Novi Sad J. Math.
Vol. 34, No. 2, 2004, 17-26
Proc. Novi Sad Algebraic Conf. 2003
(eds. I. Dolinka, A. Tepavčević)

ON THE MINIMAL LENGTH OF SEMIGROUP PRESENTATIONS

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Abstract. We define the length of a semigroup presentation and related ideas. Theoretical results and bounds are presented, gained while investigating the lengths of groups defined by semigroup presentations. Minimal length semigroup presentations are obtained in certain cases.

AMS Mathematics Subject Classification (2000): 20F05, 20M05

Key words and phrases: Group presentations, semigroup presentations, semigroup presentations for groups, minimal length presentations

1. Introduction

The relationships between group and semigroup presentations for groups have been much studied, see for example [1], [9] and [14]. As a byproduct of these investigations the semigroup efficiency of groups has also been discussed. It is shown in [6] that a finite group is efficient as a group if, and only if, it is efficient as a semigroup. A topic of recent interest in computational group theory concerns short (and possibly minimal) length presentations for certain finite groups. In this paper we present some results on the question of the minimal length of (efficient) semigroup presentations for groups on a minimal generating set. In particular we obtain minimal length semigroup presentations for certain groups $PSL(2,p)$, p an odd prime.

2. Definitions

Let A be an alphabet. We denote by A^+ the free semigroup on A consisting of all non-empty words over A, and by $\mathrm{F}(A)$ the free group of all freely reduced words over $A \cup A^{-1} \cup \{\varepsilon\}$, where A^{-1} is an alphabet whose elements represent the inverses of elements of A and ε represents the empty word. A *semigroup presentation* is an ordered pair $\langle\ A \mid R\ \rangle$, where $R \subseteq A^+ \times A^+$. In a presentation $\langle\ A \mid R\ \rangle$ we call A the *generating set* and R the *relations*. If both A and R are finite, we have a *finite (semigroup) presentation.* A semigroup S is said to be *defined by the semigroup presentation* $\langle\ A \mid R\ \rangle$ if $S \cong A^+/\rho$, where ρ

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is the congruence on A^+ generated by R. Replacing A^+ by $\mathrm{F}(A)$ in the above definitions gives the notion of a *(finite) group presentation* and of a *group defined by a (group) presentation.*

Given the semigroup relation (r, s) where $r = a_1 a_2 \ldots a_m$ and $s = b_1 b_2 \ldots b_n$ with $a_i \in A$ and $b_i \in A$, the *length of* (r, s), written $|(r, s)|$, is $m + n$. By the *length of the semigroup presentation* $\langle\, A \mid R \,\rangle$ we shall mean

$$\sum_{(r,s)\in R} |(r, s)|.$$

When considering the group relation (r, s) with $r = a_1 a_2 \ldots a_m$, $s = b_1 b_2 \ldots b_n$, $a_i \in A \cup A^{-1} \cup \varepsilon$ and $b_i \in A \cup A^{-1} \cup \varepsilon$, it is usual to let $|\varepsilon| = 0$. The analogous group definitions also hold.

The above definitions hold for a wide variety of presentations, for example, monoid presentations and inverse semigroup presentations; see [13] for definitions.

The *deficiency* of a finite presentation $\langle\, A \mid R \,\rangle$ is $|R| - |A|$ and the *semigroup deficiency* of a finitely presented semigroup T, denoted by $\mathrm{def}_S(T)$, is the minimum deficiency over all finite semigroup presentations defining T:

$$\mathrm{def}_S(T) = \min\{\text{deficiency of } \mathcal{P} \mid \mathcal{P} \text{ is a finite semigroup presentation that defines } T\}.$$

Likewise the *group deficiency* of a finitely presented group K is the minimum deficiency over all finite group presentations defining K, written $\mathrm{def}_G(K)$.

