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WEAKER UNIVERSALITIES IN SEMIGROUP VARIETIES[1]

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Abstract. A variety $\mathbb{V}$ has an alg-universal n-expansion if the addition of n nullary operations to algebras from $\mathbb{V}$ produces an alg-universal category. It is proved that any semigroup variety $\mathbb{V}$ containing a semigroup that is neither an inflation of a completely simple semigroup nor an inflation of a semilattice of groups has an alg-universal 3-expansion. We say that a variety $\mathbb{V}$ is var-relatively alg-universal if for some proper subvariety $\mathbb{W}$ of $\mathbb{V}$ there is a faithful functor F from the category of all digraphs and compatible mappings into $\mathbb{V}$ such that $\operatorname{Im}(Ff)$ belongs to $\mathbb{W}$ for no compatible mapping f and if $f : F\mathbf{G} \to F\mathbf{G}'$ is a homomorphism where $\mathbf{G}$ and $\mathbf{G}'$ are digraphs then either $\operatorname{Im}(f)$ belongs to $\mathbb{W}$ or $f = Fg$ for a compatible mapping $g : \mathbf{G} \to \mathbf{G}'$. For a cardinal $\alpha \geq 2$, a variety $\mathbb{V}$ is α-determined if any set $\mathcal{A}$ of $\mathbb{V}$-algebras of cardinality α such that the endomorphism monoids of A and B are isomorphic for all $A, B \in \mathcal{A}$ contains distinct isomorphic algebras. Similar sufficient conditions for a semigroup variety $\mathbb{V}$ to be α-determined for no cardinal α or var-relatively alg-universal are given. These results are proved by an analysis of three specific semigroup varieties.

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1 Introduction

For an object A of a category $\mathbb{K}$, let $\operatorname{End}(A)$ denote the endomorphism monoid of A, and let $\operatorname{Mon}(\mathbb{K})$ be the class of all monoids $\mathbf{M}$ isomorphic to $\operatorname{End}(A)$ for some $\mathbb{K}$-object A. A $\mathbb{K}$-object A is called rigid if $\operatorname{End}(A)$ consists of the identity endomorphism alone. We say that a category $\mathbb{K}$ is monoid-universal

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if Mon($\mathbb{K}$) consists of all monoids. The class Mon($\mathbb{K}$) describes a representative power of the 'structure' of $\mathbb{K}$-objects. We can naturally generalize these notions. Let Cat($\mathbb{K}$) denote the family of all categories $\mathbb{L}$ such that there exists a full embedding $\Phi : \mathbb{L} \to \mathbb{K}$. Observe that if $\mathbb{K}$ is a concrete category then any category in Cat($\mathbb{K}$) is concrete. We restrict ourselves to the investigation of concrete categories and thus we say that a concrete category $\mathbb{K}$ is universal if Cat($\mathbb{K}$) consists of all concrete categories and $\mathbb{K}$ is algebraically universal (or alg-universal) if any category of algebras (of a given similarity type) and all their homomorphisms belongs to Cat($\mathbb{K}$). Clearly, Cat($\mathbb{K}$) describes a representative power of $\mathbb{K}$ more fully than Mon($\mathbb{K}$). The folklore observation below shows the key feature of alg-universality, the 'localized' selection of morphisms common to all categories of algebras. Specifically, a concrete category $\mathbb{L}$ belongs to Cat($\mathbb{K}$) for any alg-universal category $\mathbb{K}$ whenever

> there exists a cardinal α such that for any mapping $f : |A| \to |B|$ of the underlying sets $|A|$ and $|B|$ of any $\mathbb{L}$-objects A and B such that for every subset $Z \subseteq |A|$ of cardinality α there is an $\mathbb{L}$-morphism $g : A \to B$ such that f and the underlying mapping of g coincide on Z, there exists an $\mathbb{L}$-morphism $h : A \to B$ whose underlying mapping is f.

Examples and basic properties of universal, alg-universal and monoid-universal categories can be found in the monograph by A. Pultr and V. Trnková [31]. We recall several facts about the relations between these notions.

Theorem 1.1 (Hedrlín–Kučera Theorem) *[31] Any alg-universal category is universal if and only if the class of all measurable cardinals is a set.* □

This result was strengthened by L. Kučera and A. Pultr in [28] and [31], where they proved that the existence of a full embedding of the category dual to $\mathbf{A}(\Delta)$ into the category $\mathbf{A}(\Delta')$ implies that the class of all measurable cardinals is a set (we recall that $\mathbf{A}(\Delta)$ is the variety of all algebras of a given similarity type Δ).

Theorem 1.2 (Hedrlín–Sichler Theorem) *[19] and [31] If $\mathbb{K}$ is an alg-universal category then for every monoid $\mathbf{M}$ and every cardinal α there exists a set $\mathcal{C}$ of $\mathbb{K}$-objects of cardinality α such that End(A) is isomorphic to $\mathbf{M}$ for all $A \in \mathcal{C}$ and there exists no $\mathbb{K}$-morphism between any two distinct $A, B \in \mathcal{C}$.*

If $\mathbb{K}$ is universal then for every monoid $\mathbf{M}$ there exists a proper class $\mathcal{S}$ of $\mathbb{K}$ objects such that End(A) is isomorphic to $\mathbf{M}$ for all $A \in \mathcal{C}$ and there exists no $\mathbb{K}$-morphism between any two distinct $A, B \in \mathcal{C}$.

If $\mathbb{K}$ is alg-universal then for every monoid $\mathbf{M}$ there exists a proper class $\mathcal{C}$ of non-isomorphic $\mathbb{K}$-objects such that End(A) is isomorphic to $\mathbf{M}$ for all $A \in \mathcal{C}$. □

Thus any alg-universal category is monoid universal. The next theorem by J. Rosický resolves the reverse implication.

Theorem 1.3 *[32] There exists a concrete, complete, cocomplete, well-powered and co-well-powered monoid-universal category that is not alg-universal.* □

The category of Theorem 1.3 is artificially constructed, and it is an open question whether there exists some natural monoid-universal category that is not alg-universal. Thus it is not known whether there exists a monoid-universal variety (or quasivariety) that is not alg-universal.

Alg-universality and universality of a concrete category are independent of its forgetful functor. Next we connect these notions. Let $\mathbf{A}(1,1)$ be the variety of all unary algebras with two unary operations endowed by its natural forgetful functor $U : \mathbf{A}(1,1) \to \mathbb{SET}$. We say that a concrete category $\mathbb{K}$ (with an underlying functor $V : \mathbb{K} \to \mathbb{SET}$) is ff-alg-universal if there exists a full embedding $\Phi : \mathbf{A}(1,1) \to \mathbb{K}$ such that F sends any finite algebra of $\mathbf{A}(1,1)$ to a $\mathbb{K}$-object whose underlying set is finite. If there exists a set functor G with $U \circ F = G \circ V$, we say that $\mathbb{K}$ is strongly alg-universal.

Adams–Dziobiak theorem below demonstrates the importance of these notions. According to Sapir [33], a quasivariety $\mathbb{Q}$ of algebras of finite type is called Q-universal if for any quasivariety $\mathbb{V}$ of algebras of a finite type, the inclusion-ordered lattice $L(\mathbb{V})$ of all subquasivarieties of $\mathbb{V}$ is a quotient of a sublattice of the inclusion-ordered lattice $L(\mathbb{Q})$ of all subquasivarieties of $\mathbb{Q}$.

Theorem 1.4 (Adams–Dziobiak Theorem) *[3] Any ff-alg-universal quasivariety of finite type is Q-universal.* □

M. E. Adams and W. Dziobiak asked whether the assumptions of their theorem can be weakened. Some restrictions on its possible generalizations were shown by J. Sichler and the second author of this paper in [27]. The basic characterization of ff-alg-universality was given in [24].

This paper investigates the representative power of semigroup structure in the dependence on semigroup varieties. The initial result is due to Z. Hedrlín and J. Lambek, who proved that the variety $\mathbb{SEM}$ of all semigroups is alg-universal [18]. This result was generalized by J. Sichler and the second author of this paper as follows.

Theorem 1.5 *[23] and [26] For any semigroup variety $\mathbb{V}$ the following conditions are equivalent:*

1. *$\mathbb{V}$ is monoid-universal;*
2. *$\mathbb{V}$ is alg-universal;*
3. *$\mathbb{V}$ is strongly alg-universal;*
4. *there exists an infinite rigid semigroup $S \in \mathbb{V}$;*
5. *for some prime p there exists a semigroup $S \in \mathbb{V}$ such that $End(S)$ is the cyclic group of order p;*
6. *$\mathbb{V}$ contains all commutative semigroups and for any $n > 1$ the identity $(xy)^n = x^n y^n$ fails in $\mathbb{V}$.* □

If the condition (6) fails then it is easy to see that $\mathbb{V}$ is not alg-universal. Indeed, if $\mathbb{V}$ does not contain all commutative semigroups then any $S \in \mathbb{V}$ contains an idempotent element $s \in S$ and a constant mapping with the value s is an endomorphism of S. If $\mathbb{V}$ satisfies the identity $(xy)^n = x^n y^n$ for some $n > 1$ then the mapping $x \mapsto x^n$ is an endomorphism of any $S \in \mathbb{V}$. Thus if a semigroup variety $\mathbb{V}$ fails the condition (6), then there exist only finitely many non-isomorphic rigid semigroups in $\mathbb{V}$ and, by Theorem 1.2, $\mathbb{V}$ is not alg-universal. The aim of this paper is to study the representative power of semigroup varieties if we avoid these 'trivial homomorphisms' by means of disregarding them or eliminating them by additional operations. We show that from either point of view, for many semigroup varieties the existence of these 'trivial homomorphisms' is the only reason why they are not alg-universal. Only for the semigroup varieties near the bottom of the inclusion-ordered lattice of semigroup varieties does the semigroup structure substantially restrict homomorphisms, these varieties are far away from alg-universal varieties. Next we formalize these ideas.

Two objects A and B of a category $\mathbb{K}$ are <u>equimorphic</u> if $\mathrm{End}(A)$ and $\mathrm{End}(B)$ are isomorphic. For a cardinal $\alpha \geq 2$, we say that a category $\mathbb{K}$ is α-<u>determined</u> if any set of equimorphic $\mathbb{K}$-objects of a cardinality α contains at least two isomorphic objects. By the Hedrlín-Sichler Theorem, no alg-universal category is α-determined for any cardinal α and, indeed, we can say that any α-determined category is far away from any alg-universal category. This motivates an effort to find examples of α-determined categories and to characterize α-determined categories for some cardinal α. The review paper [1] summarizes results concerning determinacy, while the theory of determined categories was developed in [22]. B. M. Schein ([34] and [35]) has initialized the investigation of α-determinacy for band varieties. The results about α-determinacy for band varieties are recalled in the last section.

Next we formalize a way of disregarding 'trivial morphisms'. For a category $\mathbb{K}$, a class $\mathcal{Z}$ of $\mathbb{K}$-morphisms is called an <u>ideal</u> if for $\mathbb{K}$-morphisms $f : A \to B$ and $g : B \to C$ we have $g \circ f \in \mathcal{Z}$ whenever $f \in \mathcal{Z}$ or $g \in \mathcal{Z}$. If $\mathcal{Z}$ is an ideal of $\mathbb{K}$ then we say that a functor $\Phi : \mathbb{L} \to \mathbb{K}$ is a $\mathcal{Z}$-<u>full</u> <u>embedding</u> whenever

> Φ is faithful and $\Phi f \notin \mathcal{Z}$ for every $\mathbb{L}$-morphism f;
> if $f : \Phi A \to \Phi B$ is a $\mathbb{K}$-morphism for $\mathbb{L}$-objects A and B then either $f \in \mathcal{Z}$ or $f = \Phi g$ for an $\mathbb{L}$-morphism $g : A \to B$.

Thus Φ is a full embedding exactly when Φ is a $\mathcal{Z}$-full embedding for the empty ideal $\mathcal{Z} = \emptyset$. We say that a category $\mathbb{K}$ is $\mathcal{Z}$-<u>relatively</u> <u>alg</u>-<u>universal</u> for an ideal $\mathcal{Z}$ if there exists a $\mathcal{Z}$-full embedding Φ from an alg-universal category into $\mathbb{K}$. If Φ also preserves finite objects then $\mathbb{K}$ is $\mathcal{Z}$-<u>relatively</u> ff-<u>alg</u>-<u>universal</u>. To apply these notions to algebras we describe a way to determine ideals. For a class $\mathcal{C}$ of $\mathbb{K}$-objects, let $\mathcal{Z}_{\mathcal{C}}$ denote the class of all $\mathbb{K}$-morphisms $f : A \to B$ such that there exist $C \in \mathcal{C}$ and $\mathbb{K}$-morphisms $g : A \to C$ and $h : C \to B$ with $f = h \circ g$. It is easy to verify that $\mathcal{Z}_{\mathcal{C}}$ is an ideal in $\mathbb{K}$. If $\mathbb{W}$ is a proper subvariety of a variety $\mathbb{V}$ such that $\mathbb{V}$ is $\mathcal{Z}_{\mathbb{W}}$-relatively alg-universal (or $\mathcal{Z}_{\mathbb{W}}$-relatively ff-alg-universal) then we say that $\mathbb{V}$ is $\mathbb{W}$-relatively alg-universal (or $\mathbb{W}$-relatively ff-alg-universal). We say that a variety $\mathbb{V}$ is var-relatively universal (or var-relatively ff-universal) if $\mathbb{V}$ is $\mathbb{W}$-relatively alg-universal (or $\mathbb{W}$-relatively ff-alg-universal) for some proper subvariety $\mathbb{W}$ of $\mathbb{V}$, and $\mathbb{V}$ is weakly var-relatively universal if $\mathbb{V}$ is $\mathcal{Z}_{\mathcal{A}}$-relatively alg-universal where $\mathcal{A}$ is the union of all proper subvarieties of $\mathbb{V}$. Speaking informally, if $\mathbb{V}$ is a var-relatively universal variety then for some proper subvariety $\mathbb{W}$ of $\mathbb{V}$, the homomorphisms between algebras from $\mathbb{V}$ can be divided into two disjoint classes. The first class consists of homomorphisms factorizing through an algebra in $\mathbb{W}$ and this class of homomorphism shows that $\mathbb{V}$ is not alg-universal. And the second class of homomorphisms contains an alg-universal subcategory as a 'full subcategory'.

The second way of avoiding certain algebra homomorphisms is their elimination by means of expanding the similarity type of the variety $\mathbb{V}$ by additional operations. For example, the variety of lattices consists of expansions of certain semilattices by the second semilattice operation satisfying some identities. The preservation of the second operation eliminates some semilattice homomorphisms. The added operations bring additional structure and clearly nullary operations code a least amount of structure. Therefore we shall investigate only expansions by nullary operations. We say that a variety $\mathbb{V}$ has an alg-universal k-nullary expansion (an alg-universal k-expansion) where k is a cardinal if the variety $\mathbb{V}$ with added k nullary operations is alg-universal. An ff-alg-universal k-expansion is defined analogously. Clearly, a variety $\mathbb{V}$ is alg-universal exactly when $\mathbb{V}$ has an alg-universal 0-expansion.

Alg-universal expansions were studied for band varieties [11] and for the variety of distributive lattices [6] and [20], var-relative universality was studied for band varieties [12], for varieties of distributive p-algebras [5] and [7], for varieties of distributive dp-algebras [25] and [27], and for varieties of Heyting algebras [21]. The results for band varieties are recalled in the last section. Any variety known to be var-relatively universal has an alg-universal k-expansion for some finite k. This motivates the following conjecture.

