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A FIRST-ORDER LOGIC FOR MULTI-ALGEBRAS

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Abstract. We present a complete first-order proof system for complex algebras of multi-algebras of a fixed signature, which is based on a language whose single primitive relation is singular inclusion, i.e., restricted set inclusion with the domain consisting only of one-element sets. This proof system is then adapted for multi-algebras by relativizing both free and bounded variables in formulas to singletons.

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1. Introduction

In the literature, various more or less elaborated logical systems appropriate for multi-algebras can be found — see [2, 10, 11, 13, 14, 16, 17, 20, 21]; this list of references by no means is complete. Much work in this area is done by computer scientists.

As a rule, multi-algebras which find use in computer science applications (for example, in studies of nondeterministic computational models) are many-sorted; moreover, carrier sets for some sorts may be empty. To simplify technicalities, we deal here only with one-sorted multi-algebras, but we still do not rule out algebras with an empty carrier. In virtue of this latter circumstance, we must reckon with certain specific difficulties when adapting the ordinary first-order logic for multi-algebras. One possible way to take account of empty structures in logic is shifting from classical to the so-called universal, or inclusive, logic (see e.g. [12]); this is the approach of [17]. We shall take here some other way and move from multi-algebras to their complex algebras, which can be treated as ordinary (and non-empty) algebras. The language of multi-algebras admits a reinterpretation making it a language of complex algebras as well, and axiomatizing complex algebras in a first-order logic is not a real problem. The mentioned reinterpretation gives rise to a certain translation (transformation) of formulas in such a way that, in particular, a formula is valid in the class of all multi-algebras (including the empty one) if and only if its translation is valid in the class of all complex algebras and, hence, provable in the logic of complex algebras. (From the viewpoint of complex algebras, the translation of a formula

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is actually its relativization to singletons, or individuals.) Therefore, our logic for multi-algebras is, in fact, a logic of complex algebras used in a non-standard manner.

The standard tool in universal algebra for dealing with the ordinary algebras is equational theories, and atomic formulas of more powerful logic for such algebras are also equations. Of course, an equation is treated there as expressing identity of the values of the terms involved in it. When moving to partial algebras, this concept of an equation should be weakened or generalized somehow; see, e.g., Sect. 3.2 of [1] for an overview of some kinds of partial equations actually used by various authors.

Equations as identities have also been studied in multi-algebras and complex algebras [5, 7, 18]; then, however, the case in point is equality of sets rather than identity of individuals. It was demonstrated in [14] that expressive power of the standard first-order logic with equality is insufficient for treating some special questions concerning multi-algebras. The weak commutative and weak distributive laws for certain group-like and ring-like multi-algebras (see e.g. [19]) may be written as equations if an equation formed by two terms is interpreted as a "weak equality", asserting that their value sets overlap. Some authors have exploited also inclusion as the primitive relation rather than any kind of equality [2, 10, 13, 14, 16]. In [11, 20, 21], inclusion is complemented with identity of individuals (i.e. elements of the carrier of the multi-algebra under consideration). This kind of identity is definable in terms of inclusion in full first-order logic; on the contrary, it actually increases the expressive power of the quantifier-free language used by these authors.

Another non-traditional relation—singular inclusion, which holds between two sets if and only if the first of these is a singleton which happens to be a subset of the second one—was tested as the single primitive relation in the language for multi-algebras in [3, 4], for structures with relations and multi-operations, in [17], and for structures that can be regarded as power structures of these, in Part III of [16]. Identity of individuals, as well as equality, overlapping and inclusion of sets, are definable in terms of singular inclusion in first-order logic; one more advantage of this relation consists in its conceptual similarity to the membership relation.

In this paper, we choose singular inclusion, or "epsilon-relation" to be the primitive relation in the first-order language for multi-algebras. Remarkably, "epsilon-equations" reduce to the so called existence equations [1] in partial algebras and to the traditional identity equations in ordinary algebras. Using singular inclusion in logic comes back to S. Leśniewski's Ontology—see [15].

2. Preliminaries: multi-algebras and their ε-language

Through the paper, we keep fixed a set Ω of operation symbols (possibly, 0-ary). Where U is a set, we write $\mathcal{P}U$ for the power set of U.