A bound for the deficiency of a finite semigroup T or a finite group K is given by the rank of the second integral homology, H_2, of T or K:

$$\mathrm{def}_S(T) \geq \mathrm{rank}(H_2(T)) \quad \text{and} \quad \mathrm{def}_G(K) \geq \mathrm{rank}(H_2(K)).$$

A finite semigroup or group is called *efficient* if it attains this lower bound and is called *inefficient* otherwise. Thus there are at first sight two notions of efficiency for finite groups, namely the group and semigroup efficiency. In fact, rather surprisingly, these two notions coincide, as the following result shows; see [6] for a proof.

Theorem 1. *If* $\langle\, A \mid R \,\rangle$ *is a finite group presentation for a group* K *such that* $|R| \geq |A|$*, then there exists a semigroup presentation* $\langle\, B \mid Q \,\rangle$ *which also defines* K *and satisfies* $|Q| - |B| = |R| - |A|$*.*

This leads us to obtain:

Theorem 2. *Let* K *be a finitely presented group. If* $\mathrm{def}_G(K) \geq 0$ *(in particular if* K *is finite) then* $\mathrm{def}_S(K) = \mathrm{def}_G(K)$*. Otherwise if* $\mathrm{def}_G(K) < 0$ *then* $\mathrm{def}_S(K) > \mathrm{def}_G(K)$*.*

This suggests the following open question:

Question : Let K be a finitely presented group defined by the group presentation $\langle\, A \mid R \,\rangle$, with $\mathrm{def}_G(K) \geq 0$. Can we always find a semigroup presentation $\langle\, B \mid T \,\rangle$ for K with $|B| = |A|$ and $\mathrm{def}_G(K) = \mathrm{def}_S(K)$?

Note: We can always find B such that $|B| \leq |A| + 1$ and $\mathrm{def}_G(K) = \mathrm{def}_S(K)$; see [6]. In particular in [6] such a question was raised for the group $PSU(3,3)$. For some progress and interesting comments on this question see [11].

3. Results

It is given in [8] that an efficient group presentation for $PSL(2,p), p$ an odd prime, is:

(1) $$\langle\, x, y \quad \mid \quad x^2 = 1,\ (xy)^3 = 1,\ (xy^4xy^{(p+1)/2})^2 y^p = 1 \,\rangle.$$

It follows from the proof of Theorem 1 that an efficient semigroup presentation for $PSL(2,p)$, p an odd prime is:

$$\begin{aligned}\langle\, x, y \quad \mid \quad & y^3xy^{(p+1)/2}xy^4xy^{(p+1)/2+p}xyx = x,\\ & xy^4xy^{(p+1)/2}xy^4xy^{(p+1)/2+p} = (xy)^3,\\ & xy^4xy^{(p+1)/2}xy^4xy^{(p+1)/2+p}x^2y^5xy^{(p+1)/2}xy^4xy^{(p+1)/2+p}x = y \,\rangle.\end{aligned}$$
(2)

However it has been shown in [2] that, using the above group presentation, a nicer semigroup presentation for $PSL(2,p)$, p an odd prime, is:

(3) $$\langle\, x, y \quad \mid \quad x^3 = x,\ yxyxy = x,\ (xy^4xy^{(p+1)/2})^2 y^{p+1} = y \,\rangle\,.$$

Note that this is an efficient semigroup presentation for $PSL(2,p), p$ an odd prime.

A topic of current interest in computational group theory is the investigation of minimal length presentations for various groups (see, for example, [5]). Short, and indeed minimal length, presentations on minimal generating pairs are considered in that paper. We may ask the same question about short (and, indeed minimal length) (efficient) semigroup presentations for the groups $PSL(2,p)$, p an odd prime.

