_{k+1}\right)G''_{\eta+}\right). \end{aligned}$$

With

$$S^1_k = b_1\left(\bar{y}'_{k-1} - y'_{k-1}\right) + b_2\left(\bar{y}'_{k+1} - y'_{k+1}\right),$$

$$S_k^2 = -hb_1\left(\bar{y}'_{k-1} - y'_{k-1}\right) + Hb_2\left(\bar{y}'_{k+1} - y'_{k+1}\right),$$

$$S_k^3 = \frac{1}{2}\left(h^2 b_1\left(\bar{y}'_{k-1} - y'_{k-1}\right)G''_{\eta-} + H^2 b_2\left(\bar{y}'_{k+1} - y'_{k+1}\right)G''_{\eta+}\right)$$

we have

$$S_k = G_k S_k^1 + G'_k S_k^2 + S_k^3.$$

Since

$$K = \alpha \bar{f}_{k-1} + \beta \bar{f}_{k+1} = \alpha y''_{k-1} + \beta y''_{k+1} + \Sigma_k$$

where

$$\begin{aligned}\Sigma_k \quad = \quad & \alpha\left(G_{k-1}\left(\bar{y}'_{k-1} - y'_{k-1}\right) + H^*_{k-1}\left(\bar{y}'_{k-1} - y'_{k-1}\right)^2\right) \\ & +\beta\left(G_{k+1}\left(\bar{y}'_{k+1} - y'_{k+1}\right) + H^*_{k+1}\left(\bar{y}'_{k+1} - y'_{k+1}\right)^2\right)\end{aligned}$$

we can write

$$T_k = G_k T_k^1 + T_k^2 + T_k^3,$$

with

$$T_k^1 = b_0\left(\bar{y}'_k - y'_k + \alpha y''_{k-1} + \beta y''_{k+1}\right),$$

$$T_k^2 = b_0 G_k \Sigma_k,$$

$$T_k^3 = b_0 H_k^*\left(\bar{y}'_k - y'_k + K\right)^2.$$

Now we obtain

$$\tau_k = R_k + S_k + T_k = R_k + G_k S_k^1 + G'_k S_k^2 + S_k^3 + G_k T_k^1 + T_k^2 + T_k^3$$

$$\tau_k = R_k + G_k\left(S_k^1 + T_k^1\right) + G'_k S_k^2 + S_k^3 + T_k^2 + T_k^3. \tag{3.4}$$

3.1 Error term R_k

In [5] one can see that for some $r_1, r_2, r_3 \in I$

$$\begin{aligned}R_k \quad = \quad & \frac{(H-h)^2}{12} y_k^{(4)} + \frac{(H-h)\left(7h^2 - 5hH + 7H^2\right)}{180} y_k^{(5)} \\ & - \frac{h^5 + H^5}{360\,(h+H)} y^{(6)}(r_1) \\ & + \frac{h^4\,(2h-H)\,y^{(6)}(r_2) + H^4\,(2H-h)\,y^{(6)}(r_3)}{144\,(h+H)}\end{aligned} \tag{3.5}$$

3.2 Error term $S_k^1 + T_k^1$

Using Lemma 6 and Lemma 4 we have

$$\begin{aligned} S_k^1 &= b_1\left(-\frac{h(h+H)}{6}y_k^{(3)} + \frac{h(h+H)}{24}(2h-H)\,y_k^{(4)} + \Theta_k^5\right) + \\ &\quad b_2\left(-\frac{H(h+H)}{6}y_k^{(3)} - \frac{H(h+H)}{24}(2H-h)\,y_k^{(4)} + \Upsilon_k^5\right) \end{aligned}$$

and after some calculations we have

$$\begin{aligned} S_k^1 &= -\frac{(h+H)(b_1h+b_2H)}{6}y_k^{(3)} + \frac{(h+H)\,b_1h(2h-H) - b_2H(2H-h)}{24}y_k^{(4)} \\ &\quad + b_1\Theta_k^5 + b_2\Upsilon_k^5 \end{aligned}$$

From Lemma 2 it follows

$$\bar{y}_k' - y_k' = \frac{H-h}{2}y_k'' + \frac{h^2-hH+H^2}{6}y^{(3)}(x_k) + \frac{(H-h)\left(h^2+H^2\right)}{24}y_k^{(4)} + \Omega_k^5.$$