Definition 1. An m-ary *multi-operation* on a set U is any mapping of type $U^m \to \mathcal{P}U$. A *multi-algebra* of the signature Ω is a system $\boldsymbol{U} := (U, \omega_{\boldsymbol{U}})_{\omega\in\Omega}$, where U is a set, the *carrier* of U, and each $\omega_{\boldsymbol{U}}$ is a multi-operation on U whose arity is determined by ω.

There will not be a need in this paper to distinguish between a singleton $\{u\}$ and its single element u; therefore, we may consider U as a subset of $\mathcal{P}U$, and it then makes sense to regard the membership relation $\in$ as a relation *on* $\mathcal{P}U$ rather than *between* U *and* $\mathcal{P}U$. In particular, then every ordinary Ω-algebra on U can also be treated as a multi-algebra. Likewise, a partial algebra is then just a multi-algebra in which operations are of type $U^m \to U \cup \{\emptyset\}$.

Now let X be a denumerable set of variables, and let $\boldsymbol{T} := (T, \omega_{\boldsymbol{T}})_{\omega\in\Omega}$ be the Ω-algebra of terms freely generated by X. A term t is *linear in* x if the variable x has just one occurrence in t. The result of substitution of a term s for x in t is denoted by $[s/x]t$. In a multi-algebra $\boldsymbol{U}$, every assignment φ: $X \to U$ can be extended to a mapping $\tilde{\varphi}$: $T \to \mathcal{P}U$ as follows:

- if t is a variable x, then $u \in \tilde{\varphi}(t)$ iff $u = \varphi(x)$,
- if t is a compound term of the form $\omega t_1 t_2 \dots t_m$, then $u \in \tilde{\varphi}(t)$ iff $u \in \omega_{\boldsymbol{U}}(v_1, v_2, \dots, v_m)$ for some $v_1 \in \tilde{\varphi}(t_1), v_2 \in \tilde{\varphi}(t_2), \dots, v_m \in \tilde{\varphi}(t_m)$.

Let, furthermore, $\mathcal{L}^\varepsilon$ be the first order language over T with a single binary relation symbol ε. So atomic formulas of $\mathcal{L}^\varepsilon$ are of the form $s \mathrel{\varepsilon} t$, where s and t are terms. Such a formula is supposed to be *satisfied* in a multi-algebra $\boldsymbol{U}$ by an assignment φ if and only if $\tilde{\varphi}(s) \in \tilde{\varphi}(t)$. For compound formulas, satisfaction is defined in the usual way.

A multi-algebra $\boldsymbol{U}$ is said to be a *multi-algebra model* or, in short, an *m-model* of a formula f if f is valid in $\boldsymbol{U}$, i.e. is satisfied by all assignments in U. In particular, in an empty algebra, where there are no assignments at all, every open formula as well as the universal closure of a formula counts as valid, while the existential closure of a formula is invalid. Given a set of formulas Γ and a formula f, we write $\Gamma \models_m f$ to mean that f is valid in all m-models of Γ.

Proposition 1. (a) *The following formulas are valid in any multi-algebra for all terms* $r, s, t, t_1, \dots, t_m \in T$*:*

(ε1): $s \mathrel{\varepsilon} t \to s \mathrel{\varepsilon} s$,

(ε2): $s \mathrel{\varepsilon} t \wedge t \mathrel{\varepsilon} r \to s \mathrel{\varepsilon} r$,

(ε3): $s \mathrel{\varepsilon} t \wedge t \mathrel{\varepsilon} r \to t \mathrel{\varepsilon} s$,

(ε4): $\forall z(z \mathrel{\varepsilon} s \leftrightarrow z \mathrel{\varepsilon} t) \wedge s \mathrel{\varepsilon} r \to t \mathrel{\varepsilon} r$,
where z *does not occur in* s *and* t*,*

(ε5): $s \mathrel{\varepsilon} \omega t_1 \dots t_m \leftrightarrow \exists x_1 \dots x_m (x_1 \mathrel{\varepsilon} t_1 \wedge \dots \wedge x_m \mathrel{\varepsilon} t_m \wedge s \mathrel{\varepsilon} \omega x_1 \dots x_m)$,
where the variables x_i *are all distinct and occur neither in* s *nor any of the terms* t_j*.*

(b) *A multi-algebra is nonempty, a partial algebra, or a total algebra (with univalent operations) if and only if the following formulas are respectively valid in it for all terms* s, t*:*

(ε7): $\exists x\, x \varepsilon x$,

(ε8): $s \varepsilon t \to t \varepsilon s$,

(ε9): $t \varepsilon t$.