Conjecture 1.6 *If a variety $\mathbb{V}$ is var-relatively universal then there exists a cardinal α such that $\mathbb{V}$ has an alg-universal α-expansion.*

Since there exists a band variety with an alg-universal 2-expansion that is not var-relatively universal, see [11] and [12], it appears that the existence of an alg-universal α-expansion is weaker than var-relative universality. Further, there exists a 3-determined variety of bands having an alg-universal 2-expansion [10] and [11], see also the last section. Thus the determinacy and the existence of alg-universal expansion are not disjunctive. Furthermore, there exists an n-determined finitely generated variety of dp-algebras (here n is a natural number) that is weakly var-relatively universal, see [25] and [27], but a relation of var-relative universality to determinacy is an open problem.

For a semigroup $\mathbf{S}$, let $\mathrm{Var}(\mathbf{S})$ denote the least semigroup variety containing the semigroup $\mathbf{S}$. We recall that $\mathbf{T} \in \mathrm{Var}(\mathbf{S})$ if and only if $\mathbf{T}$ is a quotient semigroup of a subdirect power $\mathbf{T}'$ of $\mathbf{S}$. And $\mathbf{T}'$ is a subdirect power of $\mathbf{S}$ if and only if for every pair of distinct elements x and y of $\mathbf{T}'$ there exists a semigroup homomorphism $f : \mathbf{T}' \to \mathbf{S}$ with $f(x) \neq f(y)$.

Consider semigroups $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_3$ given by the following multiplication tables

M_1	1	a	0
1	1	a	0
a	0	0	0
0	0	0	0

M_2	a	b	c	0
a	0	c	0	0
b	c	0	0	0
c	0	0	0	0
0	0	0	0	0

M_3	d	a	b	c
d	a	a	a	b
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c

Let $\mathbf{M}_2'$ be the subsemigroup of $\mathbf{M}_2$ on the set $\{a, 0\}$ and let $\mathbf{M}_3'$ be the subsemigroup of $\mathbf{M}_3$ on the set $\{a, b, d\}$. We prove the following theorem

Theorem 1.7 *The variety Var*$(\mathbf{M}_1)$ *has an ff-alg-universal* 3*-expansion, it is* 3*-determined and it is not var-relatively universal.*

The variety Var$(\mathbf{M}_2)$ *has an ff-alg-universal* 1*-expansion, it is* α*-determined for no cardinal* α *and it is Var*$(\mathbf{M}_2')$*-relatively ff-alg-universal.*

The variety Var$(\mathbf{M}_3)$ *has an ff-alg-universal* 2*-expansion, it is* α*-determined for no cardinal* α *and it is Var*$(\mathbf{M}_3')$*-relatively ff-alg-universal.*

The second section contains auxiliary technical statements concerning graphs. These statements are exploited in the constructions presented in subsequent sections. The third section is devoted to the variety $\mathrm{Var}(\mathbf{M}_1)$ and it contains the proof of the first statement of Theorem 1.7. The fourth section concerns the variety $\mathrm{Var}(\mathbf{M}_2)$ and aims to prove the second statement of Theorem 1.7. The third statement of Theorem 1.7 is proved in the fifth section which studies the variety $\mathrm{Var}(\mathbf{M}_3)$. The last section discusses consequences of Theorem 1.7 for semigroup varieties. We deduce sufficient conditions under which a semigroup variety $\mathbb{V}$ fulfills one of the following statements:

1. $\mathbb{V}$ has an alg-universal n-expansion for some natural number n;
2. $\mathbb{V}$ is var-relatively ff-universal;
3. $\mathbb{V}$ is α-determined for no cardinal α.

Several open problems are formulated and given in this section. The relation between these notions and Q-universality is also discussed.

In this paper any semilattice is viewed as a join semilattice. Thus the induced partial order on the underlying set of the semilattice is given by $s \leq t$ if and only if $st = t$. Further we identify the natural number n with the set $\{0, 1, \dots, n-1\}$ of all natural numbers less than n.

2 Graphs

First we investigate undirected graphs. For a set X, let $\mathfrak{P}_2 X$ denote the set of all doubletons of X. An <u>undirected</u> <u>graph</u> is a pair (V, E) where V is a non-empty set and $\emptyset \neq E \subseteq \mathfrak{P}_2 V$. A mapping $f : V \to W$ is called a <u>compatible mapping</u> from an undirected graph (V, E) to an undirected graph (W, D) if $\{f(v), f(w)\} \in D$ for all $\{v, w\} \in E$ (thus $f(v) \neq f(w)$ for $\{v, w\} \in E$). If $\mathbf{G} = (V, E)$ is an undirected graph then elements of V are called <u>vertices</u> and elements of E are called <u>edges</u>. Let $\mathbb{GRA}$ denote the category of all undirected graphs and compatible mappings. We recall

Theorem 2.1 *[31] $\mathbb{GRA}$ is ff-alg-universal.* $\square$

Next, for an undirected graph $\mathbf{G} = (V, E)$ define special sets that are used in the fourth section. Define a set $U_0(\mathbf{G}) = V \times \{0, 1, 2\} \cup X$ where $X = \{x_i \mid i \in 9\}$ is disjoint with $V \times \{0, 1, 2\}$ and let us define four subsets of $\mathfrak{P}_2(U_0(\mathbf{G}))$ by

$$\begin{aligned}
P_0(\mathbf{G}) = {} & \{\{x_i, x_{i+1}\} \mid i \in 8\} \cup \{\{x_0, x_4\}\} \cup \\
& \{\{x_0, (v, 0)\}, \{x_8, (v, 2)\} \mid v \in V\} \cup \{\{(v, i), (v, i+1)\} \mid v \in V, i \in 2\}, \\
P_1(\mathbf{G}) = {} & \{\{x_i, x_j\} \mid i, j \in 9,\ i \neq j,\ \{x_i, x_j\} \notin P_0(\mathbf{G})\} \cup \\
& \{\{x_i, (v, 0)\} \mid v \in V,\ i \in 9 \setminus \{0\}\} \cup \{\{x_i, (v, 1)\} \mid v \in V,\ i \in 9\} \cup \\
& \{\{x_i, (v, 2)\} \mid v \in V,\ i \in 8\}, \\
P_2(\mathbf{G}) = {} & \{\{(v, i), (w, j)\} \mid \{v, w\} \in E,\ i, j \in 3,\ |i - j| \leq 1\} \cup \{\{(v, 0), (w, 2)\} \mid v, w \in V\}, \\
P_3(\mathbf{G}) = {} & \{\{(v, i), (w, j)\} \mid v, w \in V,\ v \neq w,\ \{v, w\} \notin E,\ i, j \in 3,\ |i - j| \leq 1\}.
\end{aligned}$$

For $v \in V$ and $i \in 3$ define $p(v, i) = v$ and for $v, w \in V$ and $i, j \in 3$ with $v \neq w$ define $p(\{(v, i), (w, j)\}) = \{v, w\}$. Set $R(\mathbf{G}) = \mathfrak{P}_2 V \setminus E$. Next we give basic properties of these sets that are exploited by the constructions presented in the fourth section.

Lemma 2.2 *Let $\mathbf{G} = (V, E) \in \mathbb{GRA}$ be an undirected graph. Then we have:*

1. *The graph $(U_0(\mathbf{G}), P_0(\mathbf{G}))$ is the union of the cycle $(x_0, x_1, x_2, x_3, x_4, x_0)$ of length 5 and of the cycles*
$$(x_0, x_4, x_5, x_6, x_7, x_8, (v, 2), (v, 1), (v, 0), x_0)$$
of length 9 for all $v \in V$. The graph $(U_0(\mathbf{G}), P_0(\mathbf{G}))$ contains exactly one cycle of length 5 and it is the shortest cycle of $(U_0(\mathbf{G}), P_0(\mathbf{G}))$. Any cycle of length 9 in $(U_0(\mathbf{G}), P_0(\mathbf{G}))$ has a form $(x_0, x_4, x_5, x_6, x_7, x_8, (v, 2), (v, 1), (v, 0), x_0)$ for some $v \in V$.
2. *If $(y_0, y_1, \ldots, y_{m-1})$ is a cycle from the graph $(U_0(\mathbf{G}), P_0(\mathbf{G}))$ and if $i, j \in m - 1$ are distinct then $\{y_i, y_j\} \notin P_3(\mathbf{G})$.*
3. *$P_0(\mathbf{G})$, $P_1(\mathbf{G})$, $P_2(\mathbf{G})$ and $P_3(\mathbf{G})$ are pairwise disjoint sets and $\bigcup_{i=0}^{3} P_i(\mathbf{G}) = \mathfrak{P}_2(U_0(\mathbf{G}))$.*

4. *If* $\{u, w\} \in P_3(\mathbf{G})$ *then* $p(\{u, w\})$ *is defined and* $p(\{u, w\}) = \{p(u), p(w)\} \in R(\mathbf{G})$.

Proof. These statements follow from the definitions of sets $P_i(\mathbf{G})$ for $i = 0, 1, 2, 3$, by a direct verification. □

Lemma 2.3 *Let* $A = \{a_i \mid i \in m\}$ *be a set of size* m *for* $m > 3$ *and let* $f : A \to B$ *be a mapping with* $|B| < m$. *Then there exist* $i, j \in m$ *such that* $i - j \not\equiv -1, 0, 1 \bmod m$ *and either* $f(a_i) = f(a_j)$ *or* $\{f(a_i), f(a_j)\} = \{f(a_k), f(a_{k+1 \bmod m})\}$ *for some* $k \in m$.

Proof. If there exists $b \in B$ such that $f^{-1}(b)$ is not a singleton and $f^{-1}(b) \neq \{a_i, a_{i+1 \bmod m}\}$ for all $i \in m$ then there exist distinct $i, j \in m$ such that $|i-j| \not\equiv 1 \bmod m$ and $f(a_i) = f(a_j) = b$ and the statement is true. Since $|B| < m$ there exists $b \in B$ such that $f^{-1}(b)$ is not a singleton. By the first part of the proof, we can assume that $f^{-1}(b) = \{a_i, a_{i+1 \bmod m}\}$ for some $i \in m$. Since $m > 3$ we have $i+2-i \equiv 2 \bmod m$ and thus if we set $j \equiv i+2 \bmod m$ and $k \equiv i+1 \bmod m$ then $a_j = a_{k+1 \bmod m}$ and $f(a_i) = f(a_k)$. Thus the statement is proved. □

Next we prove an auxiliary statement about directed graphs. First we recall several basic notions about directed graphs. A directed graph (a digraph) is an ordered pair $\mathbf{G} = (X, R)$ where X is a set and $R \subseteq X \times X$. If (X, R) and (Y, S) are directed graphs then a mapping $f : X \to Y$ is called a compatible mapping from (X, R) to (Y, S) if $(f(x), f(y)) \in S$ for all $(x, y) \in R$. If $\mathbf{G} = (X, R)$ is a directed graph then elements of X are called nodes of $\mathbf{G}$ and elements of R are called arcs of $\mathbf{G}$. An arc $(x, x) \in R$ is called a loop. For $R \subseteq X \times X$, let us define $\mathrm{Op}(R) = \{(y, x) \mid (x, y) \in R\}$. A directed graph $\mathbf{G} = (X, R)$ is called strongly connected if for every pair of nodes $x, y \in X$ there exists a sequence $x = x_0, x_1, \ldots, x_n = y$ such that $(x_i, x_{i+1}) \in R$ for all $i = 0, 1, \ldots, n-1$. Let $\mathbb{DG}$ denote the category of all directed graphs and compatible mappings.

Next we introduce an auxiliary category $\mathbb{DG}(2)$. A triple (X, R_1, R_2), where X is a set and $R_1, R_2 \subseteq X \times X$ are subsets, is an object of $\mathbb{DG}(2)$ and morphisms of $\mathbb{DG}(2)$ from (X, R_1, R_2) to (X', R_1', R_2') are all mappings $f : X \to X'$ such that $f : (X, R_i) \to (X', R_i')$ is a compatible mapping for $i = 1, 2$. We say that $(X, R_1, R_2) \in \mathbb{DG}(2)$ is weakly connected if for every pair of distinct nodes $x, y \in X$ there exists a sequence $x = x_0, x_1, \ldots, x_n = y$ such that $(x_i, x_{i+1}) \in R_1 \cup \mathrm{Op}(R_1) \cup R_2 \cup \mathrm{Op}(R_2)$ for all $i = 0, 1, \ldots, n-1$. First we show a folklore statement about $\mathbb{DG}(2)$.

Proposition 2.4 *There exists a full embedding* $\Omega : \mathbb{GRA} \to \mathbb{DG}(2)$ *such that*

1. $\Omega\mathbf{G}$ *is weakly connected for all undirected graphs* $\mathbf{G}$*;*
2. *for any undirected graph* $\mathbf{G}$*, the underlying set of* $\Omega\mathbf{G}$ *is finite if and only if the underlying set of* $\mathbf{G}$ *is finite;*
3. *if* $\Omega\mathbf{G} = (X, R_1, R_2)$ *for an undirected graph* $\mathbf{G}$*, then* R_1 *is a symmetric relation (i.e.,* $(x, y) \in R_1 \Leftrightarrow (y, x) \in R_1$*),* R_2 *is an antisymmetric relation (i.e.,* $(x, y) \in R_2 \Rightarrow (y, x) \notin R_2$*) and neither* R_1 *nor* R_2 *have any loops;*

4. *for every undirected graph* $\mathbf{G}$ *there exist two distinct nodes* $v_{\mathbf{G}}$ *and* $w_{\mathbf{G}}$ *of* $\Omega\mathbf{G}$ *such that* $\Omega f(v_{\mathbf{G}}) = v_{\mathbf{G}'}$ *and* $\Omega f(w_{\mathbf{G}}) = w_{\mathbf{G}'}$ *for any compatible mapping* $f : \mathbf{G} \to \mathbf{G}' \in \mathbb{GRA}$.

Proof. Let $\mathbf{G} = (V, E)$ be an undirected graph. Choose distinct elements $v_{\mathbf{G}}$ and $w_{\mathbf{G}}$ with $v_{\mathbf{G}}, w_{\mathbf{G}} \notin V$ and define $\Omega\mathbf{G} = (X_{\mathbf{G}}, R_{1,\mathbf{G}}, R_{2,\mathbf{G}})$ where $X_{\mathbf{G}} = V \cup \{v_{\mathbf{G}}, w_{\mathbf{G}}\}$, $R_{1,\mathbf{G}} = \{(u,t) \mid \{u,t\} \in E\}$ and $R_{2,\mathbf{G}} = \{(u, v_{\mathbf{G}}) \mid u \in V\} \cup \{(u, w_{\mathbf{G}}) \mid u \in V\} \cup \{(v_{\mathbf{G}}, w_{\mathbf{G}})\}$. For a compatible mapping $f : \mathbf{G} \to \mathbf{G}'$ define $\Omega f : X_{\mathbf{G}} \to X_{\mathbf{G}'}$ such that $\Omega f(v) = f(v)$ for all $v \in V$, $\Omega f(v_{\mathbf{G}}) = v_{\mathbf{G}'}$ and $\Omega f(w_{\mathbf{G}}) = w_{\mathbf{G}'}$. It is easy to see that $\Omega f : \Omega\mathbf{G} \to \Omega\mathbf{G}'$ is a morphism of $\mathbb{DG}(2)$, thus Ω is an embedding. A verification of statements (1), (2), (3), and (4) of Ω is straightforward. It remains to prove that Ω is full. Let $\mathbf{G} = (V, E)$ and $\mathbf{G}' = (V', E')$ be undirected graphs and let $f : \Omega\mathbf{G} \to \Omega\mathbf{G}'$ be a morphism of $\mathbb{DG}(2)$. Observe that

(i) if $x \in V$ then $(z, x) \in R_{2,\mathbf{G}}$ for no $z \in X_{\mathbf{G}}$;

(ii) $(x, v_{\mathbf{G}}) \in R_{2,\mathbf{G}}$ for all $x \in V$ and $(v_{\mathbf{G}}, w_{\mathbf{G}}) \in R_{2,\mathbf{G}}$;

(iii) $(x, w_{\mathbf{G}}) \in R_{2,\mathbf{G}}$ for all $x \in X_{\mathbf{G}} \setminus \{w_{\mathbf{G}}\}$, and $(w_{\mathbf{G}}, z) \in R_{2,\mathbf{G}}$ for no $z \in X_{\mathbf{G}}$.