For some $\eta_{-5}^k,\ \eta_{+5}^k$ we have

$$\alpha y_{k-1}'' + \beta y_{k+1}'' = (\alpha+\beta)\,y_k'' + (-\alpha h + \beta H)\,y_k^{(3)} + \frac{\alpha h^2 + \beta H^2}{2}y_k^{(4)} + \rho_k,$$

$$\rho_k = \frac{1}{6}\left(-\alpha h^3 y^{(5)}\left(\eta_{-5}^k\right) + \beta H^3 y^{(5)}\left(\eta_{+5}^k\right)\right).$$

Now it follows that

$$T_k^1 = b_0\left(\bar{y}_k' - y_k' + \alpha y_{k-1}'' + \beta y_{k+1}''\right)$$

and

$$\begin{aligned} T_k^1 &= b_0\left(\frac{H-h}{2} + \alpha + \beta\right)y_k'' + b_0\left(\frac{h^2-hH+H^2}{6} - \alpha h + \beta H\right)y^{(3)}(x_k) \\ &\quad + b_0\left(\frac{(H-h)\left(h^2+H^2\right)}{24} + \frac{\alpha h^2+\beta H^2}{2}\right)y_k^{(4)} + b_0\left(\Omega_k^5 + \rho_k\right). \end{aligned}$$

$$\begin{aligned} S_k^1 + T_k^1 &= b_0\left(\frac{H-h}{2} + \alpha + \beta\right)y_k'' \\ &\quad - \left(\frac{(h+H)(b_1h+b_2H)}{6} - b_0\left(\frac{h^2-hH+H^2}{6} + \beta H - \alpha h\right)\right)y^{(3)}(x_k) \\ &\quad + \frac{(h+H)\left(b_1h(2h-H) - b_2H(2H-h)\right)}{24}y_k^{(4)} \\ &\quad + \frac{b_0}{2}\left(\frac{(H-h)\left(h^2+H^2\right)}{12} + \alpha h^2 + \beta H^2\right)y_k^{(4)} \\ &\quad + b_1\Theta_k^5 + b_2\Upsilon_k^5 + b_0\left(\Omega_k^5 + \rho_k\right). \end{aligned}$$

Since

$$\alpha+\beta=\frac{h-H}{2}$$

and

$$\frac{(h+H)\left(b_1 h+b_2 H\right)}{6}=b_0\left(\frac{h^2-hH+H^2}{6}-\alpha h+\beta H\right)$$

we obtain

$$\begin{aligned} S_k^1+T_k^1 &= \frac{(h+H)\left(b_1 h\,(2h-H)-b_2 H\,(2H-h)\right)}{24} y_k^{(4)} \\ &+b_0\left(\frac{(H-h)\left(h^2+H^2\right)}{24}+\frac{\alpha h^2+\beta H^2}{2}\right) y_k^{(4)} \\ &+b_1 \Theta_k^5+b_2 \Upsilon_k^5+b_0\left(\Omega_k^5+\rho_k\right).\end{aligned}$$

3.3 Error term S_k^2

Using Lemma 6 and Lemma 4 we get

$$\begin{aligned} S_k^2 &= hb_1\left(\frac{h\,(h+H)}{6} y_k^{(3)}-\frac{h\,(h+H)}{24}(2h-H)\, y_k^{(4)}-\Theta_k^5\right) \\ &+Hb_2\left(-\frac{H\,(h+H)}{6} y_k^{(3)}-\frac{H\,(h+H)}{24}(2H-h)\, y_k^{(4)}+\Upsilon_k^5\right)\end{aligned}$$

and

$$\begin{aligned} S_k^2 &= \frac{(h+H)\left(b_1 h^2-b_2 H^2\right)}{6} y_k^{(3)} \\ &-\frac{(h+H)\left(h^2\,(2h-H)\, b_1+H^2\,(2H-h)\, b_2\right)}{24} y_k^{(4)}-hb_1 \Theta_k^5+Hb_2 \Upsilon_k^5.\end{aligned}$$

3.4 Error term S_k^3

By Lemma 5 and Lemma 3

$$\bar{y}_{k-1}^{\prime}-y_{k-1}^{\prime}=\Theta_k^3, \quad \bar{y}_{k+1}^{\prime}-y_{k+1}^{\prime}=\Upsilon_k^3$$

$$S_k^3=\frac{1}{2}\left(h^2 b_1\left(\bar{y}_{k-1}^{\prime}-y_{k-1}^{\prime}\right) G_{\eta-}^{\prime\prime}+H^2 b_2\left(\bar{y}_{k+1}^{\prime}-y_{k+1}^{\prime}\right) G_{\eta+}^{\prime\prime}\right)$$

$$S_k^3=\frac{1}{2}\left(h^2 b_1 \Theta_k^3 G_{\eta-}^{\prime\prime}+H^2 b_2 \Upsilon_k^3 G_{\eta+}^{\prime\prime}\right)$$

