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A NOTE ABOUT THE ORDER OF AN EXTENSIVE MAP

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Abstract. We define the order ν of an extensive map f from a complete lattice to itself : ν is the smallest of the ordinals α such that $f \circ f^\alpha = f^\alpha$. If f is a preclosure, f^ν is the closure generated by f. We also examine the case of the extensive map which sends an $\wedge$-closed subset A of a complete lattice L towards the $\wedge$-closed subset of the A-semi-closed points of L.

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1. The order of an extensive map

In the whole text to follow, the **greek letters** will designate the **ordinal numbers**. Besides, we will work in a **complete lattice L**. Let us denote by 1 the identity map : $L \to L$.

The map f on L (i.e. $L \to L$) is said to be **extensive** if and only if $1 \leq f$. The set E of the extensive maps is an $\wedge$-closed subset of the complete lattice L^L : then it is, for itself, a complete lattice. Note that, if H is a non-empty subset of E, $\vee H$ is in E (and thus is the induced supremum). Clearly, $\{0\} \cup E$ is a $\vee$-closed subset of L^L.

Now, let us define the ordinal power f^α of an extensive map f, inductively : $f^0 = 1, f^{\alpha+1} = f \circ f^\alpha$ and, for a non-null limit ordinal α, $f^\alpha = \vee\{f^\beta, \beta < \alpha\}$. The sequence $f^0, f, \ldots, f^\alpha, \ldots$ is increasing. So, in account of the cardinality, there exists an ordinal α such that $f^{\alpha+1} = f^\alpha$. The smallest of these α will be denoted by $\mathbf{n}(\mathbf{f})$ and will be called the **order** of f. Clearly, $n(f) < |L| + 1$.

Proposition 1.1. *If f is an extensive map and $n(f) = \nu$, $f^\beta f^\nu = f^\nu$ for all β.*

Proof. We will work by transfinite induction. When $\beta = 0$, $f^\beta f^\nu = f^\nu$. Suppose that, for all $\gamma < \beta, f^\gamma f^\nu = f^\nu$. If β is a limit-ordinal, $f^\beta f^\nu = (\vee\{f^\gamma, \gamma <$

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$\beta\})f^\nu = \vee\{f^\gamma f^\nu, \gamma < \beta\} = f^\nu$. Otherwise, $\beta = \delta + 1$ for a certain δ and $f^\beta f^\nu = f f^\delta f^\nu = f f^\nu = f^\nu$. □

For a given $x \in L$, let $\tau = r(f,x)$ be the smallest α such that $f^\alpha(x) = f^{\alpha+1}(x) : \alpha \leq n(f)$. By induction on γ, we get $f^\gamma(x) = f^\tau(x)$ for all $\gamma \geq \tau$.

2. Application to the preclosures

A **preclosure** on L is an increasing extensive map. The preclosures constitute an $\wedge$-closed subset P of L^L. The **closures** (i.e. the idempotent preclosures) constitute an $\wedge$-closed subset F of L^L. The closure $\overline{\mathbf{p}}$ generated by the preclosure p is the closure $\wedge(F \cap \uparrow p)$ (i.e. the smallest closure c such that $p \leq c$). Note that $\{0\} \cup P$ is $\vee$-closed in L^L. Note that P is a semi-group for the composition of maps.

Proposition 2.1. *If ν is the order of the preclosure p, the closure generated by p is p^ν.*

Proof. Obvious for $\nu = 0$. Suppose $\nu > 0$ (i.e. $p > 1$). By Prop. 1.1, p^ν is a closure. But, by an easy induction on α, $p^\alpha \leq \overline{p}$ for all α. Then $p \leq p^\nu \leq \overline{p}$, and therefore $\overline{p} \leq p^\nu \leq \overline{p}$. □

Clearly (by induction on γ), $p^\gamma = p^\nu$ for all $\gamma \geq \nu$. Since, for $\alpha > 0$, $p^\alpha \in F$ implies $p^\alpha \leq pp^\alpha \leq p^\alpha p^\alpha = p^\alpha$, it results that, for $\alpha > 0$, $p^\alpha \in F \Leftrightarrow \alpha \geq \nu$.