On the other hand, the well-known axioms $\forall x f \to [t/x]f$ and $[t/x]f \to \exists x f$ of standard first-order logic need not be valid even in a non-empty multi-algebra. In particular, (ε9) does not imply (ε7), and, say, (ε1) is not a consequence of its particular case $x \varepsilon y \to y \varepsilon x$ in the field of multi-algebras. Furthermore, $\forall x\, x \varepsilon x \to \exists x\ \varepsilon x$ is an example of a theorem of standard first-order logic which is not valid in an empty multi-algebra.

Nevertheless, the formulas listed in (a) turn out to be characteristic, in a sense, for the epsilon-relation. The reader may consult [12] for the general concept of a first-order theory.

Definition 2. An *epsilon-theory* is any first-order theory in $\mathcal{L}^\varepsilon$ having all formulas (ε1)–(ε5) among its theorems. The least such a theory is called *epsilon-logic*.

Our aim in this paper is to axiomatize the consequence relation $\models_m$ in epsilon-logic. The above observations show that this cannot be done by means of standard first-order logic directly. However, the objective can be achieved in a roundabout way.

Let us associate with every formula f another formula $\mathrm{at}(f)$ as follows. First, we inductively define an auxiliary transformation * on the set of all formulas. It suffices to consider only connectives $\neg, \to$ and the quantifier $\forall$:

$$\begin{aligned} {}^*(s \varepsilon t) &:= s \varepsilon t, \\ {}^*(\neg f) &:= \neg({}^*f), \\ {}^*(f \to g) &:= {}^*f \to {}^*g, \\ {}^*(\forall x\, f) &:= \forall x\, (x \varepsilon x \to {}^*f). \end{aligned}$$

Next, $\mathrm{at}(f)$ is set to be the formula $x_1 \varepsilon x_1 \wedge \cdots \wedge x_n \varepsilon x_n \to {}^*f$, where $x_1, \ldots, x_n$ is the list of all free variables of f in the alphabetical order (or ${}^*f$ itself, if f is closed). Likewise, $\mathrm{at}(\Gamma)$ stands for the set of all at-transforms of members of Γ.

Note that the formula $x \varepsilon x$ is valid in all multi-algebras, even in the empty one. The formula $f \leftrightarrow \mathrm{at}(f)$ also is valid in all multi-algebras for every f. Motivation of the transformation at() will be clarified in the next section.

The following theorem, which is the main result of the paper, is a consequence of the soundness and completeness theorem for epsilon-logic proved in Sect. 5.

Theorem 2. *A formula f is valid in all ma-models of a set of formulas Γ if and only if* $\mathrm{at}(f)$ *is derivable from* $\mathrm{at}(\Gamma)$ *in epsilon-logic.*

3. From multi-algebras to complex algebras

The term 'complex algebra' has diverse meanings in the literature. In [9, 7], it meant what is called below a full complex algebra—a certain algebra whose carrier is a powerset of some set. Some authors have used it for algebras consisting only of the non-void subsets of a set; see, e.g., [5]. We follow here the extended use of this term in [6].

Definition 3. An m-ary operation o on $\mathcal{P}U$ is said to be *cumulative* if

> $u \in o(A_1, \ldots, A_m)$ if and only if $u \in o(u_1, \ldots, u_m)$ for some $u_1 \in A_1, \ldots, u_m \in A_m$.

A *complex algebra on U* is an Ω-algebra on a nonempty subset of $\mathcal{P}U$ including U with all operations cumulative. A complex algebra is said to be *full* if this subset coincides with $\mathcal{P}U$.

Thus, an ordinary algebra, i.e. algebra with total univalent operations, is a complex algebra in this sense. On the other hand, any complex algebra is a subalgebra of a full complex algebra.