3.5 Error term T_k^2

$$T_k^2 = b_0 G_k \Sigma_k,$$

where

$$\Sigma_k = G_k \Sigma_k^1 + G_k' \Sigma_k^2 + \Sigma_k^3,$$

$$\Sigma_k^1 = \alpha\left(\bar{y}_{k-1}' - y_{k-1}'\right) + \beta\left(\bar{y}_{k+1}' - y_{k+1}'\right),$$

$$\Sigma_k^2 = -h\alpha\left(\bar{y}_{k-1}' - y_{k-1}'\right) + H\beta\left(\bar{y}_{k+1}' - y_{k+1}'\right),$$

$$\begin{aligned}\Sigma_k^3 &= \frac{1}{2}\left(h^2\alpha\left(\bar{y}_{k-1}' - y_{k-1}'\right)G_{\eta-}'' + H^2\beta\left(\bar{y}_{k+1}' - y_{k+1}'\right)G_{\eta+}''\right)\\ &= \frac{1}{2}\left(h^2\alpha\Theta_k^3 G_{\eta-}'' + H^2\beta\Upsilon_k^3 G_{\eta+}''\right)\end{aligned}$$

Using Lemma 6 and Lemma 4 we find

$$\begin{aligned}\Sigma_k^1 &= \alpha\left(-\frac{h(h+H)}{6}y_k^{(3)} + \frac{h(h+H)(2h-H)}{24}y_k^{(4)} + \Theta_k^5\right) +\\ &\quad \beta\left(-\frac{H(h+H)}{6}y_k^{(3)} - \frac{H(h+H)(2H-h)}{24}y_k^{(4)} + \Upsilon_k^5\right)\end{aligned}$$

and after some simplifications we have

$$\begin{aligned}\Sigma_k^1 &= -\frac{(h+H)(\alpha h + \beta H)}{6}y_k^{(3)} + \frac{(h+H)(\alpha h(2h-H) - \beta H(2H-h))}{24}y_k^{(4)}\\ &\quad +\alpha\Theta_k^5 + \beta\Upsilon_k^5\end{aligned}$$

Using Lemma 6 and Lemma 4 we find

$$\begin{aligned}\Sigma_k^2 &= \left(\frac{\alpha h^2(h+H)}{6} - \frac{\beta H^2(h+H)}{6}\right)y_k^{(3)}\\ &\quad -\left(\frac{\alpha h^2(h+H)(2h-H)}{24} + \frac{\beta H^2(h+H)(2H-h)}{24}\right)y_k^{(4)}\\ &\quad -h\alpha\Theta_k^5 + H\beta\Upsilon_k^5\end{aligned}$$

and

$$\begin{aligned}\Sigma_k^2 &= \frac{(h+H)\left(\alpha h^2 - \beta H^2\right)}{6}y_k^{(3)} - \frac{(h+H)\left(h^2(2h-H)\alpha + H^2(2H-h)\beta\right)}{24}y_k^{(4)}\\ &\quad -h\alpha\Theta_k^5 + H\beta\Upsilon_k^5.\end{aligned}$$

3.6 Error term T_k^3

$$T_k^3 = b_0 H_k^* \left(\bar{y}_k' - y_k' + K\right)^2 .$$

$$K = \alpha \bar{f}_{k-1} + \beta \bar{f}_{k+1} = \alpha y_{k-1}'' + \beta y_{k+1}'' + \Pi_k$$

$$\begin{aligned} \Pi_k &= \alpha G_{k-1}^* \left(\bar{y}_{k-1}' - y_{k-1}'\right) + \beta G_{k+1}^* \left(\bar{y}_{k+1}' - y_{k+1}'\right) \\ &= \alpha G_{k-1}^* \Theta_k^3 + \beta G_{k+1}^* \Upsilon_k^3 \end{aligned}$$

where, for some δ_-, δ_+,

$$G_{k-1}^* = G\left(x_{k-1}, y_{k-1}, \delta_-\right), \quad G_{k+1}^* = G\left(x_{k+1}, y_{k+1}, \delta_+\right),$$

$$K = (\alpha + \beta)\, y_k'' + (-\alpha h + \beta H)\, y_k^{(3)} + \frac{\alpha h^2}{2} y^{(4)}\left(\mu_-\right) + \frac{\beta H^2}{2} y^{(4)}\left(\mu_+\right) + \Pi_k$$

$$\bar{y}_k' - y_k' = \frac{H-h}{2} y_k'' + \Omega_k^3.$$

$$\begin{aligned} \bar{y}_k' - y_k' + K &= \left(\frac{H-h}{2} + \alpha + \beta\right) y_k'' + (-\alpha h + \beta H)\, y_k^{(3)} + \frac{\alpha h^2}{2} y^{(4)}\left(\mu_-\right) \\ &\quad + \frac{\beta H^2}{2} y^{(4)}\left(\mu_+\right) + \Omega_k^3 + \alpha G_{k-1}^* \Theta_k^3 + \beta G_{k+1}^* \Upsilon_k^3 \end{aligned}$$