The length of the efficient semigroup presentation (2) is:

$$2p + 14 + 1 + 2p + 13 + 6 + 4p + 29 + 1 = 8p + 64$$

while the length of (3) is:

$$3 + 1 + 5 + 1 + 2p + 14 + 1 = 2p + 25.$$

Are there other nice semigroup presentations for $PSL(2,p)$, p an odd prime? One such group presentation, for $PSL(2,p)$, p an odd prime, given by Sidki in [15] is:

$$\langle\, x, y \mid x^p = 1, y^p = 1, (xy)^2 = 1, (x^2y^{(p+1)/2})^2 = 1, (x^4y^{1/4})^2 = 1 \,\rangle$$

where $1/4 \equiv (p+1)/4 \bmod p$ or $(3p+1)/4 \bmod p$ as appropriate. From this presentation it is possible to deduce the following efficient group presentation for $PSL(2,p)$, p an odd prime; see [7]:

$$\langle\, x, y \mid x^p = (xy)^2,\ y^p = (x^2y^{(p+1)/2})^2, (x^4y^{1/4})^2 = 1 \,\rangle\ .$$

Theorem 3. *A semigroup presentation $\mathcal{P}$ for $PSL(2,p), p$ an odd prime, is:*

$$\langle\, x, y \mid x^{p+1} = x,\ y^p = x^p,\ yxyxy = y,\ (x^2y^{(p+1)/2})^2 = x^p,\ (x^4y^{1/4})^2 = y^p \,\rangle\ .$$

Proof. We will show that x^p is a left identity and each generator has a left inverse with respect to x^p.

Now x^p acts as a left identity since

$$\begin{aligned} x^p x &= \underline{x^{p+1}} = x, \\ x^p\underline{y} &= \underline{x^p y}xyxy = y\underline{x^{p+1}}yxy = \underline{yxyxy} = y. \end{aligned}$$

Also, it is easy to see that $x^{p-1}x = x^p$ and $y^{p-1}y = \underline{y^p} = x^p$. So each generator has a left inverse.

Thus, by a result from [2], $\mathcal{P}$ defines a group which is isomorphic to that defined by the group presentation

$$\langle\, x, y \quad | \quad x^p = 1, y^p = 1, (xy)^2 = 1, (x^2y^{(p+1)/2})^2 = 1, (x^4y^{1/4})^2 = 1 \,\rangle.$$

□

Above we have discussed the length of a presentation. It should be noted that both the group and semigroup presentations based on Sidki's result, see Theorem 3, are, in general, considerably longer than those given in (1) and (3).

Now from the proof of Theorem 3 we may obtain the following lemma:

Lemma 4. *Let S be a semigroup with generators x, y. If $x^{k+1} = x$, $y^i = x^k$, $uxv = y$ hold in S, where k and i are positive integers, and $u, v \in \{x, y\}^*$, then S defines a group.*

Proof. As in the last proof we will show that x^k is a left identity and that left inverses exist with respect to this left identity. Let $u, v \in \{x, y\}^*$. Then

$$\begin{aligned} x^k x &= \underline{x^{k+1}} = x, \\ \underline{x^k}uxv &= \underline{y^i u}xv = u\underline{y^i}xv = u\underline{x^{k+1}}v = uxv. \end{aligned}$$

Also $x^{k-1}x = x^k$ and $y^{i-1}y = \underline{y^i} = x^k$. So the result holds. □

We also may obtain the following related result.

Lemma 5. *Let S be a semigroup with generators x, y. If $x^i y = y, y^j x = x$ and $ux = vy$, where $i, j > 1$ and $u, v \in \{x, y\}^*$, then S defines a group.*

Proof. The proof is similar to that given above. We show that y^j is a left identity and inverse.

The fact that y^j is a left identity for y follows from noting that $y^j \underline{y} = \underline{y^j x} x^{i-1} y = \underline{x^i} y = y$.

To show that there is a left inverse of x we proceed as follows. First note that $y^j = y^{j-1} y$. Using the relations $x^i y = y$ and $y^j x = x$ we may create the word vy, but this will create a prefix $w \in \{x, y\}^*$, so that we eventually have $y^j = w\underline{vy} = wux$. So the left inverse of x is wu. □

From this we obtain:

Corollary 6.
Let G be a group defined by the group presentation

$$\mathcal{P} \quad = \quad \langle\, x, y \mid x^i = 1, y^j = 1, R \,\rangle,$$

where $|R| \neq 0$. Then we have

$$\text{semigroup length of } G \leq \text{group length of } \mathcal{P} + \triangle + \text{replacements},$$

where replacements are the increase in length obtained by replacing x^{-1} with x^{i-1} and y^{-1} with y^{j-1}.