Note that an operation on $\mathcal{P}U$ is cumulative only if it is additive in each argument and normal (i.e. satisfies the condition $o(\emptyset, \ldots, \emptyset) = \emptyset$). Hence, every complex algebra on U whose carrier happens to be a Boolean subalgebra of $\mathcal{P}U$ is essentially a Boolean algebra with operators [9].

Definition 4. Let $\boldsymbol{U}$ be a multi-algebra. A *complex algebra of* $\boldsymbol{U}$ is any complex algebra $\boldsymbol{V}$ on U such that, for all $\omega \in \Omega$, $\omega_{\boldsymbol{U}}$ is the restriction of $\omega_{\boldsymbol{V}}$. The smallest complex algebra of $\boldsymbol{U}$ is said to be the *completion* of $\boldsymbol{U}$.

A *valuation* (of terms) in a complex algebra $\boldsymbol{V}$ is a homomorphism from $\boldsymbol{T}$ to $\boldsymbol{V}$. For example, if $\boldsymbol{U}$ is a multi-algebra, then the extension $\tilde{\varphi}$ of any assignment φ in U is a valuation in the completion of $\boldsymbol{U}$. Again, an atomic formula $s \; \varepsilon \; t$ of $\mathcal{L}^\varepsilon$ is *satisfied* by a valuation μ in a complex algebra $\boldsymbol{V}$ if $\mu(s) \in \mu(t)$. A formula f of $\mathcal{L}^\varepsilon$ is valid in $\boldsymbol{V}$ if it is satisfied by all valuations in $\boldsymbol{V}$. If it is the case, $\boldsymbol{V}$ is said to be a *complex algebra model* or, in short, a *ca-model* of f.

The following proposition is a close analogue of Proposition 1.

Proposition 3. *All formulas* $(\varepsilon 1)$–$(\varepsilon 5)$ *are valid in any complex algebra. A complex algebra is a ca-model of any of formulas* $(\varepsilon 6)$–$(\varepsilon 8)$ *if and only if the corresponding multi-algebra is an m-model of it.*

In general, the connection between validation in complex algebras and in multi-algebras is more involved.

Proposition 4. *Suppose that $\boldsymbol{U}$ is some multi-algebra. The following conditions on a formula f are equivalent:*
(a) *f is valid in $\boldsymbol{U}$,*
(b) $\mathrm{at}(f)$ *is valid in some complex algebra of $\boldsymbol{U}$,*
(c) $\mathrm{at}(f)$ *is valid in all complex algebras of $\boldsymbol{U}$.*

It is this result that motivates our interest in complex algebras, as well as in the transformation at() described in the previous section.

4. From complex algebras to epsilon-algebras

A complex algebra $\boldsymbol{V} := (V, \omega_{\boldsymbol{V}})_{\omega\in\Omega}$ may be considered as a structure of kind $(\boldsymbol{V}, \in)$ with the membership relation on V explicitly pointed out. It is profitable to consider also abstract structures of this type.

Definition 5. An *epsilon-algebra* is a structure of type $(\boldsymbol{A}, \epsilon)$, where $\boldsymbol{A}$ is an Ω-algebra and ϵ is a binary relation on A. It is said to be a *complex epsilon-algebra* if $\boldsymbol{A}$ is a complex algebra and ϵ is the membership relation on A.

Actually, just epsilon-algebras is the natural kind of structures for the language $\mathcal{L}^{\varepsilon}$, validation of formulas in them being defined in the standard way. It is easily seen that a complex epsilon-algebra $(\boldsymbol{V}, \in)$ is a model of some formula f in the standard sense if and only if $\boldsymbol{V}$ is a ca-model of f.

In the rest of the section, we shall characterize the class of all isomorphic copies of complex epsilon-algebras. An isomorphism between epsilon-algebras $(\boldsymbol{A}, \epsilon)$ and $(\boldsymbol{A}', \epsilon')$ is defined to be an isomorphism $i\colon\ \boldsymbol{A} \to \boldsymbol{A}'$ such that $a \,\epsilon\, b$ in $\boldsymbol{A}$ if and only if $ia \,\epsilon'\, ib$ in $\boldsymbol{A}'$.