Since f is a compatible mapping from $(X_{\mathbf{G}}, R_{2,\mathbf{G}})$ to $(X_{\mathbf{G}'}, R_{2,\mathbf{G}'})$, we have $f(V) \subseteq V'$, $f(v_{\mathbf{G}}) = v_{\mathbf{G}'}$ and $f(w_{\mathbf{G}}) = w_{\mathbf{G}'}$. Since f is also a compatible mapping from $(X_{\mathbf{G}}, R_{1,\mathbf{G}})$ to $(X_{\mathbf{G}'}, R_{1,\mathbf{G}'})$ we deduce that the domain-range restriction g of f to V and V' is a compatible mapping from $\mathbf{G}$ to $\mathbf{G}'$ with $\Omega g = f$. The proof is complete. □

Next we construct a full embedding of $\mathbb{DG}(2)$ into $\mathbb{DG}$, using a standard šíp-construction, see [29] or [31]. First we give several auxiliary notions and facts. We say that $T \subseteq X$ is a <u>triangle</u> in a digraph $\mathbf{G} = (X, R)$ if $|T| = 3$ and $(t, t') \in R$ or $(t', t) \in R$ for every pair of distinct nodes t and t' of T. We say that a set $U \subseteq X$ is <u>triangle</u> <u>connected</u> in a digraph $\mathbf{G} = (X, R)$ if for every node $u \in U$ there exists a triangle T in $\mathbf{G}$ with $u \in T \subseteq U$ and for distinct triangles T and T' in $\mathbf{G}$ with $T \cup T' \subseteq U$ there exists a sequence of triangles $T = T_1, T_2, \ldots, T_k = T'$ in U with $|T_i \cap \left(\bigcup_{j<i} T_j\right)| \geq 2$ for every $i = 2, 3, \ldots, k$. Any set $U \subseteq X$ is called a <u>triangle</u> <u>component</u> if it is a maximal triangle connected subset of X. It is easy to see that any large enough triangle connected set is a subset of a triangle component and if U_1 and U_2 are distinct triangle components of a digraph $\mathbf{G}$ and if $u, v \in U_1 \cap U_2$ are distinct nodes then there exists no triangle T in $\mathbf{G}$ with $u, v \in T$ (in contrary, $T \subseteq U_1 \cap U_2$ and by the maximality of triangle components $U_1 = U_2$ – a contradiction). If $f : (X, R) \to (X', R')$ is a compatible mapping between digraphs and if (X', R') has no loops then $f(U)$ is triangle connected in (X', R') for every triangle component U of (X, R).

Consider digraphs $\mathbf{D}_1$ and $\mathbf{D}_2$ from Figure 1.

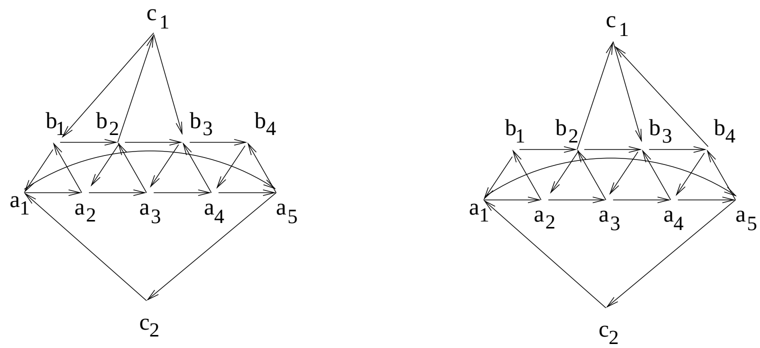

Figure 1. The digraphs $\mathbf{D}_1$ and $\mathbf{D}_2$.

Thus $\mathbf{D}_1 = (D, C_1)$ and $\mathbf{D}_2 = (D, C_2)$ where

$$D = \{a_i \mid i = 1,2,3,4,5\} \cup \{b_i \mid i = 1,2,3,4\} \cup \{c_1, c_2\},$$

and $C_1 = C \cup \{(c_1, b_1)\}$, $C_2 = C \cup \{(b_4, c_1)\}$ for

$$C = \{(a_1,a_2),(a_2,a_3),(a_3,a_4),(a_4,a_5),(b_1,a_1),(a_2,b_1),(b_2,a_2),(a_3,b_2),(b_3,a_3), (a_4,b_3),(b_4,a_4),(a_5,b_4),(b_1,b_2),(b_3,b_2),(b_3,b_4),(b_2,c_1),(c_1,b_3),(a_1,a_5), (a_5,c_2),(c_2,a_1)\}.$$

Lemma 2.5 *The graphs $\mathbf{D}_1$ and $\mathbf{D}_2$ satisfy the following conditions:*

1. *The digraphs $\mathbf{D}_1$ and $\mathbf{D}_2$ are strongly connected without loops and (a_2, a_4), $(a_4, a_2) \notin C_1 \cup C_2$.*
2. *If $(s,t) \in C_j$ is an arc for $j \in \{1,2\}$ then there exists u with $(t,u),(u,s) \in C_j$.*
3. *If $(s,t),(t,u),(s,u) \in C_j$ for some $j \in \{1,2\}$ then $s = b_3$, $t = a_3$ and $u = b_2$.*
4. *The set D is triangle connected in $\mathbf{D}_i$ for every $i \in \{1,2\}$.*
5. *If $f : \mathbf{D}_i \to \mathbf{D}_j$ is a compatible mapping for $i, j \in \{1,2\}$ then $i = j$ and f is the identity mapping.*
6. *If $f : \mathbf{D}_i \to (D, Op(C_j))$ is a compatible mapping for $i, j \in \{1,2\}$ then $i \neq j$, $f(a_k) = a_{6-k}$ for all $k = 1,2,\ldots,5$, $f(b_k) = b_{5-k}$ for all $k = 1,2,3,4$ and $f(c_k) = c_k$ for all $k = 1,2$.*

Proof. The statements (1)–(4) are clear, see Figure 1. We prove the statement (5). Assume that $i, j \in \{1,2\}$ and $f : \mathbf{D}_i \to \mathbf{D}_j$ is a compatible mapping. Then the statement (3) implies that $f(a_3) = a_3$, $f(b_2) = b_2$ and $f(b_3) = b_3$ because, by (1), $\mathbf{D}_j$ has no loops. In both $\mathbf{D}_i$ and $\mathbf{D}_j$, (a_2, a_3, b_2) is the unique cycle of length 3 containing the arc (a_3, b_2), therefore $f(a_2) = a_2$. For the same reason, we obtain $f(a_4) = a_4$ and $f(c_1) = c_1$. Since $(a_1, a_2) \in C_i$ we deduce that $f(a_1) \in \{a_1, b_2\}$ and, by the dual reason $f(a_5) \in \{b_3, a_5\}$. From the arc $(a_1, a_5) \in C_i$ it follows that $f(a_1) = a_1$ and $f(a_5) = a_5$ because $(b_2, b_3), (b_2, a_5), (a_1, b_3) \notin C_j$.

Since (a_1, a_2, b_1), (a_4, a_5, b_4) and (a_1, a_5, c_2) are the only cycles of length 3 in $\mathbf{D}_j$ containing arcs (a_1, a_2), (a_4, a_5) and (a_1, a_5) from C_i respectively, we find that $f(b_1) = b_1$, $f(b_4) = b_4$ and $f(c_2) = c_2$. Thus f is the identity mapping and since $\mathbf{D}_i$ is not a subgraph of $\mathbf{D}_j$ for $i \neq j$, we conclude that $i = j$ and (5) is proved.

The proof of (6) is dual to the proof of (5). Assume that $f : (D, C_i) \to (D, \mathrm{Op}(C_j))$ is a compatible mapping for $i, j \in \{1, 2\}$. By (3), observe that if $(s,t), (s,u), (t,u) \in \mathrm{Op}(C_j)$ then $s = b_2$, $t = a_3$ and $u = b_3$. Thus $f(b_3) = b_2$, $f(a_3) = a_3$ and $f(b_3) = b_2$. The unique cycles of length 3 containing arcs (b_3, b_2), (b_3, a_3) and (a_3, b_2) imply that $f(c_1) = c_1$, $f(a_2) = a_4$ and $f(a_4) = a_2$ (analogously to the proof of (5)). Using the cycle (a_1, a_5, c_2) in C_i and C_j and by arguments dual to those in the proof of (5), conclude that $f(a_1) = a_5$, $f(a_5) = a_1$, $f(b_1) = b_4$, $f(b_4) = b_1$, and $f(c_2) = c_2$. From $(c_1, b_1) \in C_1$, $(c_1, b_1) \notin C_2$, $(b_4, c_1) \in C_2$, $(b_4, c_1) \notin C_1$ we obtain that $(b_1, c_1) \in \mathrm{Op}(C_1)$, $(b_1, c_1) \notin \mathrm{Op}(C_2)$, $(c_1, b_4) \in \mathrm{Op}(C_2)$, $(c_1, b_4) \notin \mathrm{Op}(C_1)$. This demonstrates that $i \neq j$ and completes the proof of (6). $\square$

Next we prove the main theorem for digraphs.

Theorem 2.6 *There exists an ff-alg-universal full subcategory $\mathbb{DG}_s$ of $\mathbb{DG}$ such that*

1. *any digraph $\mathbf{G} \in \mathbb{DG}_s$ is strongly connected and has no loops;*
2. *for any digraph $\mathbf{G} = (X, R) \in \mathbb{DG}_s$ and for any arc $(x, y) \in R$ there exist a node $z \in X$ and arcs $(y, z), (z, x) \in R$;*
3. *for any digraph $\mathbf{G} = (X, R) \in \mathbb{DG}_s$ there exist two distinct nodes $a_{\mathbf{G}}, b_{\mathbf{G}} \in X$ such that $f(a_{\mathbf{G}}) = a_{\mathbf{G}'}$ and $f(b_{\mathbf{G}}) = b_{\mathbf{G}'}$ for any compatible mapping $f : \mathbf{G} \to \mathbf{G}' \in \mathbb{DG}_s$;*
4. *for any digraph $\mathbf{G} = (X, R) \in \mathbb{DG}_s$ neither $(a_{\mathbf{G}}, b_{\mathbf{G}})$ nor $(b_{\mathbf{G}}, a_{\mathbf{G}})$ is an arc of $\mathbf{G}$;*
5. *if $(X, R), (X', R') \in \mathbb{DG}_s$ then there exists no compatible mapping $f : (X, R) \to (X', Op(R'))$.*

Proof. We shall construct a functor $\Lambda : \mathbb{DG}(2) \to \mathbb{DG}$ such that $\Lambda \circ \Omega$ is a full embedding from $\mathbb{GRA}$ preserving finiteness and the full subcategory $\mathbb{DG}_s$ consisting of digraphs $\Lambda \circ \Omega \mathbf{G}$ for all undirected graphs $\mathbf{G}$ satisfies the conditions (1)–(5). Let $\mathbf{G} = (X, R_1, R_2)$ be an object of $\mathbb{DG}(2)$. Define $Y' = (D \times \{1\} \times R_1) \cup (D \times \{2\} \times R_2)$ and

$$Q' = \{((u, i, (x, y)), (v, i, (x, y))) \mid i \in \{1, 2\},\, (u, v) \in C_i,\, (x, y) \in R_i\}.$$

By Lemma 2.5(4), $T \subseteq Y'$ is a triangle component of (Y', Q') if and only if $T = D \times \{i\} \times \{(x, y)\}$ for some $i \in \{1, 2\}$ and some $(x, y) \in R_i$. Let $\sim$ be the least equivalence such that for all $x \in X$

(i) $(a_2, i, (x, y)) \sim (a_2, j, (x, z))$ for all $(x, y) \in R_i$, $(x, z) \in R_j$ and $i, j \in \{1, 2\}$;

(ii) $(a_4, i, (y, x)) \sim (a_4, j, (z, x))$ for all $(y, x) \in R_i$, $(z, x) \in R_j$ and $i, j \in \{1, 2\}$;

(iii) $(a_2, i, (x, y)) \sim (a_4, j, (z, x))$ for all $(x, y) \in R_i$, $(z, x) \in R_j$ and $i, j \in \{1, 2\}$.

Let $[u]$ denote the class of $\sim$ containing $u \in Y'$. By a direct calculation, $(a_i, j, (x, y)) \sim (a_{i'}, j', (x', y'))$ if and only if one of the following possibilities occurs: $i = i' = 2$, $j, j' \in \{1, 2\}$ and $x = x'$ or $i = i' = 4$, $j, j' \in \{1, 2\}$ and $y = y'$ or $i = 2$, $i' = 4$, $j, j' \in \{1, 2\}$ and $x = y'$ or $i = 4$, $i' = 2$, $j, j' \in \{1, 2\}$ and $y = x'$. Hence we can identify an element $x \in X$ with the class of $\sim$ containing one of the following elements:

$(a_2, 1, (x, y))$ for some $(x, y) \in R_1$;
$(a_2, 2, (x, y))$ for some $(x, y) \in R_2$;
$(a_4, 1, (y, x))$ for some $(y, x) \in R_1$;
$(a_4, 2, (y, x))$ for some $(y, x) \in R_2$.

Observe that this convention is correct because distinct elements of Y' belong to the same class of $\sim$ if and only if for a fixed $x \in X$ they satisfy one of the conditions. Define $Y = Y'/\sim$ and $Q = \{([u], [v]) \mid (u, v) \in Q'\}$. Then $X \subseteq Y$. Observe that

(•) $|[u] \cap (D \times \{i\} \times \{(x, y)\})| \leq 1$ for all $u \in Y'$, all $i \in \{1, 2\}$ and $(x, y) \in R_i$,

(•) if $u, v \in Y'$ are distinct and $[u]$ and $[v]$ are not singletons, then (by Lemma 2.5(1)), $(u, v), (v, u) \notin Q'$ and thus $([u], [v]), ([v], [u]) \notin Q$; hence there exists no triangle T in (Y, Q) with $[u], [v] \in T$,

(•) for every $i \in \{1, 2\}$ and every $(x, y) \in R_i$ there exist exactly two more-element classes of $\sim$ intersecting $D \times \{i\} \times \{(x, y)\}$.

As a consequence (Y, Q) satisfies these conditions:

(Y, Q) has no loops,
a subset $T \subseteq Y$ is a triangle component of (Y, Q) if and only if $T = \{[u] \mid u \in D \times \{i\} \times \{(x, y)\}\}$ for some $i \in \{1, 2\}$ and some $(x, y) \in R_i$,
for $i \in \{1, 2\}$, and $(x, y) \in R_i$, the induced subgraph of (Y, Q) on the set $D \times \{i\} \times \{(x, y)\}$ is isomorphic to $\mathbf{D}_i$,
any arc from Q belongs to a triangle component of (Y, Q),
for every $(x, y) \in Q$ there exists $z \in Y$ with $(y, z), (z, x) \in Q$
$(x, y) \in Q$ for no $x, y \in X \subseteq Y$.