We have:

Corollary 7. *Let $K = \langle x, y \rangle$ be a group generated by elements of finite order. By $\ell_{2,S}(K)$ we shall mean the minimal semigroup length of K when we consider K to be generated by only two semigroup generators; $\ell_{2,G}(K)$ has the obvious definition. Let $o(x)$ and $o(y)$ be the orders of the generators of K. Then*

$$\ell_{2,S}(K) \quad \leq \quad \ell_{2,G}(K) + o(x) + o(y) + 4 + \text{replacements}$$

The obvious n-generator generalizations also hold.

Although not an efficient presentation, the length of the presentation

$$\langle\, x, y \mid x^{p+1} = x,\ y^p = x^p,\ yxyxy = y,\ (x^2 y^{(p+1)/2})^2 = x^p,\ (x^4 y^{1/4})^2 = y^p \,\rangle\,.$$

is $(13p + 43)/2$ or $(15p + 43)/2$.

A better result is obtained by considering the efficient group presentation for $PSL(2, p)$, p an odd prime, given by Renshaw (see [4] and [16]):

$$\langle\, x, y \mid x^p = (y^4 x^{(p+1)/2})^2 = 1,\ xyx = yxy \,\rangle\,.$$

Here the length of the (efficient) group presentation is $2p + 15$.

Before presenting the result we will need the following lemmas.

Lemma 8. *Let S_p be the semigroup defined by the presentation*

$$\langle\, x, y \quad | \quad x^p y = y, xyx = yxy, (y^4 x^{(p+1)/2})^2 x = x \,\rangle.$$

Then y^p is central in S_p.

Proof. We have

$$\begin{aligned} y^p x &= y^{p-1}\underline{y}x = y^{p-1}x^{p-1}\underline{xyx} = y^{p-1}x^{p-2}\underline{xyx}y \\ &= y^{p-1}x^{p-3}\underline{xyx}y^2 \\ &= \ldots \\ &= y^p x y^p = y^{p-1}\underline{yxy}y^{p-1} = y^{p-2}\underline{yxy}xy^{p-1} = y^{p-3}\underline{yxy}x^2 y^{p-1} \\ &= xy\underline{x^p y}y^{p-2} = xy^p. \end{aligned}$$

Thus the result holds. □

It is interesting to note that the last proof relies only on the relations $x^p y = y$ and $xyx = yxy$.

Lemma 9. *Let S_p be the semigroup defined by the semigroup presentation*

$$\langle\, x, y \quad | \quad x^p y = y, xyx = yxy, (y^4 x^{(p+1)/2})^2 x = x \,\rangle.$$

Then $y = yx^p$ also holds in S_p.