Because of

$$\frac{H-h}{2} + \alpha + \beta = 0$$

we have

$$\begin{aligned} \bar{y}_k' - y_k' + K &= (-\alpha h + \beta H)\, y_k^{(3)} + \frac{\alpha h^2}{2} y^{(4)}\left(\mu_-\right) + \frac{\beta H^2}{2} y^{(4)}\left(\mu_+\right) \\ &\quad + \Omega_k^3 + \alpha G_{k-1}^* \Theta_k^3 + \beta G_{k+1}^* \Upsilon_k^3 \end{aligned}$$

3.7 Truncation error τ_k

From now on we shall assume that our mesh has the following properties:

$$H_{\max} \le h_{\min}\left(1 + M h_{\min}\right)$$

where

$$H_{\max} = \max\left\{h_k : k = 1, 2, \ldots, n\right\}, \qquad h_{\min} = \min\left\{h_k : k = 1, 2, \ldots, n\right\}.$$

Such a mesh is called almost equidistant, see [7].

For simplicity, we have defined for a fixed k

$$h = x_k - x_{k-1}, \qquad H = x_{k+1} - x_k.$$

Since our mesh is almost equidistant, it then holds

$$|H-h| \le Mh^2. \tag{3.6}$$

Here and throughout this section M, sometimes subscripted, denotes a generic constant, independent of the number n of discretization subintervals that will be used to solve (1.1) numerically.

The truncation error τ_k is given by (3.1). Simplification of local truncation error terms was done using *Mathematica*. Only final expressions for each of the terms will be given here.

3.7.1 Error term R_k

The error term, given by (3.5), is obviously $\mathcal{O}(H_{\max}^4)$, if $|H-h| \le Mh^2$, since

$$|R_k| \le \frac{|H-h|^2}{12} M_4 + \frac{7\,|H-h|\,(H+h)^2}{180} M_5 + \frac{(H+h)^4}{60} M_6 \le M H_{\max}^4.$$

3.7.2 Error term S_k^1

$$\begin{aligned} S_k^1 \;=\;& \frac{-h^2+h\,H-H^2}{18} y_k^{(3)} + \frac{(h-H)\,\left(4\,h^2-h\,H+4\,H^2\right)}{144} y_k^{(4)} \\ &+\frac{h\,(H-2\,h)\,\left(5\,h^4+5\,h^3\,H+H^4\right)}{720\,(h+H)^2} y^{(5)}\,(\theta_k) \\ &+\frac{h^4\,(2\,h-H)\,(2\,h+H)}{720\,(h+H)^2} y^{(5)}\left(\theta_{k,0}^-\right) \\ &+\frac{H\,(h-2\,H)\,\left(h^4+5\,h\,H^3+5\,H^4\right)}{720\,(h+H)^2} y^{(5)}\,(\tau_k) \\ &+\frac{(2H-h)\,H^4\,(h+2\,H)}{720\,(h+H)^2} y^{(5)}\left(\tau_{k,0}^+\right) \end{aligned}$$

3.7.3 Error term T_k^1

$$\begin{aligned} T_k^1 \;=\;& \frac{h^2-h\,H+H^2}{18} y^{(3)}\,(x_k) + \frac{(h-H)\,\left(h^2+24\,h\,H+H^2\right)}{144} y_k^{(4)} \\ &+\frac{h^4-h^3\,H+h^2\,H^2-h\,H^3+H^4}{144} y^{(5)}\,(\omega_k) \\ &-\frac{h^3\,\left(h^2+4\,h\,H-4\,H^2\right)}{72\,(h+H)} y^{(5)}\left(\eta_{-5}^k\right) - \frac{H^3\,\left(-4\,h^2+4\,h\,H+H^2\right)}{72\,(h+H)} y^{(5)}\left(\eta_{+5}^k\right). \end{aligned}$$