Further, if (X, R_1, R_2) is weakly connected then (Y, Q) is strongly connected because, by Lemma 2.5(1), $\mathbf{D}_1$ and $\mathbf{D}_2$ are strongly connected (indeed, for any $x \in X$, $x \in \{[u] \mid u \in D \times \{i\} \times \{(x, y)\}\}$ and $x \in \{[u] \mid u \in D \times \{j\} \times \{(z, x)\}\}$ for all $i \in \{1, 2\}$ and $(x, y) \in R_i$ and for all $j \in \{1, 2\}$ and $(z, x) \in R_j$). Let us define $\Lambda(X, R_1, R_2) = (Y, Q)$. We can summarize: if (X, R_1, R_2) is weakly connected then $\Lambda(X, R_1, R_2)$ satisfies the statements (1), (2) and if $a_{\mathbf{G}}$ and $b_{\mathbf{G}}$ are chosen from X then the statement (4) holds.

For a morphism $f : (X, R_1, R_2) \to (V, S_1, S_2)$ of $\mathbb{DG}(2)$ define a mapping $\Lambda f : Y \to Z$ where $\Lambda(X, R_1, R_2) = (Y, Q)$, $\Lambda(V, S_1, S_2) = (Z, T)$ by

$$\Lambda f([(d, i, (x, y))]) = [(d, i, (f(x), f(y)))]$$

for all $d \in D$, $i \in \{1, 2\}$ and $(x, y) \in R_i$. It is easy to see that Λf is a correctly defined compatible mapping from (Y, Q) to (Z, T). Furthermore, $\Lambda f(X) \subseteq V$ and $\Lambda f(x) = f(x)$ for all $x \in X$, and hence we conclude that $\Lambda : \mathbb{DG}(2) \to \mathbb{DG}$ is a faithful functor.

Next we prove that Λ is full on the full subcategory of $\mathbb{DG}(2)$ formed by all objects of $\mathbb{DG}(2)$ without isolated nodes. Let (X, R_1, R_2) and (V, S_1, S_2) be objects of $\mathbb{DG}(2)$ such that for every $x \in X$ there exists $(x, y) \in R_1 \cup R_2$ or $(z, x) \in R_1 \cup R_2$. Assume that $f : (Y, Q) \to (Z, T)$ is a compatible mapping of digraphs for $\Lambda(X, R_1, R_2) = (Y, Q)$ and $\Lambda(V, S_1, S_2) = (Z, T)$. From the description of triangle components and from Lemma 2.5(4) and (5) it follows that for every $i \in \{1, 2\}$ and for every arc $(x, y) \in R_i$ there exists an arc $(x', y') \in S_i$ such that $f([(d, i, (x, y))]) = [(d, i, (x', y'))]$ for all $d \in D$. Thus $f(X) \subseteq V$ and the domain-range restriction g of f on X and V satisfies $(g(x), g(y)) \in S_i$ for all $(x, y) \in R_i$ and $i \in \{1, 2\}$. Therefore g is a compatible mapping from (X, R_i) to (V, S_i) for $i = 1, 2$, and whence $g : (X, R_1, R_2) \to (V, S_1, S_2)$ is a morphism of $\mathbb{DG}(2)$ with $\Lambda g = f$.

Let $\mathbb{DG}_s$ be a full subcategory of $\mathbb{DG}$ whose objects form the class

$$\{\Lambda(\Omega\mathbf{G}) \mid \mathbf{G} \in \mathbb{GRA} \text{ is an undirected graph}\}.$$

For an undirected graph $\mathbf{G}$, let $a_{\Lambda(\Omega\mathbf{G})} = v_{\mathbf{G}}$ and $b_{\Lambda(\Omega\mathbf{G})} = w_{\mathbf{G}}$. Then, by the foregoing considerations, Theorem 2.1 and Proposition 2.4, $\mathbb{DG}_s$ is an ff-alg-universal category satisfying statements (1)–(4).

It remains to prove (5). Let $\mathbf{G}_1 = (V, E)$ and $\mathbf{G}_2 = (W, F)$ be undirected graphs and let us denote $\Omega\mathbf{G}_1 = (X, R_1, R_2)$, $\Lambda(\Omega\mathbf{G}_1) = (Y, Q)$, $\Omega\mathbf{G}_2 = (U, S_1, S_2)$, and $\Lambda(\Omega\mathbf{G}_2) = (Z, T)$. Assume that $f : (Y, Q) \to (Z, \mathrm{Op}(T))$ is a compatible mapping. By the definition of Ω, we obtain

(i) $X = V \cup \{v, w\}$, $U = W \cup \{v, w\}$ for distinct v and w such that $v, w \notin V \cup W$;
(ii) R_1 and S_1 are symmetric relations, i.e. $(x, y) \in R_1$ if and only if $(y, x) \in R_1$ and $(x, y) \in S_1$ if and only if $(y, x) \in S_1$;
(iii) R_2 and S_2 are antisymmetric relations, i.e. if $(x, y) \in R_2$ then $(y, x) \notin R_2$ and if $(x, y) \in S_2$ then $(y, x) \notin S_2$.

For every $(x, y) \in R_1$, $\{[(d, 1, (x, y))] \mid d \in D\}$ is a triangle component of (Y, Q). Since (Z, T) has no loops, $\{f([(d, 1, (x, y))]) \mid d \in D\}$ is triangle connected, and Lemma 2.5(6) implies that there exists $(x', y') \in S_2$ such that $f([(a_k, 1, (x, y))]) = [(a_{6-k}, 2, (x', y'))]$ for all $k = 1, 2, \ldots, 5$. Since $x = [(a_2, 1, (x, y))]$, $y = [(a_4, 1, (x, y))]$, $x' = [(a_2, 2, (x', y'))]$, and $y' = [(a_4, 2, (x', y'))]$ we find that $f(x) = y'$ and $f(y) = x'$. Since R_1 is a symmetric relation we conclude that $(y, x) \in R_1$ and, for an analogous reason, there exists $(x'', y'') \in S_2$ such that $f([(a_k, 1, (y, x))]) = [(a_{6-k}, 2, (x'', y''))]$ for all $k = 1, 2, \ldots, 5$. Since $y = [(a_2, 1,\ (y, x))]$ and $x = [(a_4, 1, (y, x))]$, we deduce that $f(y) = y'' = x'$ and $f(x) = x'' = y'$. Whence $(x', y'), (y', x') \in S_2$ and this is a contradiction because S_2 is an antisymmetric relation. The proof of (5) is complete. $\square$

3. The variety $\mathbf{Var}(\mathbf{M}_1)$

For a semigroup $\mathbf{S} = (S, \cdot)$, let $r(\mathbf{S})$ denote the union of all subsemigroups of $\mathbf{S}$ that are groups. For $s \in S$, let us define $\mathcal{L}(\mathbf{S})_s = \{x \in S \mid \exists u \in S,\ xu = s\}$ and $\mathcal{R}(\mathbf{S})_s = \{x \in S \mid \exists u \in S,\ ux = s\}$. A subset $A \subseteq S$ is closed if it has these properties:

(c1) $ab \in A$ for $a, b \in A$;
(c2) if $a^2 \in A$ for some $a \in S$ then $a \in A$;
(c3) if $ab, b \in A$ for $a \in S$ then $a \in A$, if $ba, b \in A$ for $a \in S$, then $a \in A$.

By straightforward verification, we obtain

Lemma 3.1 *The semigroup $\mathbf{M}_1$ satisfies the identities $x^2y = xy$, $x^2y^2 = y^2x^2$ and $x^2y^2 = (xy)^2$.* $\square$

Next we recall several notions useful in a characterization of endomorphism monoids, see [22] or [25]. For a semigroup $\mathbf{S}$ and for $f, g \in \mathrm{End}(\mathbf{S})$ we write

$f \succeq g$ if $f \circ g = g$ and $f \circ f = f$,
$f \asymp g$ if $f \succeq g \succeq f$;

and we say that f covers g if f and g are idempotents such that $f \succeq g$, $f \not\asymp g$ and for $h \in \mathrm{End}(\mathbf{S})$ with $f \succeq h \succeq g$ we have either $f \asymp h$ or $h \asymp g$. Clearly, if $f, g \in \mathrm{End}(\mathbf{S})$ then

$f \succeq g$ if and only if f is idempotent and $\mathrm{Im}(g) \subseteq \mathrm{Im}(f)$;
$f \asymp g$ if and only if f and g are idempotent and $\mathrm{Im}(f) = \mathrm{Im}(g)$;
f covers g if and only if f and g are idempotents, $\mathrm{Im}(g) \subsetneq \mathrm{Im}(f)$ and for every idempotent $h \in \mathrm{End}(\mathbf{S})$ with $\mathrm{Im}(g) \subseteq \mathrm{Im}(h) \subseteq \mathrm{Im}(f)$ we have either $\mathrm{Im}(f) = \mathrm{Im}(h)$ or $\mathrm{Im}(g) = \mathrm{Im}(h)$.

The following proposition gives several important properties of semigroups $\mathbf{S}$ satisfying the identities from Lemma 3.1.

Proposition 3.2 *Let $\mathbf{S} = (S, \cdot)$ be a semigroup satisfying identities $x^2y = xy$ and $x^2y^2 = y^2x^2 = (xy)^2$. Then*

1. *$r(\mathbf{S}) = \{s \in S \mid s^2 = s\}$ is a semilattice and a left ideal of $\mathbf{S}$;*

2. *if* $s \in S \setminus r(\mathbf{S})$ *then either* s *is irreducible in* $\mathbf{S}$ *or* $\{x \in S \mid xs = s\} = \mathcal{L}(\mathbf{S})_s$ *is a closed set in* $\mathbf{S}$ *with* $\mathcal{L}(\mathbf{S})_s \cap r(\mathbf{S}) \neq \emptyset$*;*
3. *if* $u \in \mathcal{R}(\mathbf{S})_s$ *for some* $s \in S \setminus r(\mathbf{S})$ *then* $u \in S \setminus r(\mathbf{S})$*,* $\mathcal{L}(\mathbf{S})_u \subseteq \mathcal{L}(\mathbf{S})_s$*,* $\mathcal{R}(\mathbf{S})_u \subseteq \mathcal{R}(\mathbf{S})_s$*,* $tu \in \mathcal{R}(\mathbf{S})_s$ *for all* $t \in \mathcal{L}(\mathbf{S})_s$ *and either* $u = s$ *or* $s \notin \mathcal{R}(\mathbf{S})_u$*;*
4. *the mapping* $f : S \to S$ *given by* $f(s) = s^2$ *for all* $s \in S$ *is an idempotent endomorphism of* $\mathbf{S}$ *with* $Im(f) = r(\mathbf{S})$*;*
5. *a constant mapping* f *of* S *with the value* $s \in S$ *is an endomorphism of* $\mathbf{S}$ *if and only if* $s \in r(\mathbf{S})$*;*
6. $Im(f)$ *is a doubleton for an idempotent endomorphism* $f \in End(\mathbf{S})$ *if and only if* $1 \leq |\{g \in End(\mathbf{S}) \mid g \text{ is constant and } f \text{ covers } g\}| \leq 2$*;*
7. *if* $X \subseteq r(\mathbf{S})$ *is a two-element chain then there exists an idempotent endomorphism* f *of* $\mathbf{S}$ *such that* $Im(f) = X$ *and* f *covers exactly two constant endomorphisms of* $\mathbf{S}$*; conversely, if* f *is an idempotent endomorphism of* $\mathbf{S}$ *that covers exactly two constant endomorphisms of* $\mathbf{S}$ *then* $Im(f) \subseteq r(\mathbf{S})$ *is a two-element chain;*
8. *if* $s \in S \setminus r(\mathbf{S})$ *is irreducible then there exists an idempotent endomorphism* f *of* $\mathbf{S}$ *such that* $Im(f) = \{s, s^2\}$*,* $f^{-1}(s) = \{s\}$ *and* f *covers exactly one constant endomorphism of* $\mathbf{S}$*; conversely, if* f *is an idempotent endomorphism of* $\mathbf{S}$ *that covers exactly one constant endomorphism of* $\mathbf{S}$ *then there exists an irreducible* $s \in S \setminus r(\mathbf{S})$ *such that* $Im(f) = \{s, s^2\}$ *and any* $t \in f^{-1}(s)$ *is irreducible;*
9. *if* $s \in S \setminus r(\mathbf{S})$ *is reducible then for every* $t \in \mathcal{L}(\mathbf{S})_s \cap r(\mathbf{S})$ *there exists an idempotent endomorphism* f *of* $\mathbf{S}$ *such that* $Im(f) = \{s, s^2, t\}$*,* $f^{-1}(s) = \mathcal{R}(\mathbf{S})_s$*,* $f^{-1}(t) = \mathcal{L}(\mathbf{S})_s$ *and if* f *covers an idempotent endomorphism* g *of* $\mathbf{S}$ *then* $Im(g) = \{s^2, t\}$*; conversely, if* f *is an idempotent endomorphism of* $\mathbf{S}$ *such that any idempotent endomorphisms* g_1 *and* g_2 *of* $\mathbf{S}$ *covered by* f *satisfy* $g_1 \asymp g_2$ *and* $Im(g_1)$ *is a doubleton, then there exists a reducible* $s \in S \setminus r(\mathbf{S})$ *and* $t \in \mathcal{L}(\mathbf{S})_s \cap r(\mathbf{S})$ *such that* $Im(f) = \{s, s^2, t\}$*,* $f^{-1}(t) = \bigcup\{\mathcal{L}(\mathbf{S})_u \mid f(u) = s\}$*, and* $f^{-1}(s) = \bigcup\{\mathcal{R}(\mathbf{S})_u \mid f(u) = s\} \subseteq S \setminus r(\mathbf{S})$*;*
10. *if* f *is an idempotent endomorphism of* $\mathbf{S}$ *such that there exists a reducible* $s \in S \setminus r(\mathbf{S})$ *with* $s \in Im(f)$ *then* $s^2 \in r(\mathbf{S}) \cap Im(f)$ *and* $\mathcal{L}(\mathbf{S})_s \cap Im(f) \neq \emptyset$*; in particular,* $|Im(f) \cap r(\mathbf{S})| \geq 2$*.*

Proof. The identity $x^2y = xy$ implies the identity $x^3 = x^2$. Thus any subsemigroup of $\mathbf{S}$ that is a group is a singleton and hence $r(\mathbf{S}) = \{s \in S \mid s^2 = s\} = \{s^2 \mid s \in S\}$. From $x^2y^2 = y^2x^2$ we conclude that $r(\mathbf{S})$ is a semilattice. Finally, the identities $(xy)^2 = x^2y^2$ and $x^2y = xy$ imply $xy^2 = x^2y^2 = (xy)^2$. Hence $r(\mathbf{S})$ is a left ideal of $\mathbf{S}$, and (1) is proved.