Proof. We can see that

$$\begin{aligned} y &= x^p y = x^{p-1}\underline{x}y = x^{p-1}y^4x^{(p+1)/2}yyy\underline{y}x^{(p+1)/2}xy \\ &= x^{p-1}y^4x^{(p+1)/2}y^3x^{p-1}\underline{xyx}x^{(p+1)/2}y \\ &= x^{p-1}y^4x^{(p+1)/2}y^3x^{p-1}y\underline{xyx}x^{(p+1)/2-1}y \\ &= x^{p-1}y^4x^{(p+1)/2}y^3x^{p-1}yy\underline{xyx}x^{(p+1)/2-2}y \\ &= \ldots \\ &= x^{p-1}y^4x^{(p+1)/2}y^3x^{p-1}y^{(p+1)/2}x\underline{yxy} \\ &= x^{p-1}y^4x^{(p+1)/2}y^3x^{p-1}y^{(p+1)/2-1}\underline{y}xxyx \\ &= x^{p-1}y^4x^{(p+1)/2}y^3x^{p-1}y^{(p+1)/2-1}x^{p-1}\underline{xyx}xyx \\ &= x^{p-1}y^4x^{(p+1)/2}y^3x^{p-1}y^{(p+1)/2-1}x^{p-1}yx\underline{yxy}x \\ &= x^{p-1}y^4x^{(p+1)/2}y^3x^{p-1}y^{(p+1)/2-1}x^{p-2}\underline{xyx}xyx^2 \\ &= x^{p-1}y^4x^{(p+1)/2}y^3x^{p-1}y^{(p+1)/2-1}x^{p-2}yx\underline{yxy}x^2 \\ &= x^{p-1}y^4x^{(p+1)/2}y^3x^{p-1}y^{(p+1)/2-1}x^{p-3}\underline{xyx}xyx^3 \\ &= \ldots \end{aligned}$$

$$\begin{aligned}
&= x^{p-1}y^4x^{(p+1)/2}y^3x^{p-1}y^{(p+1)/2-1}xyxxyx^p \\
&= x^{p-1}y^4x^{(p+1)/2}y^3x^{p-1}y^{(p+1)/2-2}\underline{yxy}xxyx^p \\
&= x^{p-1}y^4x^{(p+1)/2}y^3x^{p-1}y^{(p+1)/2-3}\underline{yxy}xxxyx^p \\
&= \dots \\
&= x^{p-1}y^4x^{(p+1)/2}y^3x^{p-1}xyx^{(p+1)/2+1}yx^p \\
&= x^{p-1}y^4x^{(p+1)/2}y^3x^pyx^{(p+1)/2}xyx^p \\
&= x^{p-1}\underline{y^4x^{(p+1)/2}y^4x^{(p+1)/2+1}}yx^p = \underline{x^py}x^p = yx^p.
\end{aligned}$$

Thus the result holds. □

Now we may obtain the following theorem:

Theorem 10. *A semigroup presentation for $PSL(2,p), p$ an odd prime, is:*

$$\langle\, x, y \mid x^py = y,\ xyx = yxy,\ (y^4x^{(p+1)/2})^2x = x \,\rangle\ .$$

Proof. Firstly we show that x^p is a left identity in the semigroup S_p. The equality $x^py = y$ is one of the defining relations. To show $x^px = x$ we note that:

$$\begin{aligned}
x^p\underline{x} &= \underline{x^py}y^3x^{(p+1)/2}y^4x^{(p+1)/2}x \\
&= \underline{y^4x^{(p+1)/2}y^4x^{(p+1)/2}x} = x.
\end{aligned}$$

So x^p is a left identity of S_p.

We note that this result together with Lemma 9 shows that x^p is also a right identity of S_p. Thus S_p has one unique identity, namely x^p.

Using Lemmas 8 and 9 we show that y^p is a left identity and then we will have the equality $x^p = y^p$ (this follows since $x^p = y^px^p = y^p$).

$$\begin{aligned}
y^py &= y\underline{y}y^{p-1} = yx^p\underline{y^p} = yxy^px^{p-1} \\
&= \underline{yxy}y^{p-1}x^{p-1} = x\underline{yxy}y^{p-2}x^{p-1} \\
&= \dots = x^{p-1}xyxx^{p-1} = x^p\underline{yx^p} \\
&= \underline{x^py} = y.
\end{aligned}$$