3.7.4 Error term $S_k^1 + T_k^1$

$$\begin{aligned}
S_k^1 + T_k^1 \quad = \quad & \frac{(h-H)\,\left(5\,h^2+23h\,H+5\,H^2\right)}{144} y_k^{(4)} \\
& + \frac{h\,(H-2\,h)\,\left(5\,h^4+5\,h^3\,H+H^4\right)}{720\,(h+H)^2} y^{(5)}\,(\theta_k) \\
& + \frac{h^4\,(2\,h-H)\,(2\,h+H)}{720\,(h+H)^2} y^{(5)}\left(\theta_{k,0}^-\right) \\
& + \frac{H\,(h-2\,H)\,\left(h^4+5\,h\,H^3+5\,H^4\right)}{720\,(h+H)^2} y^{(5)}\,(\tau_k) \\
& + \frac{(2H-h)\,H^4\,(h+2\,H)}{720\,(h+H)^2} y^{(5)}\left(\tau_{k,0}^+\right) \\
& + \frac{h^4-h^3\,H+h^2\,H^2-h\,H^3+H^4}{144} y^{(5)}\,(\omega_k) \\
& - \frac{h^3\,\left(h^2+4\,h\,H-4\,H^2\right)}{72\,(h+H)} y^{(5)}\left(\eta_{-5}^k\right) \\
& - \frac{H^3\,\left(-4\,h^2+4\,h\,H+H^2\right)}{72\,(h+H)} y^{(5)}\left(\eta_{+5}^k\right).
\end{aligned}$$

$$\left|S_k^1 + T_k^1\right| \le \frac{|h-H|\,\,(h+\,H)^2}{12} M_4 + \frac{(h+H)^4}{10} M_5 \le M H_{\max}^4.$$

3.7.5 Error term S_k^2

$$\begin{aligned}
S_k^2 \quad = \quad & \frac{(h-H)\,\left(2\,h^2+h\,H+2\,H^2\right)}{36} y_k^{(3)} \\
& + \frac{-2\,h^4+2\,h^3\,H-h^2\,H^2+2\,h\,H^3-2\,H^4}{72} y_k^{(4)} \\
& + \frac{h^2\,(2\,h-H)\,\left(5\,h^4+5\,h^3\,H+H^4\right)}{720\,(h+H)^2} y^{(5)}\,(\theta_k) \\
& - \frac{h^5\,(2\,h-H)\,(2\,h+H)}{720\,(h+H)^2} y^{(5)}\left(\theta_{k,0}^-\right) \\
& - \frac{H^2\,(2\,H-h)\,\left(h^4+5\,h\,H^3+5\,H^4\right)}{720\,(h+H)^2} y^{(5)}\,(\tau_k) \\
& + \frac{(2\,H-h)\,H^5\,(h+2\,H)}{720\,(h+H)^2} y^{(5)}\left(\tau_{k,0}^+\right).
\end{aligned}$$

$$\left|S_k^2\right| \leq \frac{|h-H|\,(H+h)^2}{18} M_3 + \frac{(H+h)^4}{36} M_4 + \frac{(H+h)^5}{48} M_5 \leq M H_{\max}^4.$$

3.7.6 Error term S_k^3

$$\begin{aligned} S_k^3 &= \frac{h^3\,(H-2\,h)\,\left(3\,h^2+3\,h\,H+H^2\right)}{72\,(h+H)^2} y^{(3)}(\theta_k)\,G''_{\eta-} \\ &+ \frac{h^4\,(2\,h-H)\,(2\,h+H)}{72\,(h+H)^2} y^{(3)}\left(\theta_{k,0}^-\right) G''_{\eta-} \\ &+ \frac{H^3\,(h-2H)\,\left(h^2+3\,h\,H+3\,H^2\right)}{72\,(h+H)^2} y^{(3)}(\tau_k)\,G''_{\eta+} \\ &+ \frac{(2H-h)\,H^4\,(h+2\,H)}{72\,(h+H)^2} y^{(3)}\left(\tau_{k,0}^+\right) G''_{\eta+} \end{aligned}$$