To prove (2), assume that $s \in S \setminus r(\mathbf{S})$ is reducible. If $xu = s$ for some $u, x \in S$ then

$$s = xu = x^2u = x(xu) = xs$$

and hence $\mathcal{L}(\mathbf{S})_s \subseteq \{x \in S \mid xs = s\}$. It is clear that $\{x \in S \mid xs = s\} \subseteq \mathcal{L}(\mathbf{S})_s$ and hence $\{x \in S \mid xs = s\} = \mathcal{L}(\mathbf{S})_s$. To prove that $\mathcal{L}(\mathbf{S})_s$ is closed we observe that

if $x, y \in \mathcal{L}(\mathbf{S})_s$ then $(xy)s = x(ys) = xs = s$ and (c1) is true;
if $x^2 \in \mathcal{L}(\mathbf{S})_s$ then $xs = x^2 s = s$ and (c2) is true;
if $a, ba \in \mathcal{L}(\mathbf{S})_s$ then $bs = b(as) = (ba)s = s$ and $b \in \mathcal{L}(\mathbf{S})_s$;
if $a, ab \in \mathcal{L}(\mathbf{S})_s$ then, by (c1), $a^2 \in \mathcal{L}(\mathbf{S})_s$ and $(ab)^2 = b^2a^2 \in \mathcal{L}(\mathbf{S})_s$; hence, by the foregoing step, $b^2 \in \mathcal{L}(\mathbf{S})_s$ and, by (c2), we obtain $b \in \mathcal{L}(\mathbf{S})_s$, thus (c3) is true.

Since s is reducible we deduce that $\mathcal{L}(\mathbf{S})_s \neq \emptyset$, thus there exists $y \in \mathcal{L}(\mathbf{S})_s$ and, by (c2), $y^2 \in r(\mathbf{S}) \cap \mathcal{L}(\mathbf{S})_s$. Thus $r(\mathbf{S}) \cap \mathcal{L}(\mathbf{S})_s \neq \emptyset$ and the proof of (2) is complete.

To prove (3), assume that $u \in \mathcal{R}(\mathbf{S})_s$ for some $s \in S \setminus r(\mathbf{S})$. Then there exists $t \in S$ with $tu = s$, and, by (1), $u \notin r(\mathbf{S})$. If $x \in \mathcal{L}(\mathbf{S})_u$ then, by (2), $xu = u$ and hence

$$xs = x^2 s = x^2 tu = x^2 t^2 u = t^2 x^2 u = t^2 u = ts = s,$$

therefore $x \in \mathcal{L}(\mathbf{S})_s$ and $\mathcal{L}(\mathbf{S})_u \subseteq \mathcal{L}(\mathbf{S})_s$. If $x \in \mathcal{R}(\mathbf{S})_u$ then $yx = u$ for some $y \in S$ and hence $tyx = tu = s$. Therefore $x \in \mathcal{R}(\mathbf{S})_s$ and hence $\mathcal{R}(\mathbf{S})_u \subseteq \mathcal{R}(\mathbf{S})_s$. If $x \in \mathcal{L}(\mathbf{S})_s$ then

$$s = xs = x^2 s = x^2 tu = x^2 t^2 u = t^2 x^2 u = t^2 xu,$$

and thus $xu \in \mathcal{R}(\mathbf{S})_s$. Finally, if $s \in \mathcal{R}(\mathbf{S})_u$ then $u = vs$ for some $v \in S$ and hence

$$s = tu = t^2 u = t^2 vs = t^2 v^2 s = v^2 t^2 s = v^2 s = vs = u.$$

Whence either $s = u$ or $s \notin \mathcal{R}(\mathbf{S})_u$ and (3) is proved.

(4) follows from the identity $a^2b^2 = (ab)^2$ and (1).

(5) is a consequence of (1).

Next we define special endomorphisms of $\mathbf{S}$ proving the first statements of (7), (8) and (9). If $X \subseteq r(\mathbf{S})$ is a doubleton subsemigroup of $\mathbf{S}$ then X is a two-element chain in the semilattice $r(\mathbf{S})$ (we recall that, by a convention, a partial ordering on $r(\mathbf{S})$ is defined so that $x \leq y$ just when $xy = yx = y$ for $x, y \in r(\mathbf{S})$). Assume that $\{x < y\} = X$. Define a mapping $f : S \to S$ by

$$f(z) = \begin{cases} x & \text{if } z^2 \leq x, \\ y & \text{if } z^2 \not\leq x. \end{cases}$$

It is easy to verify that f is an idempotent endomorphism of $\mathbf{S}$. Clearly, f covers exactly two constant endomorphisms of $\mathbf{S}$. If $s \in S \setminus r(\mathbf{S})$ is irreducible then obviously the mapping $f : S \to S$ given by

$$f(z) = \begin{cases} s & \text{if } z = s, \\ s^2 & \text{if } z \in S \setminus \{s\} \end{cases}$$

is an idempotent endomorphism of $\mathbf{S}$ with $\operatorname{Im}(f) = \{s, s^2\}$ and $f^{-1}(s) = \{s\}$. It is easy to see that f covers exactly one constant endomorphism of $\mathbf{S}$. If

$s \in S \setminus r(\mathbf{S})$ is reducible and $t \in \mathcal{L}(\mathbf{S})_s \cap r(\mathbf{S})$ then we prove that a mapping $f : S \to S$ such that

$$f(z) = \begin{cases} s & \text{if } z \in \mathcal{R}(\mathbf{S})_s, \\ t & \text{if } z \in \mathcal{L}(\mathbf{S})_s, \\ s^2 & \text{if } z \in S \setminus (\mathcal{L}(\mathbf{S})_s \cup \mathcal{R}(\mathbf{S})_s) \end{cases}$$

is an idempotent endomorphism of $\mathbf{S}$ with $\mathrm{Im}(f) = \{s, t, s^2\}$, $f^{-1}(s) = \mathcal{R}(\mathbf{S})_s$ and $f^{-1}(t) = \mathcal{L}(\mathbf{S})_s$. To prove the correctness of the definition it suffices to prove that $\mathcal{L}(\mathbf{S})_s \cap \mathcal{R}(\mathbf{S})_s = \emptyset$. For $u \in \mathcal{L}(\mathbf{S})_s \cap \mathcal{R}(\mathbf{S})_s$ there exist $x, y \in S$ with $xu = s = uy$. By (2), $us = s$ and thus

$$s = u^2 s = u^2 xu = u^2 x^2 u = x^2 u^3 = x^2 u^2.$$

By (1), $s \in r(\mathbf{S})$ – a contradiction with $s \in S \setminus r(\mathbf{S})$. If we prove that $xy \in \mathcal{L}(\mathbf{S})_s$ implies $x, y \in \mathcal{L}(\mathbf{S}_s)$ and $xy \notin \mathcal{R}(\mathbf{S})_s$ whenever $x \notin \mathcal{L}(\mathbf{S})_s$ then, by (3), we conclude that f is an endomorphism. From $xy \in \mathcal{L}(\mathbf{S})_s$ it follows, by (c1), that $(xy)^2 \in \mathcal{L}(\mathbf{S})_s$. Since $x^2(xy)^2 = x^2x^2y^2 = x^2y^2 = (xy)^2$ we conclude, by (c3), that $x^2 \in \mathcal{L}(\mathbf{S})_s$. Whence, by (c2), $x \in \mathcal{L}(\mathbf{S})_s$ and, by (c3), also $y \in \mathcal{L}(\mathbf{S})_s$. If $xy \in \mathcal{R}(\mathbf{S})_s$ then $txy = s$ for some $t \in S$. Thus $tx \in \mathcal{L}(\mathbf{S})_s$ and, by the foregoing step, $x \in \mathcal{L}(\mathbf{S})_s$, and thus $xy \notin \mathcal{R}(\mathbf{S})_s$ whenever $x \notin \mathcal{L}(\mathbf{S})_s$. Whence f is a correctly defined idempotent endomorphism of $\mathbf{S}$. Clearly, if f covers an idempotent endomorphism g of $\mathbf{S}$ then $\mathrm{Im}(g) = \{t, s\} \subseteq r(\mathbf{S})$.

Observe that if $s \in \mathrm{Im}(f) \setminus r(\mathbf{S})$ is reducible for an idempotent $f \in \mathrm{End}(\mathbf{S})$, then $s^2 \in \mathrm{Im}(f)$ and $\mathrm{Im}(f) \cap \mathcal{L}(\mathbf{S})_s \cap r(\mathbf{S}) \neq \emptyset$. Indeed, by (2), $s = tu$ for some $t \in r(\mathbf{S})$ and $u \in S$ and hence $f(t) \in \mathcal{L}(\mathbf{S})_s \cap r(\mathbf{S})$. Note that, by (2), $s^2 \notin \mathcal{L}(\mathbf{S})_s$ and thus $|\mathrm{Im}(f) \cap r(\mathbf{S})| \geq 2$. Thus (10) is proved.

To prove (6), observe that $\mathrm{Im}(f)$ is a subsemigroup of $\mathbf{S}$. Thus if $\mathrm{Im}(f)$ is a doubleton then $1 \leq |\{g \in \mathrm{End}(\mathbf{S}) \mid g \text{ is constant and } f \text{ covers } g\}| \leq 2$. To prove the reverse implication, assume that $f \in \mathrm{End}(\mathbf{S})$ is idempotent with $1 \leq |\{g \in \mathrm{End}(\mathbf{S}) \mid g \text{ is constant and } f \text{ covers } g\}| \leq 2$. Then f is non-constant. First assume that $|r(\mathbf{S}) \cap \mathrm{Im}(f)| \geq 2$. Since $\mathrm{Im}(f) \cap r(\mathbf{S})$ is a subsemigroup of $\mathbf{S}$ for every $x \in \mathrm{Im}(f) \cap r(\mathbf{S})$ there exists a two-element chain X with $x \in X \subseteq \mathrm{Im}(f) \cap r(\mathbf{S})$. By the foregoing construction, there exists an idempotent endomorphism $f' \in \mathrm{End}(\mathbf{S})$ with $\mathrm{Im}(f') = X$. Hence $g \preceq f' \preceq f$ for every constant endomorphism $g \in \mathrm{End}(\mathbf{S})$ with $g \preceq f$ and $\{x\} = \mathrm{Im}(g)$. Thus $\mathrm{Im}(f') = \mathrm{Im}(f) = r(\mathbf{S}) \cap \mathrm{Im}(f)$ is a doubleton. Assume that $|r(\mathbf{S}) \cap \mathrm{Im}(f)| = 1$. Since f is non-constant there exists $s \in \mathrm{Im}(f) \setminus r(\mathbf{S})$. Clearly $\{s^2\} = r(\mathbf{S}) \cap \mathrm{Im}(f)$. If s is irreducible, then, by the foregoing construction, there exists an idempotent endomorphism $f' \in \mathrm{End}(\mathbf{S})$ with $\mathrm{Im}(f') = \{s, s^2\} \subseteq \mathrm{Im}(f)$. Analogously to the above, we deduce that $\mathrm{Im}(f') = \mathrm{Im}(f)$. If s is reducible then, (10) implies that $|\mathrm{Im}(f) \cap r(\mathbf{S})| \geq 2$ – a contradiction; the proof of (6) is complete.

The first statement of (7) and/or of (8) follows from the foregoing construction. The second statement of (7) and/or of (8) follows from (6), (10) and from the fact that $\mathrm{Im}(f)$ is a subsemigroup of $\mathbf{S}$ for every $f \in \mathrm{End}(\mathbf{S})$. Thus (7) and (8) are proved.

The first statement of (9) follows from the foregoing construction. Conversely, assume that $f \in \mathrm{End}(\mathbf{S})$ is an idempotent endomorphism such that any idempotent endomorphisms g_1 and g_2 of $\mathbf{S}$ covered by f satisfy $g_1 \asymp g_2$ and $\mathrm{Im}(g_1)$ is a doubleton. Let $g \in \mathrm{End}(\mathbf{S})$ be an idempotent endomorphism covered by f. First we prove that $\mathrm{Im}(g) \subseteq r(\mathbf{S})$. Assume the contrary. Since $\mathrm{Im}(g)$ is a doubleton, by (8), there exists an irreducible s with $\{s\} = \mathrm{Im}(g) \setminus r(\mathbf{S})$. If $\mathrm{Im}(f) \cap r(\mathbf{S}) \neq \mathrm{Im}(g) \cap r(\mathbf{S})$, then there exists $x \in \big(\mathrm{Im}(f) \cap r(\mathbf{S})\big) \setminus \mathrm{Im}(g)$ such that $\{x, s^2\}$ is a chain (because $\mathrm{Im}(f) \cap r(\mathbf{S})$ is a subsemilattice of $r(\mathbf{S})$). By (7), there exists an idempotent endomorphism g' of $\mathbf{S}$ with $\mathrm{Im}(g') = \{x, s^2\}$. Since $g'(s) = s^2$ and s is irreducible, the mapping

$$f'(z) = \begin{cases} g'(z) & \text{if } z \neq s, \\ s & \text{if } z = s \end{cases}$$

is an idempotent endomorphism of $\mathbf{S}$ with $\mathrm{Im}(g) \subseteq \mathrm{Im}(f') \subseteq \mathrm{Im}(f)$. Then $\mathrm{Im}(f) = \mathrm{Im}(f')$ because f covers g and $\mathrm{Im}(g) \neq \mathrm{Im}(f')$ – this is a contradiction because $\mathrm{Im}(g) \neq \mathrm{Im}(g')$ and f covers g'. If $\mathrm{Im}(f) \cap r(\mathbf{S}) = \mathrm{Im}(g) \cap r(\mathbf{S})$ then, by (10), any $u \in \mathrm{Im}(f) \setminus \{s^2\}$ is irreducible and $u^2 = s^2$. Choose $u \in \mathrm{Im}(f) \setminus \mathrm{Im}(g)$. Then, clearly, the mapping

$$f'(z) = \begin{cases} u & \text{if } z = u, \\ s & \text{if } z = s, \\ s^2 & \text{if } z \in S \setminus \{u, s\} \end{cases}$$

is an idempotent endomorphism of $\mathbf{S}$ with $\mathrm{Im}(g) \subseteq \mathrm{Im}(f') \subseteq \mathrm{Im}(f)$. Hence $\mathrm{Im}(f) = \mathrm{Im}(f')$ and, by (8), there exists an idempotent endomorphism g' of $\mathbf{S}$ such that f covers g' and $\mathrm{Im}(g) \neq \mathrm{Im}(g') = \{u, s^2\}$ – a contradiction. Whence $\mathrm{Im}(g) \subseteq r(\mathbf{S})$ and, by (7), $\mathrm{Im}(g)$ is a two-element chain. If there exists a three-element chain $X = \{u < v < w\}$ with $\mathrm{Im}(g) \subseteq X \subseteq \mathrm{Im}(f)$ then the mapping

$$f'(z) = \begin{cases} u & \text{if } f(z)^2 \leq u, \\ v & \text{if } f(z)^2 \leq v \text{ and } f(z)^2 \not\leq u, \\ w & \text{if } f(z)^2 \not\leq v \end{cases}$$

is an idempotent endomorphism of $\mathbf{S}$ with $\mathrm{Im}(g) \subseteq \mathrm{Im}(f') \subseteq \mathrm{Im}(f)$. Thus $\mathrm{Im}(f) = \mathrm{Im}(f')$; but there exist three distinct two-element chains contained in $\mathrm{Im}(f)$ and (7) yields a contradiction. In what follows, we assume that $\mathrm{Im}(g) = \{u < v\}$. Then there exists no $x \in \big(\mathrm{Im}(f) \cap r(\mathbf{S})\big) \setminus \mathrm{Im}(g)$ comparable to u (else there exists a three element chain X with $\mathrm{Im}(g) \subseteq X \subseteq \mathrm{Im}(f)$). If there exists $x \in \big(\mathrm{Im}(f) \cap r(\mathbf{S})\big) \setminus \mathrm{Im}(g)$ then $x < v$ and, by a direct verification, the mapping