We also have

$$\begin{aligned}
y^p\underline{x} &= y^py^4x^{(p+1)/2}y^4x^{(p+1)/2}x = y^4x^{(p+1)/2-1}y^pxy^4x^{(p+1)/2+1} \\
&= y^4x^{(p+1)/2-1}y^{p-1}\underline{yxy}y^3x^{(p+1)/2+1} \\
&= y^4x^{(p+1)/2-1}y^{p-2}\underline{yxy}xy^3x^{(p+1)/2+1} \\
&= y^4x^{(p+1)/2-1}y^{p-3}\underline{yxy}x^2y^3x^{(p+1)/2+1} \\
&= \dots
\end{aligned}$$

$$
\begin{aligned}
&= y^4x^{(p+1)/2-1}\underline{yxy}x^{p-1}y^3x^{(p+1)/2+1} \\
&= y^4x^{(p+1)/2}y\underline{x^p}y^3x^{(p+1)/2+1} \\
&= \underline{x^p}(y^4x^{(p+1)/2})^2x \\
&= \underline{(y^4x^{(p+1)/2})^2x} = x.
\end{aligned}
$$

So y^p is a left identity and thus $y^p = x^p$.

Next we show that left inverses of the generators exist. Obviously, y^{p-1} is the left inverse of y, while $x^{p-1}y^p$ will give the desired result for the generator x as $x^{p-1}\overline{y^p}x = x^py^p = \underline{x^py}y^{p-1} = y^p$ as required.

So S_p defines a group with identity x^p and the relation $x^p = y^p$. Using this information we can see that the presentation defines a group isomorphic to that defined by the group presentation

$$\langle\ x, y \quad | \quad x^p = 1, xyx = yxy, (y^4x^{(p+1)/2})^2 = 1\ \rangle.$$

which is Renshaw's presentation for $PSL(2, p)$, see [4] and [16]. □

Corollary 11 *The length of this (efficient) semigroup presentation is* $2p + 19$.

4. Conclusion

It should be noted that in [5] the minimal length of group presentations on minimal generating sets is considered for the non-abelian simple groups of order $< 10^5$. However the minimal length of presentations for a family of groups is not considered. In this paper we have shown that, for three presentations for the simple groups $PSL(2, p)$, p an odd prime, their semigroup presentations have lengths $8p + 64$ (presentation (2)), $2p + 25$ (presentation (3)) and $2p + 19$ (Theorem 10). These semigroup presentations were arrived at after examining some group presentations for $PSL(2, p)$ whose lengths are $2p + 21$, $2p + 21$ and $2p + 15$ respectively. This compares with a proved best possible length of 12 for a group presentation for $PSL(2, 5)$ given by

$$\langle x, y \mid x^3 = 1, y^5 = 1, (xy)^2 = 1\ \rangle,$$

see [5]. Based on this presentation, we may obtain a semigroup presentation of length 17 given by

$$\langle\ x, y \mid x^3y = y, y^5x = x, yxy = x^2\ \rangle.$$

This compares with a best length of 29 given by the results above. Using a program written in the GAP computational algebra package, see [10], based on ideas from [3], [11] and [12] we have proved that $PSL(2, 5)$ does indeed have a minimal semigroup length of 17.

We have also found that $PSL(2,7)$ has a minimal semigroup presentation

$$\langle\, x, y \mid x^4y = y, y^3x = x, xyxyx = yxy^2xy \,\rangle.$$

of length 22. We are currently using both theoretical and computational techniques in order to find a family of minimal length presentations for $PSL(2,p)$, p an odd prime.

We must note here that sometimes the minimal length semigroup presentation of a group can equal the minimal length group presentation of the group. An example of this is $SL(2,5)$ defined by the minimal length (semigroup or group) presentation

$$\langle x, y \mid x = yx^2y, y = xy^4x \,\rangle,$$

see [12].

Four questions arise for the group $PSL(2,p)$, p an odd prime.

Question 1. What is the minimal length of a group presentation for $PSL(2,p)$, p an odd prime?

Question 2. What is the minimal length of an efficient group presentation for $PSL(2,p)$, p an odd prime?

Question 3. What is the minimal length of a semigroup presentation for $PSL(2,p)$, p an odd prime?

Question 4. What is the minimal length of an efficient semigroup presentation for $PSL(2,p)$, p an odd prime?

Acknowledgments

The authors would like to thank Dr. George Havas and Dr. H. Ayık for some useful discussions.

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