$$\left|S_k^3\right| \leq \frac{5\,(H+h)^4}{36} M_5 M_G \leq M H_{\max}^4.$$

3.7.7 Error term T_k^2

$$T_k^2 = b_0 G_k \left(G_k \Sigma_k^1 + G'_k \Sigma_k^2 + \Sigma_k^3\right),$$

where

$$\begin{aligned} \Sigma_k^1 &= \frac{(H-h)\,\left(h^2+9\,h\,H+H^2\right)}{60} y_k^{(3)} \\ &+ \frac{2\,h^4+11\,h^3\,H-24\,h^2\,H^2+11\,h\,H^3+2\,H^4}{240} y_k^{(4)} \\ &- \frac{h\,\left(h^2+4\,h\,H-4\,H^2\right)\,\left(5\,h^4+5\,h^3\,H+H^4\right)}{1200\,(h+H)^2} y^{(5)}(\theta_k) \\ &+ \frac{h^4\,(2\,h+H)\,\left(h^2+4\,h\,H-4\,H^2\right)}{1200\,(h+H)^2} y^{(5)}\left(\theta_{k,0}^-\right) \\ &+ \frac{H\,\left(-4\,h^2+4\,h\,H+H^2\right)\,\left(h^4+5\,h\,H^3+5\,H^4\right)}{1200\,(h+H)^2} y^{(5)}(\tau_k) \\ &- \frac{H^4\,(h+2\,H)\,\left(-4\,h^2+4\,h\,H+H^2\right)}{1200\,(h+H)^2} y^{(5)}\left(\tau_{k,0}^+\right) \end{aligned}$$

$$\left|\Sigma_k^1\right| \leq \frac{|H-h|\,(h+H)^2}{12} M_3 + \frac{(h+H)^2}{60} M_4 + \frac{7\,(H+h)^5}{300} M_5 \leq M H_{\max}^4.$$

$$\begin{aligned}\Sigma_k^2 \;=\;& \frac{h^4+4\,h^3\,H-8\,h^2\,H^2+4\,h\,H^3+H^4}{60}y_k^{(3)}\\&-\frac{(h-H)\,\left(h^2-h\,H+H^2\right)\,\left(2\,h^2+11\,h\,H+2\,H^2\right)}{240}y_k^{(4)}\\&+\frac{h^2\,\left(h^2+4\,h\,H-4\,H^2\right)\,\left(5\,h^4+5\,h^3\,H+H^4\right)}{1200\,(h+H)^2}y^{(5)}\left(\theta_k\right)\\&-\frac{h^5\,(2\,h+H)\,\left(h^2+4\,h\,H-4\,H^2\right)}{1200\,(h+H)^2}y^{(5)}\left(\theta_{k,0}^-\right)\\&+\frac{H^2\,\left(-4\,h^2+4\,h\,H+H^2\right)\,\left(h^4+5\,h\,H^3+5\,H^4\right)}{1200\,(h+H)^2}y^{(5)}\left(\tau_k\right)\\&-\frac{H^5\,(h+2\,H)\,\left(-4\,h^2+4\,h\,H+H^2\right)}{1200\,(h+H)^2}y^{(5)}\left(\tau_{k,0}^+\right)\end{aligned}$$

$$\left|\Sigma_k^2\right| \le \frac{(h+H)^4}{30}M_3+\frac{|h-H|\,(h+H)^4}{60}M_4+\frac{(h+H)^4}{30}M_5 \le MH_{\max}^4.$$

$$\begin{aligned}\Sigma_k^3 \;=\;& \frac{h^3\,\left(h^2+4\,h\,H-4\,H^2\right)\,\left(5\,h^4+5\,h^3\,H+H^4\right)}{2400\,(h+H)^2}y^{(3)}\left(\theta_k\right)G''_{\eta-}\\&+\frac{h^6\,(2\,h+H)\,\left(h^2+4\,h\,H-4\,H^2\right)}{2400\,(h+H)^2}y^{(3)}\left(\theta_{k,0}^-\right)G''_{\eta-}\\&+\frac{H^3\,\left(-4\,h^2+4\,h\,H+H^2\right)\,\left(h^4+5\,h\,H^3+5\,H^4\right)}{2400\,(h+H)^2}y^{(3)}\left(\tau_k\right)G''_{\eta+}\\&-\frac{H^6\,(h+2\,H)\,\left(-4\,h^2+4\,h\,H+H^2\right)}{2400\,(h+H)^2}y^{(3)}\left(\tau_{k,0}^+\right)G''_{\eta+}\end{aligned}$$

$$\left|\Sigma_k^3\right| \le \frac{(H+h)\left(7H^6+7h^6-Hh^5-H^5h+H^2h^4+H^4h^2\right)}{600}M_{G2} \le MH_{\max}^4.$$

$$\left|T_k^2\right| \le b_0\left|G_k\right|\left|G_k\Sigma_k^1+G'_k\Sigma_k^2+\Sigma_k^3\right| \le MH_{\max}^4.$$