$$f'(z) = \begin{cases} x & \text{if } f(z)^2 \leq x, \\ u & \text{if } f(z)^2 \leq u, \\ v & \text{if } f(z)^2 \not\leq x \text{ and } f(z)^2 \not\leq u \end{cases}$$

is an idempotent endomorphism of $\mathbf{S}$ with $\mathrm{Im}(g) \subseteq \mathrm{Im}(f') \subseteq \mathrm{Im}(f)$. Thus $\mathrm{Im}(f) = \mathrm{Im}(f')$; but there exist two distinct two-element chains contained in $\mathrm{Im}(f)$ and (7) yields a contradiction. Thus $\mathrm{Im}(f) \cap r(\mathbf{S}) = \mathrm{Im}(g)$. Choose $s \in \mathrm{Im}(f) \setminus \mathrm{Im}(g)$, then $s^2 \in \mathrm{Im}(g)$. If s is irreducible then the mapping

$$f'(z) = \begin{cases} g(z) & \text{if } z \neq s, \\ s & \text{if } z = s \end{cases}$$

is an idempotent endomorphism of $\mathbf{S}$ with $\mathrm{Im}(g) \subseteq \mathrm{Im}(f') \subseteq \mathrm{Im}(f)$. Hence $\mathrm{Im}(f) = \mathrm{Im}(f')$ and, by (8), there exists an idempotent endomorphism g' of $\mathbf{S}$ with $\mathrm{Im}(g') = \{s, s^2\}$. We obtain a contradiction because $\mathrm{Im}(g) \neq \mathrm{Im}(g')$ and f covers g'. Thus we can assume that s is reducible and, by (10), $s^2 = v$ and $u \in \mathcal{L}(\mathbf{S})_s$. It is clear that $f^{-1}(s) \subseteq S \setminus r(\mathbf{S})$. For $t \in S \setminus r(\mathbf{S})$ with $f(t) = s$ we have $f(w) = u$ for all $w \in \mathcal{L}(\mathbf{S})_t$ because $wt = t$ and $f(t') = s$ for all $t' \in \mathcal{R}(\mathbf{S})_t$ because $t = w't'$ for some $w' \in \mathcal{R}(\mathbf{S})$. On the other hand, if $f(w) = u$ and $f(t) = s$ then $s = us = f(w)f(t) = f(wt)$ and $w \in \mathcal{L}(\mathbf{S})_{wt}$, $t \in \mathcal{R}(\mathbf{S})_{wt}$ and (9) is proved. □

By Proposition 3.2(3), for a semigroup $\mathbf{S}$ satisfying the hypothesis of Proposition 3.2 we can define a partial ordering $\sqsubseteq_{\mathbf{S}}$ on the set $S \setminus r(\mathbf{S})$ so that $u \sqsubseteq v$ for $u, v \in S \setminus r(\mathbf{S})$ just when $u \in \mathcal{R}(\mathbf{S})_v$.

It is well known that any constant endomorphism of $\mathbf{S}$ is a left zero of $\mathrm{End}(\mathbf{S})$ and that any semigroup isomorphism preserves left zeros. From the definition, for a monoid isomorphism $\phi : \mathrm{End}(\mathbf{S}) \to \mathrm{End}(\mathbf{T})$ it follows that

1. f and g are endomorphisms of $\mathbf{S}$ with $f \succeq g$ if and only if $\phi(f)$ and $\phi(g)$ are endomorphisms of $\mathbf{T}$ with $\phi(f) \succeq \phi(g)$;
2. f and g are endomorphisms of $\mathbf{S}$ with $f \asymp g$ if and only if $\phi(f)$ and $\phi(g)$ are endomorphisms of $\mathbf{T}$ with $\phi(f) \asymp \phi(g)$;
3. f and g are endomorphisms of $\mathbf{S}$ such that f covers g if and only if $\phi(f)$ and $\phi(g)$ are endomorphisms of $\mathbf{T}$ such that $\phi(f)$ covers $\phi(g)$.

Combining these facts with Proposition 3.2, we immediately obtain the following corollary.

Corollary 3.3 *Let $\mathbf{S} = (S, \cdot)$ and $\mathbf{T} = (T, \cdot)$ be semigroups satisfying the identities $x^2y = xy$, $x^2y^2 = y^2x^2$ and $x^2y^2 = (xy)^2$, and let $\phi : End(\mathbf{S}) \to End(\mathbf{T})$ be a monoid isomorphism. Then*

1. *$f \in End(\mathbf{S})$ is constant if and only if $\phi(f) \in End(\mathbf{T})$ is constant;*
2. *$f \in End(\mathbf{S})$ is idempotent such that $Im(f) \subseteq r(\mathbf{S})$ is a doubleton if and only if $\phi(f) \in End(\mathbf{T})$ is idempotent such that $Im(\phi(f)) \subseteq r(\mathbf{T})$ is a doubleton;*
3. *$f \in End(\mathbf{S})$ is idempotent such that $Im(f) = \{s, s^2\}$ for some irreducible $s \in S \setminus r(\mathbf{S})$ if and only if $\phi(f) \in End(\mathbf{T})$ is idempotent such that $Im(\phi(f)) = \{t, t^2\}$ for some irreducible $t \in T \setminus r(\mathbf{T})$;*

4. *$f \in End(\mathbf{S})$ is idempotent such that $Im(f) = \{s, s^2, v\}$ for a reducible $s \in S \setminus r(\mathbf{S})$ and $v \in \mathcal{L}(\mathbf{S})_s \cap r(\mathbf{S})$ if and only if $\phi(f) \in End(\mathbf{T})$ is idempotent such that $Im(\phi(f)) = \{t, t^2, w\}$ for a reducible $t \in T \setminus r(\mathbf{T})$ and $w \in \mathcal{L}(\mathbf{T})_s \cap r(\mathbf{T})$;*
5. *$f \in End(\mathbf{S})$ is idempotent with $Im(f) = r(\mathbf{S})$ if and only if $\phi(f) \in End(\mathbf{T})$ is idempotent with $Im(\phi(f)) = r(\mathbf{T})$.* □

Theorem 3.4 *$Var(\mathbf{M}_1)$ is 3-determined but not 2-determined.*

Proof. Let $\mathcal{S}$ be a family of equimorphic semigroups from $\mathrm{Var}(\mathbf{M}_1)$. By Lemma 3.1, any semigroup in $\mathrm{Var}(\mathbf{M}_1)$ satisfies the identities $x^2y = xy$ and $x^2y^2 = (xy)^2 = y^2x^2$ and thus we can apply Proposition 3.2 and Corollary 3.3. It is clear that for any $\mathbf{S}_1, \mathbf{S}_2 \in \mathcal{S}$, there exists a monoid isomorphism $\phi_{\mathbf{S}_1,\mathbf{S}_2} : \mathrm{End}(\mathbf{S}_1) \to \mathrm{End}(\mathbf{S}_2)$ such that

> $\phi_{\mathbf{S},\mathbf{S}}$ is the identity mapping for every $\mathbf{S} \in \mathcal{S}$;
> $\phi_{\mathbf{S}_2,\mathbf{S}_3} \circ \phi_{\mathbf{S}_1,\mathbf{S}_2} = \phi_{\mathbf{S}_1,\mathbf{S}_3}$ for all $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3 \in \mathcal{S}$.

For $\mathbf{S}_1 = (S_1, \cdot), \mathbf{S}_2 = (S_2, \cdot) \in \mathcal{S}$ define a mapping $\psi_{\mathbf{S}_1,\mathbf{S}_2} : S_1 \to S_2$ as follows.

> If $s \in r(\mathbf{S}_1)$ then, by Proposition 3.2(5), the constant mapping $f_s : S_1 \to S_1$ with the value s is an endomorphism of $\mathbf{S}_1$, and, by Corollary 3.3(1), $\phi_{\mathbf{S}_1,\mathbf{S}_2}(f_s)$ is also a constant mapping, say that its value is $t \in S_2$. By Proposition 3.2(5), $t \in r(\mathbf{S}_2)$ and we define $\psi_{\mathbf{S}_1,\mathbf{S}_2}(s) = t$.
> If $s \in S_1 \setminus r(\mathbf{S}_1)$ is irreducible then, by Proposition 3.2(8), there exists an idempotent endomorphism f_s of $\mathbf{S}_1$ with $\mathrm{Im}(f_s) = \{s, s^2\}$. By Corollary 3.3(3), there exists an irreducible $t \in S_2 \setminus r(\mathbf{S}_2)$ with $\mathrm{Im}(\phi_{\mathbf{S}_1,\mathbf{S}_2}(f_s)) = \{t, t^2\}$. Define $\psi_{\mathbf{S}_1,\mathbf{S}_2}(s) = t$.
> If $s \in S_1 \setminus r(\mathbf{S}_1)$ is reducible then, by Proposition 3.2(9), for $u \in \mathcal{L}(\mathbf{S}_1)_s$ there exists an idempotent endomorphism $f_{s,u}$ of $\mathbf{S}_1$ with $\mathrm{Im}(f_{s,u}) = \{s, s^2, u\}$. By Corollary 3.3(4), and Proposition 3.2(9), there exist a reducible $t \in S_2 \setminus r(\mathbf{S}_2)$ and $v \in \mathcal{L}(\mathbf{S}_2)_t$ with $\mathrm{Im}(\phi_{\mathbf{S}_1,\mathbf{S}_2}(f_{s,u})) = \{t, t^2, v\}$. Define $\psi_{\mathbf{S}_1,\mathbf{S}_2}(s) = t$.

Since $\phi_{\mathbf{S}_1,\mathbf{S}_2}$ preserves the relation $\asymp$, the definition of $\psi_{\mathbf{S}_1,\mathbf{S}_2}$ is independent of the choice of f_s and $f_{s,u}$ (observe that $f_{s,u'} \circ f_{s,u} = f_{s,u}$ for all reducible $s \in S_1 \setminus r(\mathbf{S}_1)$ and $u, u' \in \mathcal{L}(\mathbf{S}_1)_s$). Moreover, from the bijectivity of $\phi_{\mathbf{S}_1,\mathbf{S}_2}$ it follows that $\psi_{\mathbf{S}_1,\mathbf{S}_2}$ is also a bijection and $\psi_{\mathbf{S}_1,\mathbf{S}_2}(r(\mathbf{S}_1)) = r(\mathbf{S}_2)$. Let $f \in \mathrm{End}(\mathbf{S}_1)$. If $s \in r(\mathbf{S}_1)$ then $f \circ f_s = f_{f(s)}$ and hence $\phi_{\mathbf{S}_1,\mathbf{S}_2}(f)(\psi_{\mathbf{S}_1,\mathbf{S}_2}(s)) = \psi_{\mathbf{S}_1,\mathbf{S}_2}(f(s))$. If $s \in S_1 \setminus r(\mathbf{S}_1)$ is irreducible then, by (10), either $f(s)$ is irreducible or $f(s) = f(s^2) \in r(\mathbf{S})$. Thus $f \circ f_s = f_{f(s)}$ and hence $\phi_{\mathbf{S}_1,\mathbf{S}_2}(f)(\psi_{\mathbf{S}_1,\mathbf{S}_2}(s)) = \psi_{\mathbf{S}_1,\mathbf{S}_2}(f(s))$. Finally, assume that $s \in S_1 \setminus r(\mathbf{S}_1)$ is reducible. Choose $u \in \mathcal{L}(\mathbf{S}_1)_s$. By (10), if $f(s) \in S_1 \setminus r(\mathbf{S}_1)$ then $f(s)$ is reducible and thus

$$f \circ f_{s,u} = \begin{cases} f_{f(s),f(u)} & \text{if } f(s) \in S_1 \setminus r(\mathbf{S}_1), \\ g_{f(s),f(u)} & \text{if } f(s) = f(s^2) \neq f(u), \\ f_{f(s)} & \text{if } f(s) = f(u), \end{cases}$$

where $g_{u,v}$ is an idempotent endomorphism of $\mathbf{S}_1$ with $\operatorname{Im}(g_{u,v}) = \{u, v\}$. By the definition of $\psi_{\mathbf{S}_1,\mathbf{S}_2}$ and Corollary 3.3(2), we get $\phi_{\mathbf{S}_1,\mathbf{S}_2}(f)(\psi_{\mathbf{S}_1,\mathbf{S}_2}(s)) = \psi_{\mathbf{S}_1,\mathbf{S}_2}(f(s))$. Thus

$$(a) \qquad \phi_{\mathbf{S}_1,\mathbf{S}_2}(f)(\psi_{\mathbf{S}_1,\mathbf{S}_2}(s)) = \psi_{\mathbf{S}_1,\mathbf{S}_2}(f(s)) \text{ for all } s \in S_1.$$

For $\mathbf{S} = (S, \cdot) \in \mathcal{S}$, by Proposition 3.2(4), there exists an idempotent endomorphism $h_\mathbf{S}$ of $\mathbf{S}$ with $h_\mathbf{S}(s) = s^2$ for all $s \in S$. By Proposition 3.2(1), $\operatorname{Im}(h_\mathbf{S}) = r(\mathbf{S})$ and, by [22], the subsemigroup $h_\mathbf{S}\operatorname{End}(\mathbf{S})h_\mathbf{S}$ of $\operatorname{End}(\mathbf{S})$ is isomorphic to $\operatorname{End}(r(\mathbf{S}))$. From Corollary 3.3(5) it follows that $\phi_{\mathbf{S}_1,\mathbf{S}_2}(h_{\mathbf{S}_1}\operatorname{End}(\mathbf{S}_1)h_{\mathbf{S}_1}) = h_{\mathbf{S}_2}\operatorname{End}(\mathbf{S}_2)h_{\mathbf{S}_2}$. Define an equivalence $\sim$ on $\mathcal{S}$ so that $\mathbf{S}_1 \sim \mathbf{S}_2$ just when the domain range restriction of $\psi_{\mathbf{S}_1,\mathbf{S}_2}$ to $r(\mathbf{S}_1)$ and $r(\mathbf{S}_2)$ is a semigroup isomorphism between the respective subsemigroups $r(\mathbf{S}_1)$ and $r(\mathbf{S}_2)$ of $\mathbf{S}_1$ and $\mathbf{S}_2$. According to B. M. Schein [34], $\sim$ has at most two classes. To complete the proof it suffices to show that if $\mathbf{S}_1 \sim \mathbf{S}_2$ then $\psi_{\mathbf{S}_1,\mathbf{S}_2} : \mathbf{S}_1 \to \mathbf{S}_2$ is an isomorphism. Assume that $\mathbf{S}_1 = (S_1, \cdot)$ and $\mathbf{S}_2 = (S_2, \cdot)$. Thus we must prove that

$$(b) \qquad \psi_{\mathbf{S}_1,\mathbf{S}_2}(s)\psi_{\mathbf{S}_1,\mathbf{S}_2}(t) = \psi_{\mathbf{S}_1,\mathbf{S}_2}(st) \qquad \text{for all } s, t \in S_1.$$

Assume that $s, t \in S_1$. If $t \in r(\mathbf{S}_1)$ then, by Proposition 3.2(1), $st \in r(\mathbf{S}_1)$ and hence $t^2 = t$ and $(st)^2 = st$. Since $\psi_{\mathbf{S}_1,\mathbf{S}_2}(r(\mathbf{S}_1)) = r(\mathbf{S}_2)$ and $\psi_{\mathbf{S}_1,\mathbf{S}_2}$ is an isomorphism between subsemigroups $r(\mathbf{S}_1)$ and $r(\mathbf{S}_2)$ we deduce that $\psi_{\mathbf{S}_1,\mathbf{S}_2}(s)\psi_{\mathbf{S}_1,\mathbf{S}_2}(t) = \psi_{\mathbf{S}_1,\mathbf{S}_2}(st)$ whenever $\psi_{\mathbf{S}_1,\mathbf{S}_2}(s^2) = (\psi_{\mathbf{S}_1,\mathbf{S}_2}(s))^2$. Therefore we first prove that

$$(c) \qquad \psi_{\mathbf{S}_1,\mathbf{S}_2}(s^2) = (\psi_{\mathbf{S}_1,\mathbf{S}_2}(s))^2 \qquad \text{for all } s \in S_1.$$