Definition 6. An epsilon-algebra $(\boldsymbol{A}, \epsilon)$ is said to be *proper* if the relation ϵ satisfies the following conditions:

weak reflexivity: if $a \,\epsilon\, b$, then $a \,\epsilon\, a$,

transitivity: if $a \,\epsilon\, b$ and $b \,\epsilon\, c$, then $a \,\epsilon\, c$,

weak symmetry: if $a \,\epsilon\, b$ and $b \,\epsilon\, c$, then $b \,\epsilon\, a$,

and each operation of $\boldsymbol{A}$ is *cumulative w.r.t.* ϵ:

$c \,\epsilon\, \omega_{\boldsymbol{A}}(a_1, a_2, \ldots, a_m)$ if and only if $c \,\epsilon\, \omega_{\boldsymbol{A}}(b_1, b_2, \cdots, b_m)$ for some $b_1, b_2, \ldots, b_m$ such that $b_1 \,\epsilon\, a_1, b_2 \,\epsilon\, a_2, \ldots, b_m \,\epsilon\, a_m$.

It follows from Proposition 3 that every complex epsilon-algebra is proper. It is also extensional in the sense of the next definition.

Definition 7. In an epsilon-algebra $(\boldsymbol{A}, \epsilon)$, an element a is said to be an *atom of* b if $a \,\epsilon\, b$, and an *atom* if it is an atom of some element of A. The epsilon-algebra is said to be *extensional* if two elements of A are equal whenever they have equal sets of atoms.

Theorem 5. *An epsilon-algebra is isomorphic to a complex algebra if and only if it is proper and extensional.*

Proof. We already noticed that a complex algebra is always proper and extensional. On the other hand, given a proper and extensional epsilon-algebra $(\boldsymbol{A}, \epsilon)$, we couple a complex algebra with it as follows. Let $V := \{atm(a)\colon\ a \in A\}$, where $atm(a)$ stands for the set of atoms of a:

$$b \in atm(a) \text{ if and only if } b \mathrel{\epsilon} a.$$

If ω is an m-ary operation symbol from Ω, let o_ω be the m-ary operation on V defined by

$$o_\omega(atm(a_1), atm(a_2), \ldots, atm(a_m)) := atm(\omega_{\boldsymbol{A}}(a_1, a_2, \ldots, a_m).$$

Then $\boldsymbol{V} := (V, o_\omega)_{\omega \in \Omega}$ is a complex algebra on the set of atoms of $\boldsymbol{A}$. Moreover, the mapping $a \mapsto atm(a)$ is an isomorphism from $(\boldsymbol{A}, \epsilon)$ to $(\boldsymbol{V}, \in)$. The proof of these assertions consists of several steps.

Claim C1: If $b \mathrel{\epsilon} a$, then $atm(b) = \{b\} = b$.

Suppose that, indeed, $b \mathrel{\epsilon} a$ for some a. By weak reflexivity of ϵ, then $b \in atm(b)$. If $c \in atm(b)$ too, then $b \mathrel{\epsilon} c$ (by weak symmetry) and, furthermore, $d \mathrel{\epsilon} c$ iff $d \mathrel{\epsilon} b$ (by transitivity). Now, $c = b$ by extensionality.

Thus, all atoms of $\boldsymbol{A}$ belong to V, and $\boldsymbol{V}$ is indeed a complex algebra on the set of atoms on $\boldsymbol{A}$.

Claim C2: If $atm(b)$ is a singleton, then $atm(b) = b$.

Suppose that $atm(b) = c$. Then $d \mathrel{\epsilon} c$ implies $d \mathrel{\epsilon} b$ (by transitivity), while $d \mathrel{\epsilon} b$ implies $d = c$ and (by weak reflexivity) $d \mathrel{\varepsilon} d$, and, furthermore $d \mathrel{\varepsilon} c$. Now $c = b$ by extensionality.

Claim C3: The mapping $atm\colon A \to V$ is bijective.

It is surjective by construction of V, and injective by extensionality of $(\boldsymbol{A}, \epsilon)$.

Claim C4: $b \mathrel{\epsilon} a$ if and only if $atm(b) \in atm(a)$.

If $b \mathrel{\epsilon} a$, then $atm(b) = b$ by C1. Conversely, if $atm(b) \in atm(a)$, then $atm(b)$ is a singleton and, by C2, again $atm(b) = b$.