3.7.8 Error term $T_k^3 = b_0 H_k^* \left(\bar{y}'_k - y'_k + K\right)^2$.

$$\begin{aligned}\bar{y}'_k-y'_k+K \;=\;& \frac{-h^2+h\,H-H^2}{10}y_k^{(3)}+\frac{h^2\,\left(h^2+4\,h\,H-4\,H^2\right)}{20\,(h+H)}y^{(4)}\left(\mu_-\right)\\&-\frac{H^2\,\left(H^2-4\,h^2+4\,h\,H\right)}{20\,(h+H)}y^{(4)}\left(\mu_+\right)\\&+\frac{H\,\left(H^4-h+H\right)}{6}y^{(3)}\left(\omega_k\right)\end{aligned}$$

$$+G^*_{k-1}\frac{-h\,\left(h^2+4\,h\,H-4\,H^2\right)\,\left(3\,h^2+3\,h\,H+H^2\right)}{60\,\left(h+H\right)^2}y^{(3)}\left(\theta_k\right)$$
$$+G^*_{k-1}\frac{h^2\,\left(2\,h+H\right)\,\left(h^2+4\,h\,H-4\,H^2\right)}{60\,\left(h+H\right)^2}y^{(3)}\left(\theta^-_{k,0}\right)$$
$$+G^*_{k+1}\frac{H\,\left(-4\,h^2+4\,h\,H+H^2\right)\,\left(h^2+3\,h\,H+3\,H^2\right)}{60\,\left(h+H\right)^2}y^{(3)}\left(\tau_k\right)$$
$$-G^*_{k+1}\frac{H^2\,\left(h+2\,H\right)\,\left(-4\,h^2+4\,h\,H+H^2\right)}{60\,\left(h+H\right)^2}y^{(3)}\left(\tau^+_{k,0}\right)$$

$$|\bar{y}'_k-y'_k+K|\le\frac{\left(h+H\right)^2}{5}M_3+\frac{\left(H+h\right)^3}{5}M_4+\frac{H^5}{6}M_3+\frac{\left(H+h\right)^3}{5}M_G\le MH^2_{\max}.$$
$$\left|T^3_k\right|\le b_0\left|H^*_k\right|\left|\bar{y}'_k-y'_k+K\right|^2\le MH^4_{\max}.$$

It can be easily seen that each of the error terms is $\mathcal{O}(H^4_{\max})$. Since τ_k is given by (3.4), i.e.

$$\tau_k=R_k+G_k\left(S^1_k+T^1_k\right)+G'_kS^2_k+S^3_k+T^2_k+T^3_k,$$

we can see that τ_k is also $\mathcal{O}(H^4_{\max})$, since

$$|\tau_k|\le|R_k|+|G_k|\left|S^1_k+T^1_k\right|+|G'_k|\left|S^2_k\right|+\left|S^3_k\right|+\left|T^2_k\right|+\left|T^3_k\right|\le MH^4_{\max}.$$

4. Numerical results

Obtained theoretical results are confirmed by numerical experiments. Exact solutions for both of the tested examples are known. The error is measured by

$$E_n=\left\|y^h-w^*\right\|_\infty,$$

where w^* is the solution of the discrete problem and

$$y^h=(y(x_0),y(x_1)...,y(x_n))^T,$$

where y is the exact solution of the observed problem. Also, we define in the usual way the order of convergence Ord_n for two successive values of n with respective errors E_n and E_{2n} :

$$Ord_n=\frac{\ln E_n-\ln E_{2n}}{\ln 2}.$$

The expected value of $Ord_n=4$.

Our discrete analogue $F\left(w\right)=0$ is a nonlinear system. We solve this system using the Newton-Raphson method, where a tridiagonal linear system is solved in each step. We performed the calculation in *Mathematica*. The stop criterion applied is

$$\text{maxiter} > 20 \text{ or } \left\|w^k - w^{k-1}\right\|_\infty < 10^{-12} \text{ or } \left\|F(w^k)\right\|_\infty < 10^{-12},$$

where maxiter is the iteration limit of the Newton-Raphson method and w^k is the $k-$th approximation of w^* obtained by the Newton-Raphson method.

Among the others, we have tested the following two problems:

4.1 Example 1

$$-\varepsilon y'' + x - y' = 0, \qquad y(0) = y(1) = 0, \tag{4.1}$$

where

$$y(x) = \left(\varepsilon - \frac{1}{2}\right)\frac{1 - e^{-x/\varepsilon}}{1 - e^{-1/\varepsilon}} - \varepsilon s + \frac{s^2}{2},$$

is the exact solution, [7].

In order to obtain a good approximation for the exact solution of the problem (4.1) we use nonuniform mesh that is dense in the neighborhood of the point $x = 0$, where the boundary layer appears. We considered a mesh of Bakhvalov type, [5], [7], [8]. This mesh satisfies the condition (3.6) and $H \geq h$.