If $s \in r(\mathbf{S}_1)$ then $\psi_{\mathbf{S}_1,\mathbf{S}_2}(s) \in r(\mathbf{S}_2)$ and (c) is true. If $s \in S_1 \setminus r(\mathbf{S}_1)$ is irreducible, then, by Proposition 3.2(8), there exists an idempotent endomorphism f of $\mathbf{S}_1$ with $\operatorname{Im}(f) = \{s, s^2\}$, by Corollary 3.3(3) and Proposition 3.2(8), there exists an irreducible $u \in S_2 \setminus r(\mathbf{S}_2)$ with $\operatorname{Im}(\phi(f)) = \{u, u^2\}$. By (a), $\psi_{\mathbf{S}_1,\mathbf{S}_2}(s), \psi_{\mathbf{S}_1,\mathbf{S}_2}(s^2) \in \operatorname{Im}(\phi(f))$. Since $\psi_{\mathbf{S}_1,\mathbf{S}_2}(s^2) \in r(\mathbf{S}_2)$ we conclude that $\psi_{\mathbf{S}_2,\mathbf{S}_2}(s) = u$ and $\psi_{\mathbf{S}_1,\mathbf{S}_2}(s^2) = u^2$, thus $\psi_{\mathbf{S}_1,\mathbf{S}_2}(s^2) = (\psi_{\mathbf{S}_1,\mathbf{S}_2}(s))^2$. If $s \in S_1 \setminus r(\mathbf{S}_1)$ is reducible, then, by Proposition 3.2(9), for every $u \in \mathcal{L}(\mathbf{S}_1)_s$ there exists an idempotent endomorphism f of $\mathbf{S}_1$ with $\operatorname{Im}(f) = \{s, s^2, u\}$. By Corollary 3.3(4) and Proposition 3.2(9), there exist a reducible $v \in S_2 \setminus r(\mathbf{S}_2)$ and $w \in \mathcal{L}(\mathbf{S}_2)_v$ with $\operatorname{Im}(\phi(f)) = \{v, v^2, w\}$. By (a), $\psi_{\mathbf{S}_1,\mathbf{S}_2}(s), \psi_{\mathbf{S}_1,\mathbf{S}_2}(s^2), \psi_{\mathbf{S}_1,\mathbf{S}_2}(u) \in \operatorname{Im}(\phi(f))$ and $\psi_{\mathbf{S}_1,\mathbf{S}_2}(s^2), \psi_{\mathbf{S}_1,\mathbf{S}_2}(u) \in r(\mathbf{S}_2)$, thus $\psi_{\mathbf{S}_1,\mathbf{S}_2}(s) = v$. Since $u < s^2$ and $w < v^2$ and since, by the hypothesis, $\psi_{\mathbf{S}_1,\mathbf{S}_2}$ is an isomorphism between $r(\mathbf{S}_1)$ and $r(\mathbf{S}_2)$ and therefore it is an isomorphism of posets we conclude that $\psi_{\mathbf{S}_1,\mathbf{S}_2}(s^2) = v^2$. Thus (c) is proved. Hence (b) is proved whenever $t \in r(\mathbf{S}_1)$.

Assume that $t \in S_1 \setminus r(\mathbf{S}_1)$. According to the identities $(xy)^2 = x^2y^2$ and $x^2y = xy$ we deduce that $(st)^2 = s^2t^2 = st^2$ and according to the foregoing part of the proof $(\psi_{\mathbf{S}_1,\mathbf{S}_2}(st))^2 = \psi_{\mathbf{S}_1,\mathbf{S}_2}(s)\psi_{\mathbf{S}_1,\mathbf{S}_2}(t^2)$ because $t^2 \in r(\mathbf{S}_1)$. Observe that if $s \in \mathcal{L}(\mathbf{S}_1)_u$ and $t \in \mathcal{R}(\mathbf{S}_1)_u$ for some $u \in S_1 \setminus r(\mathbf{S}_1)$ then there exists

$v \in \mathcal{L}(\mathbf{S}_1)_u$ with $vt = u$ and, by Proposition 3.2(2), $vu = u = su$. Hence

$$u = s^2u = s^2v^2t = v^2s^2t = v^2st = vst$$

and thus $st \in \mathcal{R}(\mathbf{S}_1)_u$ and, by Proposition 3.2(3), $st \notin r(\mathbf{S}_1)$ and $st \sqsubseteq_{\mathbf{S}_1} u$. From $s \in \mathcal{L}(\mathbf{S}_1)_{st}$ and $t \in \mathcal{R}(\mathbf{S}_1)_{st}$ it follows that

(d) $st \in S_1 \setminus r(\mathbf{S}_1)$ if and only if there exists $u \in S_1 \setminus r(\mathbf{S}_1)$ with $s \in \mathcal{L}(\mathbf{S}_1)_u$ and $t \in \mathcal{R}(\mathbf{S}_1)_u$; moreover, if $st \in S_1 \setminus r(\mathbf{S}_1)$ and $s \in \mathcal{L}(\mathbf{S}_1)_u$ and $t \in \mathcal{R}(\mathbf{S}_1)_u$ for some $u \in S_1 \setminus r(\mathbf{S}_1)$ then $st \sqsubseteq_{\mathbf{S}_1} u$.

From the definition of $\sqsubseteq$ it follows that if $\psi_{\mathbf{S}_1,\mathbf{S}_2}(\mathcal{R}(\mathbf{S}_1)_u) = \mathcal{R}(\mathbf{S}_2)_{\psi_{\mathbf{S}_1,\mathbf{S}_2}(u)}$ for all $u \in S_1 \setminus r(\mathbf{S}_1)$ then $u \sqsubseteq_{\mathbf{S}_1} v$ if and only if $\psi_{\mathbf{S}_1,\mathbf{S}_2}(u) \sqsubseteq \psi_{\mathbf{S}_1,\mathbf{S}_2}(v)$ for all $u, v \in S_1 \setminus r(\mathbf{S}_1)$. Thus if $\psi_{\mathbf{S}_1,\mathbf{S}_2}(\mathcal{R}(\mathbf{S}_1)_u) = \mathcal{R}(\mathbf{S}_2)_{\psi_{\mathbf{S}_1,\mathbf{S}_2}(u)}$ and $\psi_{\mathbf{S}_1,\mathbf{S}_2}(\mathcal{L}(\mathbf{S}_1)_u) = \mathcal{L}(\mathbf{S}_2)_{\psi_{\mathbf{S}_1,\mathbf{S}_2}(u)}$ for all $u \in S_1 \setminus r(\mathbf{S}_1)$, then (d) implies (b), and the proof will be complete.

We recall that $\psi_{\mathbf{S}_1,\mathbf{S}_2}(\{u \mid u \in S_1 \setminus r(\mathbf{S}_1) \text{ is reducible}\}) = \{u \mid u \in S_2 \setminus r(\mathbf{S}_2) \text{ is reducible}\}$. Let $u \in S_1 \setminus r(\mathbf{S}_1)$ be reducible. By Proposition 3.2(9), $v \in \mathcal{L}(\mathbf{S}_1)_u$ if and only if there exists an idempotent endomorphism f of $\mathbf{S}_1$ with $\operatorname{Im}(f) = \{u, u^2, v\}$. By Corollary 3.3(4), Proposition 3.2(9) and (10) and (a), then $\psi_{\mathbf{S}_1,\mathbf{S}_2}(v) \in \mathcal{L}(\mathbf{S}_2)_{\psi_{\mathbf{S}_1,\mathbf{S}_2}(u)}$ and, by the symmetry, we conclude that $\psi_{\mathbf{S}_1,\mathbf{S}_2}(\mathcal{L}(\mathbf{S}_1)_u) = \mathcal{L}(\mathbf{S}_2)_{\psi_{\mathbf{S}_1,\mathbf{S}_2}(u)}$. By Proposition 3.2(9), $w \in \mathcal{R}(\mathbf{S}_1)_u$ if and only if $f(w) = u$ for every idempotent endomorphism f of $\mathbf{S}_1$ with $u \in \operatorname{Im}(f)$ and $|\operatorname{Im}(f)| = 3$. By (a), then $\phi_{\mathbf{S}_1,\mathbf{S}_2}(f)(\psi_{\mathbf{S}_1,\mathbf{S}_2}(w)) = \psi_{\mathbf{S}_1,\mathbf{S}_2}(u)$ and, by Corollary 3.3(4) and Proposition 3.2(9), we conclude that $\psi_{\mathbf{S}_1,\mathbf{S}_2}(w) \in \mathcal{R}(\mathbf{S}_2)_{\psi_{\mathbf{S}_1,\mathbf{S}_2}(u)}$. By the symmetry, we obtain $\psi_{\mathbf{S}_1,\mathbf{S}_2}(\mathcal{R}(\mathbf{S}_1)_u = \mathcal{R}(\mathbf{S}_2)_{\psi_{\mathbf{S}_1,\mathbf{S}_2}(u)}$ and the proof of (b) is complete. Thus $\psi_{\mathbf{S}_1,\mathbf{S}_2} : \mathbf{S}_1 \to \mathbf{S}_2$ is an isomorphism and the variety $\operatorname{Var}(\mathbf{M}_1)$ is 3-determined.

Since $\operatorname{Var}(\mathbf{M}_1)$ contains all semilattices and since the variety of semilattices is not 2-determined, see [34], the proof is complete. □

Theorem 3.5 *The variety Var*$(\mathbf{M}_1)$ *is not (weakly) var-relatively universal.*

Proof. Let $\mathbf{S} = (S, \cdot) \in \operatorname{Var}(\mathbf{M}_1)$ be a semigroup generating the variety $\operatorname{Var}(\mathbf{M}_1)$. By Lemma 3.1, $\mathbf{S}$ satisfies the identities $x^2y = xy$ and $(xy)^2 = x^2y^2 = y^2x^2$ but it is not a commutative semigroup. Observe that if any element $s \in S \setminus r(\mathbf{S})$ is irreducible then $us = (us)^2 = u^2s^2 = s^2u^2 = (su)^2 = su$ for all $u \in S$, and hence $\mathbf{S}$ is a commutative semigroup. Therefore there exists a reducible $s \in S \setminus r(\mathbf{S})$. By Proposition 3.2(9), there exists an idempotent endomorphism f of $\mathbf{S}$ such that $\operatorname{Im}(f)$ is isomorphic to $\mathbf{M}_1$. Whence $\mathbf{S}$ is isomorphic to $\mathbf{M}_1$ whenever the subsemigroup $\operatorname{Im}(f)$ generates the variety $\operatorname{Var}(\mathbf{M}_1)$ only for the identity endomorphism f of $\mathbf{S}$. On the other hand, if $\operatorname{Var}(\mathbf{M}_1)$ is weakly var-relatively universal then, by Theorem 1.2, there exists a proper class of non-isomorphic semigroups such that $\operatorname{Im}(f)$ generates the variety $\operatorname{Var}(\mathbf{M}_1)$ only for the identity endomorphism f. Therefore $\operatorname{Var}(\mathbf{M}_1)$ is not weakly var-relatively universal. □

For a directed graph $\mathbf{G} = (X, R)$ let us define a groupoid $\Psi_0\mathbf{G} = (T(\mathbf{G}), \cdot)$ where $T(\mathbf{G}) = R \cup (X \times 5) \cup \{a, b, c, a', b', c', u, v, w, 0\}$ (assume $R \cap (X \times 5) = \emptyset$,

$\{a, b, c, a', b', c', u, v, w, 0\} \cap (R \cup (X \times 5)) = \emptyset$ and a, b, c, a', b', c', u, v, w, 0 are pairwise distinct elements), and

(m1) $a = a^2$, $a(x, y) = a(x, 0) = a(x, 2) = (x, 2)$ and $a(x, 1) = au = u$ for all $x \in X$ and all $(x, y) \in R$;
(m2) $b = b^2$, $b(x, y) = b(y, 1) = b(y, 3) = (y, 3)$ and $b(y, 0) = bv = v$ for all $y \in X$ and all $(x, y) \in R$;
(m3) $c = c^2$, $c(x, 0) = c(x, 1) = c(x, 4) = (x, 4)$ and $c(x, y) = cw = w$ for all $x \in X$ and all $(x, y) \in R$;
(m4) $(x, y)(x', y') = a'(x, y) = (x, y)a' = (a')^2 = a'$ for all $(x, y), (x', y') \in R$;
(m5) $(x, 0)(x', 0) = b'(x, 0) = (x, 0)b' = (b')^2 = b'$ for all $x, x' \in X$;
(m6) $(x, 1)(x', 1) = c'(x, 1) = (x, 1)c' = (c')^2 = c'$ for all $x, x' \in X$;
(m7) the product of all remaining pairs is 0.

The proposition below describes $\Psi_0\mathbf{G}$ for a directed graph $\mathbf{G}$.

Proposition 3.6 *For any digraph $\mathbf{G}$ the groupoid $\Psi_0\mathbf{G}$ belongs to the variety $Var(\mathbf{M}_1)$.*

Proof. For any directed graph $\mathbf{G} = (X, R)$ we construct a family $\mathcal{F}(\mathbf{G})$ of homomorphisms from $\Psi_0\mathbf{G}$ to $\mathbf{M}_1$ such that for any pair $\{x, y\}$ of distinct elements of $T(\mathbf{G})$ there exists $f \in \mathcal{F}(\mathbf{G})$ with $f(x) \neq f(y)$.