It follows from C3, C4 and the definition of o_ω that atm is an isomorphism. □

5. Epsilon-logic as the logic of complex algebras

We begin with a simplification of the definition of an epsilon-theory. It is worthwhile to note that the first three formulas in Proposition 6, which are particular cases of $(\varepsilon 1)$–$(\varepsilon 3)$ respectively, are the axioms the quantifier-free fragment of Leśniewski's Ontology studied in [8].

Proposition 6. *A first-order theory is an epsilon-theory if and only if the following formulas are theorems for all $t \in T$ (all variables in each formula are supposed to be distinct):*

$x \mathrel{\varepsilon} y \to x \mathrel{\varepsilon} x$,

$x \mathrel{\varepsilon} y \wedge y \mathrel{\varepsilon} z \to x \mathrel{\varepsilon} z$,

$x \mathrel{\varepsilon} y \wedge y \mathrel{\varepsilon} z \to y \mathrel{\varepsilon} x$,

$\forall u\,(u \mathrel{\varepsilon} x \leftrightarrow u \mathrel{\varepsilon} y) \wedge x \mathrel{\varepsilon} z \to y \mathrel{\varepsilon} z$,

$s \mathrel{\varepsilon} \omega t_1 \ldots t_m \leftrightarrow \exists x(x \mathrel{\varepsilon} t_i \wedge s \mathrel{\varepsilon} \omega t_1 \ldots x \ldots t_m)$,
where x *occurs neither in* s *nor any of the terms* t_j.

The next theorem explains the title of this section.

Theorem 7. *Let* $\mathcal{T}$ *be any epsilon-theory. A formula is a theorem of* $\mathcal{T}$ *if and only if it is valid in all ca-models of* $\mathcal{T}$.

Proof. Necessity of the conditions follows from Proposition 3. To prove its sufficiency, assume that a formula f is valid in all ca-models of $\mathcal{T}$. If it is not a theorem, then the theory obtained from $\mathcal{T}$ by adding the negation of f as a new axiom is consistent. It is well-known that every consistent first order theory has a model (see, e.g. [12]). In our case, the model of $\mathcal{T} + \neg f$ is a proper epsilon-algebra $(\boldsymbol{A}, \epsilon)$. Due to ($\varepsilon$5), the relation $\asymp$ defined on A by

$$a \asymp b \text{ if and only if } atm(a) = atm(b)$$

is easily seen to be a congruence of $\boldsymbol{A}$; let $\boldsymbol{A}'$ be the corresponding quotient algebra $\boldsymbol{A}/\!\asymp$. By induction on terms, $[\mu(t)] = \mu'(t)$, where μ' is the valuation in $\boldsymbol{A}'$ such that $\mu'(x) = [\mu(x)]$ for all variables x. (Every valuation in $\boldsymbol{A}'$ arises this way.) Define the relation ϵ' on A' as follows:

$$[a] \mathrel{\epsilon'} [b] \text{ if and only if } a \mathrel{\epsilon} b;$$

this definition is correct, for (ε4) is also valid in $(\boldsymbol{A}, \epsilon)$. By induction on formulas, every formula of $\mathcal{L}^\varepsilon$ is satisfied in $(\boldsymbol{A}, \epsilon)$ by a valuation μ if and only it is satisfied in $(\boldsymbol{A}', \epsilon')$ by μ'. Thus, any formula is valid in $(\boldsymbol{A}', \epsilon')$ if and only it is valid in $(\boldsymbol{A}, \epsilon)$. Henceforth, $(\boldsymbol{A}', \epsilon')$ is a proper epsilon-algebra which is extensional and, like $(\boldsymbol{A}, \epsilon)$ is a model of $\mathcal{T} + \neg f$. By Theorem 5, there is a complex epsilon-algebra $(\boldsymbol{V}, \in)$ isomorphic to $(\boldsymbol{A}', \epsilon')$; $\boldsymbol{V}$ is then a *ca*-model of $\mathcal{T} + \neg\{$. Formula f itself cannot be valid in this model—a contradiction. □

Theorem 2, a peculiar adequacy theorem for epsilon-logic with respect to multi-algebras, now follows in virtue of Proposition 4.

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