The mesh, further on called $R-$mesh, is generated by the function

$$\lambda(t) = \begin{cases} \dfrac{a\varepsilon t}{q - t}, & t \in [0, \tau] \\ \dfrac{a\varepsilon}{q - \tau}\left(\tau + \dfrac{q(t - \tau)}{q - \tau}\right), & t \in [\tau, 1] \end{cases} \tag{4.2}$$

with

$$\tau = \frac{q - \sqrt{aq\varepsilon(1 - q + a\varepsilon)}}{1 + a\varepsilon},$$

and the constants a and q satisfy

$$q \in (0,1), \quad a \in (0, q/\varepsilon), \tag{4.3}$$

so that the transition point has the property $\tau \in (1 - q, 1)$. Mesh points are given by

$$x_k = \lambda\left(\frac{k}{n}\right), \quad k = 0, 1, \ldots, n.$$

The approximations obtained by the $R-$mesh with $a = 1$ and $q = 0.96$ were tested for different values of ε and n. The results, Table 1, confirmed the order of convergence of the method.

Newton-Raphson method is used for solving the nonlinear system of equations $F(w) = 0$ with initial approximation $w^0 = (y_0(x_0), y_0(x_1), ..., y_0(x_n))^T$, where $y_0(x) = \frac{1}{2}\left(x^2 - 1\right)$ is the solution of the reduced problem $x - y' = 0$.

$n\diagdown\varepsilon$	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	
64	3.70×10^{-5}	3.70×10^{-5}	6.55×10^{-5}	1.51×10^{-4}	1.16×10^{-3}	E_n
	$-$	$-$	$-$	$-$	$-$	Ord_n
128	2.38×10^{-6}	2.43×10^{-6}	2.43×10^{-6}	2.46×10^{-6}	2.79×10^{-6}	E_n
	3.95529	4.03122	4.75120	5.94600	8.70388	Ord_n
256	1.50×10^{-7}	1.52×10^{-7}	1.53×10^{-7}	1.53×10^{-7}	1.53×10^{-7}	E_n
	3.99339	3.99337	3.99361	4.00831	4.19144	Ord_n
512	9.36×10^{-9}	9.53×10^{-9}	9.55×10^{-9}	9.55×10^{-9}	9.55×10^{-9}	E_n
	3.99888	3.99888	3.99888	3.99889	3.99889	Ord_n
1024	5.85×10^{-10}	5.96×10^{-10}	5.97×10^{-10}	5.9710^{-10}	5.96×10^{-10}	E_n
	3.99967	3.99985	3.99980	3.99960	4.00239	Ord_n

Table 1.

The average order of convergence is 4.378. We can see that for $n > 128$ the order of convergence does not change much as ε changes, and is practically 4.

4.2 Example 2

To illustrate computationally the fourth-order method we solved the following nonlinear two-point boundary value problem

$$y'' = \frac{1}{3}\left((2-x)\,e^{2(y-x\ln 2)} + \ln 2 - y'\right), \qquad y(0) = y(1) = 0,$$

with the exact solution $y(x) = \ln\frac{1}{1+x} + x\ln 2$, [4]. In this example we used an equidistant mesh and a nonequidistant mesh, generated by the following mesh generating function

$$\lambda(t) = \frac{1}{2}\left(1 - \sin\left(\frac{\pi}{2}\cos(\pi t)\right)\right). \tag{4.4}$$

In the first case the mesh points are

$$x_i = \frac{i}{n}, \qquad i = 0, 1, \ldots, n.$$

In the second case the mesh points are

$$x_i = \lambda\left(\frac{i}{n}\right), \qquad i = 0, 1, \ldots, n.$$

Newton-Raphson method is used to solve the nonlinear system of equations $F(w) = 0$ with initial approximation $w^0 = (-0.05, -0.05, ..., -0.05)^T$. The numerical results are given in Table 2.

n	Equidistant mesh	Mesh generated by (4.4)	
16	1.01×10^{-7}	7.18×10^{-6}	E_n
	–	–	Ord_n
32	6.32×10^{-9}	4.75×10^{-7}	E_n
	3.99537	3.91787	Ord_n
64	3.95×10^{-10}	3.00×10^{-8}	E_n
	3.99884	3.98495	Ord_n
128	2.47×10^{-11}	1.88×10^{-9}	E_n
	3.99968	3.99624	Ord_n
256	1.54×10^{-12}	1.18×10^{-10}	E_n
	4.00068	3.99905	Ord_n
512	9.24×10^{-14}	7.35×10^{-12}	E_n
	4.06233	3.99887	Ord_n

Table 2.

The average order of convergence for the equidistant mesh is 4.011 and for the mesh (4.4) it is 3.979.

In this case, when the problem is not singularly perturbed, both the equidistant mesh and the mesh (4.4) give practically the same results.

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