Remark 1 – Let $P_\alpha = \{p \in P / n(p) \geq \alpha\}$. We get : $P_0 = P, P_\alpha = \{p \in P / 1 < p < p^2 < \ldots < p^\alpha\}$ if $\alpha \geq 1$ and $P_1 \backslash P_2 = F \backslash \{1\}$.

Remark 2 – Let c be a closure. Put $H = \{p \in P / \overline{p} = c\}$. The set $\{0\} \cup H$ is $\vee$-closed in L^L. The map defined, for $p \in H$, by $p \mapsto n(p)$, is decreasing [Indeed, if $p \leq q$ are elements of H and $\alpha = n(p), c = p^\alpha \leq q^\alpha \leq c$, then $\alpha \geq n(q)$].

3. Application to the semi-closed points

In paragraph 3, we suppose that the complete lattice L is equipped with an **$\wedge$-closed subset A of L**. Let c be the closure $c(t) = \wedge(A \cap \uparrow t)$ linked with A. Let us call **c-closed** the points of A.

Definition 3.1. *A point s of L is said to be* **A-semi-closed** *iff it verifies one of the equivalent properties*
1/ $A \cup \{s\}$ is $\wedge$-closed in L
2/ $\forall a \in A$, $a \wedge s \in A \cup \{s\}$

Proof of : **2** $\Rightarrow$ **1**. We must prove that, whenever $H \subset A \cup \{s\}$, $\wedge H \in A \cup \{s\}$. If $s \notin H$, it is obvious. Otherwise, let us put $K = H \backslash \{s\}$. Let $I = \{k \in K / k \wedge s = s\}$ and $J = \{k \in K / k \wedge s < s\}$. We can write $\wedge H = \wedge(K \cup \{s\}) = \wedge(I \cup J \cup \{s\}) = (\wedge(I \cup \{s\})) \wedge (\wedge J)$. But $\wedge(I \cup \{s\}) = s$ and $\wedge J \in A$. Then $\wedge H = s \wedge (\wedge J)$ is in $A \cup \{s\}$. □

Proposition 3.2. *The set B of the A-semi-closed points includes A and it is also an $\wedge$-subset of L.*

Proof. Clearly, each point of A is A-semi-closed. We must prove that, if $H \subset B$, the membership $a \wedge (\wedge H) \in A \cup \{\wedge H\}$ is insured for each $a \in A$. If $\wedge H \leq a$, it is obvious. Otherwise, for each possible $h \in H$, $h \nleq a$ ($h \leq a$ would induce $\wedge H \leq a$). So, for every $h \in H$, $a \wedge h \neq h$. Since $h \in B$, $a \wedge h \in A \cup \{h\}$, and then $a \wedge h \in A$. So, $a \wedge (\wedge H) = \wedge\{a \wedge h, h \in H\} \in A$. □

Consequence. Let us apply paragraph 1 to the extensive map which sends A towards B.

Let $\mathcal{F}$ be the $\cap$-closed subset of 2^L constituted by the $\wedge$-closed subset of L : $\mathcal{F}$ is a complete lattice. We can define an extensive map k from $\mathcal{F}$ to itself by $k(A) = B$. Let σ be the order of k. Put $k^\alpha(A) = A_\alpha$. Put $A_\sigma = S$.

Now we can define a map $m : S \rightarrow \downarrow \sigma$, by $m(t) = \wedge\{\alpha / t \in A_\alpha\}$. It is easily seen that $A_\alpha = \{t \in L / m(t) \leq \alpha\}$. We get the following Galois connection $2^S \underset{A}{\overset{r}{\rightleftarrows}} (\downarrow \sigma)^*$, in which $r(X) = \bigvee\{m(t), t \in X\}$, A is the map $\alpha \mapsto A_\alpha$ and $(\downarrow \sigma)^*$ is the dual of $\downarrow \sigma$.

If c_α is the closure associated with A_α, it may be said that c_α is the α-**weakening** of c. ($c = c_0 \geq c_1 \geq c_2 \ldots \geq c_\sigma$).

Let $\tau = r(m, A)$ $(\leq \sigma)$ (Cf. the end of paragraph 1). If $\tau = 0$, $c_\alpha = c$ for all α. Otherwise, $c_0 > \ldots > c_\tau = \ldots = c_\sigma$.

4. Questions

What are the bases of a given closure c (i.e. the minimal preclosures which generate c)? What are the A such that $k(A) = A$? Is it possible that $A = k(A)$ for all A ?

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