For $x \in R \cup (X \times 2)$ define a mapping $f_x : T(\mathbf{G}) \to \{1, a, 0\}$ so that $f_x(x) = a$ and $f_x(y) = 0$ for $y \in T(\mathbf{G}) \setminus \{x\}$. Since x is irreducible, it is easy to see that $f_x : \Psi_0\mathbf{G} \to \mathbf{M}_1$ is a homomorphism. For $x \in X$ and $i = 2, 3, 4$ define mappings $f_{(x,i)} : T(\mathbf{G}) \to \{1, a, 0\}$ as follows:

$$f_{(x,2)}(z) = \begin{cases} 1 & \text{if } z = a, \\ a & \text{if } z = (x, y) \in R, \\ a & \text{if } z = (x, i) \text{ for } i = 0, 2, \\ 0 & \text{else,} \end{cases} \qquad f_u(z) = \begin{cases} 1 & \text{if } z = a, \\ a & \text{if } z = u, \\ a & \text{if } z = (y, 1) \in X \times \{1\}, \\ 0 & \text{else,} \end{cases}$$

$$f_{(x,3)}(z) = \begin{cases} 1 & \text{if } z = b, \\ a & \text{if } z = (y, x) \in R, \\ a & \text{if } z = (x, i) \text{ for } i = 1, 3, \\ 0 & \text{else,} \end{cases} \qquad f_v(z) = \begin{cases} 1 & \text{if } z = b, \\ a & \text{if } z = v, \\ a & \text{if } z = (y, 0) \in X \times \{0\}, \\ 0 & \text{else,} \end{cases}$$

$$f_{(x,4)}(z) = \begin{cases} 1 & \text{if } z = c, \\ a & \text{if } z = (x, i) \text{ for } i \in \{0, 1, 4\}, \\ 0 & \text{else,} \end{cases} \quad f_w(z) = \begin{cases} 1 & \text{if } z = c, \\ a & \text{if } z = w, \\ a & \text{if } z = (y, y') \in R, \\ 0 & \text{else.} \end{cases}$$

Since $\{a\}$, $\{b\}$ and $\{c\}$ are subgroupoids of $\Psi_0\mathbf{G}$, by (m1)-(m7) it follows that the maps $f_{(x,i)}$ for $x \in X$ and $i \in \{2, 3, 4\}$ and f_u, f_v, f_w are homomorphisms from $\Psi_0\mathbf{G}$ to $\mathbf{M}_1$. For $x \in \{a', b', c'\}$ define a mapping $f_x : T(\mathbf{G}) \to \{1, a, 0\}$ by

$$f_x(z) = \begin{cases} 1 & \text{if } z^2 = x, \\ 0 & \text{else.} \end{cases}$$

A direct calculation demonstrates that $f_x : \Psi_0\mathbf{G} \to \mathbf{M}_1$ is a homomorphism for $x \in \{a', b', c'\}$. Clearly $f_x^{-1}(a) = \{x\}$ for $x \in R \cup (X \times 2)$, $(x,i) \in f_{(x,i)}^{-1}(a) \subseteq \{(x,i)\} \cup R \cup (X \times 2)$ for $x \in X$, $i \in \{2,3,4\}$, $x \in f_x^{-1}(a) \subseteq \{x\} \cup R \cup (X \times 2)$ for $x \in \{u,v,w\}$, $f_u^{-1}(1) = \{a\}$, $f_v^{-1}(1) = \{b\}$, $f_w^{-1}(1) = \{c\}$ and $x \in f_x^{-1}(1) \subseteq \{x\} \cup R \cup (X \times 2)$ for $x \in \{a', b', c'\}$. Whence homomorphisms from $\Psi_0\mathbf{G}$ to $\mathbf{M}_1$ separate elements of $\Psi_0\mathbf{G}$. Thus $\Psi_0\mathbf{G}$ is a subdirect power of $\mathbf{M}_1$ and therefore $\Psi_0\mathbf{G} \in \mathrm{Var}(\mathbf{M}_1)$. □

If $\mathbf{G} = (X,R)$ and $\mathbf{G}' = (X', R')$ are directed graphs then for a compatible mapping $f : \mathbf{G} \to \mathbf{G}'$ define a mapping $\Psi_0 f : T(\mathbf{G}) \to T(\mathbf{G}')$ by

$$\Psi_0 f(z) = \begin{cases} z & \text{if } z \in \{a,b,c,a',b',c',u,v,w,0\}, \\ (f(x), f(y)) & \text{if } z = (x,y) \in R, \\ (f(x), i) & \text{if } z = (x,i) \text{ for } x \in X \text{ and } i \in 5. \end{cases}$$

Since f is compatible we deduce that $(f(x), f(y)) \in R'$ for all $(x,y) \in R$, and therefore $\Psi_0 f$ is correctly defined.

Lemma 3.7 *If $\mathbf{G}$ and $\mathbf{G}'$ are directed graphs and if $f : \mathbf{G} \to \mathbf{G}'$ is a compatible mapping, then $\Psi_0 f : \Psi_0\mathbf{G} \to \Psi_0\mathbf{G}'$ is a semigroup homomorphism.*

Proof. Clearly, $\Psi_0 f(R) \subseteq R'$ and $\Psi_0(X \times \{i\}) \subseteq (X' \times \{i\})$ for all $i \in 5$. Then (m1)–(m7) complete the proof because $\Psi_0 f(z) = z$ for all $z \in \{a,b,c,a',b',c',u, v,w,0\}$. □

As a consequence we immediately obtain

Statement 3.8 *Ψ_0 is a faithful functor from the category of all directed graphs and compatible mappings into Var$(\mathbf{M}_1)$.* □

Consider the category $\mathbb{DG}_s$ from Theorem 2.6. Then for every $\mathbf{G} = (X,R) \in \mathbb{DG}_s$ there exists $a_\mathbf{G} \in X$ such that $f(a_\mathbf{G}) = a_{\mathbf{G}'}$ for every compatible mapping $f : \mathbf{G} \to \mathbf{G}' \in \mathbb{DG}_s$. Define a functor Ψ_1 from $\mathbb{DG}_s$ into the 3-expansion $\mathbb{V}$ of $\mathrm{Var}(\mathbf{M}_1)$ so that for $\mathbf{G} \in \mathbb{DG}_s$ we set $\Psi_1\mathbf{G} = (\Psi_0\mathbf{G}, \xi_i, i = 0,1,2)$ where ξ_i is a nullary operation satisfying $\xi_i(0) = (a_\mathbf{G}, i+2)$ for $i = 0,1,2$. Since $\Psi_0 f(a_\mathbf{G}, i) = (a_{\mathbf{G}'}, i)$ for every compatible mapping $f : \mathbf{G} \to \mathbf{G}'$ and every $i \in 5$, we deduce that $\Psi_1 f = \Psi_0 f$ is a homomorphism from $\Psi_1\mathbf{G}$ into $\Psi_1\mathbf{G}'$ for every compatible mapping $f : \mathbf{G} \to \mathbf{G}' \in \mathbb{DG}_s$. We can summarize as follows.

Corollary 3.9 *$\Psi_1 : \mathbb{DG}_s \to \mathbb{V}$ is a faithful functor for the 3-expansion variety $\mathbb{V}$ of Var$(\mathbf{M}_1)$.* □

To prove that Ψ_1 is full consider graphs $\mathbf{G} = (X,R), \mathbf{G}' = (X', R') \in \mathbb{DG}_s$ and a homomorphism $f : \Psi_1\mathbf{G} \to \Psi_1\mathbf{G}'$.

Lemma 3.10 *$f(z) = z$ for every $z \in \{a,b,c,a',b',c',u,v,w,0\}$, $f(R) \subseteq R'$, and there exists a mapping $g : X \to X'$ such that $f(x,i) = (g(x), i)$ for all $x \in X$ and $i \in 5$.*

Proof. Observe that $x(a_{\mathbf{G}}, 2) = (a_{\mathbf{G}}, 2)$ for $x \in T(\mathbf{G})$ if and only if $x = a$, $x(a_{\mathbf{G}}, 3) = (a_{\mathbf{G}}, 3)$ for $x \in T(\mathbf{G})$ if and only if $x = b$ and $x(a_{\mathbf{G}}, 4) = (a_{\mathbf{G}}, 4)$ for $x \in T(\mathbf{G})$ if and only if $x = c$. Since f preserves nullary operations we conclude that $f(a_{\mathbf{G}}, i) = (a_{\mathbf{G}'}, i)$ for $i = 2, 3, 4$ and whence $f(a) = a$, $f(b) = b$ and $f(c) = c$. From $(a_{\mathbf{G}}, 2)^2 = 0$ it follows that $f(0) = 0$. Since $ax = (a_{\mathbf{G}}, 2)$ and $cx = (a_{\mathbf{G}}, 4)$ for $x \in T(\mathbf{G})$ if and only if $x = (a_{\mathbf{G}}, 0)$ and since $bx = (a_{\mathbf{G}}, 3)$ and $cx = (a_{\mathbf{G}}, 4)$ for $x \in T(\mathbf{G})$ if and only if $x = (a_{\mathbf{G}}, 1)$, we obtain $f(a_{\mathbf{G}}, 0) = (a_{\mathbf{G}'}, 0)$ and $f(a_{\mathbf{G}}, 1) = (a_{\mathbf{G}'}, 1)$. Since $(a_{\mathbf{G}}, 0)^2 = b'$, $(a_{\mathbf{G}}, 1)^2 = c'$, $a(a_{\mathbf{G}}, 1) = u$, and $b(a_{\mathbf{G}}, 0) = v$ we conclude that $f(b') = b'$, $f(c') = c'$, $f(u) = u$, and $f(v) = v$. Furthermore, $x^2 = b'$ and $ax = u$ for $x \in T(\mathbf{G})$ if and only if $x \in X \times \{1\}$ and $x^2 = c'$ and $bx = v$ for $x \in T(\mathbf{G})$ if and only if $x \in X \times \{0\}$. Whence $f(X \times \{i\}) \subseteq X' \times \{i\}$ for $i = 0, 1$. Note that $a(x, 0) = (x, 2)$, $c(x, 0) = c(x, 1) = (x, 4)$ and $b(x, 1) = (x, 3)$ for all $x \in X$. Then $f(a) = a$, $f(b) = b$, $f(c) = c$ and $f(X \times \{i\}) \subseteq X' \times \{i\}$ for $i = 0, 1$ imply the existence of a mapping $g : X \to X'$ such that $f(x, i) = (g(x), i)$ for all $x \in X$ and $i \in 5$. Finally, $ax \in X \times \{0\}$ and $bx \in X \times \{1\}$ for $x \in T(\mathbf{G})$ if and only if $x \in R$ and thus $f(R) \subseteq R'$. Since $z^2 = a'$ and $cz = w$ for all $z \in R$ we obtain that $f(a') = a'$ and $f(w) = w$. □

Theorem 3.11 *Ψ_1 is a full embedding of $\mathbb{DG}_s$ into the 3-expansion $\mathbb{V}$ of Var$(\mathbf{M}_1)$ which preserves finiteness.*

Proof. Let $\mathbf{G} = (X, R)$ and $\mathbf{G}' = (X', R')$ be digraphs from $\mathbb{DG}_s$ and let $f : \Psi_1\mathbf{G} \to \Psi_1\mathbf{G}'$ be a homomorphism. By Lemma 3.10, there exists a mapping $g : X \to X'$ such that $f(x, i) = (g(x), i)$ for all $x \in X$ and all $i \in 5$ and $f(z) = z$ for all $z \in \{a, b, c, a', b', c', u, v, w, 0\}$. Observe that there exists $z \in T(\mathbf{G})$ such that $az = (x, 2)$ and $bz = (y, 3)$ for $x, y \in X$ if and only if $z = (x, y) \in R$. Then $af(z) = (g(x), 2)$ and $bf(z) = (g(y), 3)$ and hence $f(z) = (g(x), g(y)) \in R'$. Thus $g : \mathbf{G} \to \mathbf{G}'$ is a compatible mapping with $\Psi_1 g = f$. Therefore, by Corollary 3.9, Ψ_1 is a full embedding. From the definition of Ψ_0 it follows that Ψ_0 and hence also Ψ_1 preserve the finiteness. □

Theorem 2.6 provides the following corollary.

Corollary 3.12 *The variety Var$(\mathbf{M}_1)$ has an ff-alg-universal 3-expansion.* □

The first statement of Theorem 1.7 is a consequence of Theorems 3.4, 3.5 and Corollary 3.12.

Remark. By Theorem 1.2 and Proposition 3.2(7), (8) and (9), the 1-expansion of Var$(\mathbf{M}_1)$ is not alg-universal because it has only finitely many non-isomorphic rigid algebras. If $\mathbf{S} = (S, \cdot, \xi_0, \xi_1)$ is a rigid algebra from the 2-expansion of Var$(\mathbf{M}_1)$ then, by Proposition 3.2, S is finite whenever either both $\xi_0(0)$ and $\xi_1(0)$ are irreducible or $\xi_0(0), \xi_1(0) \in r(\mathbf{S})$. But it is an open problem whether the 2-expansion of Var$(\mathbf{M}_1)$ is alg-universal.

4 The variety $\mathrm{Var}(\mathbf{M}_2)$

The aim of this section is to investigate the semigroup variety generated by $\mathbf{M}_2$. First we construct a functor Φ_0 from $\mathbb{GRA}$ into $\mathrm{Var}(\mathbf{M}_2)$. For an undirected graph $\mathbf{G} = (V, E) \in \mathbb{GRA}$, the sets $U_0(\mathbf{G})$, $P_i(\mathbf{G})$ for $i = 0, 1, 2, 3$, $R(\mathbf{G})$ and p were defined in the second section. We use these sets in the construction of a groupoid $\Phi_0\mathbf{G}$. The groupoid $\Phi_0\mathbf{G}$ is on the set $U_1(\mathbf{G}) = U_0(\mathbf{G}) \cup R(\mathbf{G}) \cup \{u, 0\}$ (we assume that $U_0(\mathbf{G}) \cap R(\mathbf{G}) = \emptyset$, $u, 0 \notin U_0(\mathbf{G}) \cup R(\mathbf{G})$ and $u \neq 0$) and the binary operation is defined as follows:

(m8) $tw = 0$ for $t, w \in U_1(\mathbf{G})$ such that $t = w$ or $\{t, w\} \in P_0(\mathbf{G})$ or $\{t, w\} \cap (R(\mathbf{G}) \cup \{u, 0\}) \neq \emptyset$;
(m9) $tw = u$ for distinct $t, w \in U_0(\mathbf{G}) \subseteq U(\mathbf{G}_1)$ with $\{t, w\} \in P_1(\mathbf{G}) \cup P_2(\mathbf{G})$;
(m10) $tw = \{p(t), p(w)\} = p(\{t, w\})$ for distinct $t, w \in U_0(\mathbf{G}) \subseteq U_1(\mathbf{G})$ with $\{t, w\} \in P_3(\mathbf{G})$.

From Lemma 2.2(3) it follows that the definition of the binary operation is correct.

Proposition 4.1 $\Phi_0\mathbf{G} \in \mathit{Var}(\mathbf{M}_2)$ *for every graph* $\mathbf{G} \in \mathbb{GRA}$.

Proof. Assume that $\mathbf{G} = (V, E) \in \mathbb{GRA}$. Consider the Ree's quotient $\mathbf{S} = \mathcal{C}(U_0(\mathbf{G}))/I$ of the free commutative semigroup $\mathcal{C}(U_0(\mathbf{G}))$ over the set $U_0(\mathbf{G})$ by the ideal I generated by the set $\{x^2 \mid x \in U_0(\mathbf{G})\} \cup \{xyz \mid x, y, z \in U(\mathbf{G}_0)\}$. If we identify the ideal I with zero 0 then we can write $\mathbf{S} = (U_0(\mathbf{G}) \cup \mathfrak{P}_2(U_0(\mathbf{G})) \cup \{0\}, \cdot)$. Let $\sim$ be the least equivalence on the set $U_0(\mathbf{G}) \cup \mathfrak{P}_2(U_0(\mathbf{G})) \cup \{0\}$ such that $x \sim y$ just when $x, y \in P_1(\mathbf{G}) \cup P_2(\mathbf{G})$ or $x, y \in P_0(\mathbf{G}) \cup \{0\}$ or $x, y \in P_3(\mathbf{G})$ and $p(x) = p(y)$. It is easy to see that $\sim$ is a congruence of $\mathbf{S}$ such that $\Phi_0\mathbf{G}$ is isomorphic to $\mathbf{S}/\sim$. A verification that the mapping $f_{x,y} : U_0(\mathbf{G}) \cup \mathfrak{P}_2(U_0(\mathbf{G})) \cup \{0\} \to \{a, b, c, 0\}$ defined for distinct $x, y \in U_0(\mathbf{G})$ by

$$f_{x,y}(z) = \begin{cases} a & \text{if } z = x, \\ b & \text{if } z = y, \\ c & \text{if } z = \{x, y\}, \\ 0 & \text{if } z \neq x, y, \{x, y\} \end{cases}$$

is a homomorphism from $\mathbf{S}$ to $\mathbf{M}_2$ is straightforward. Since the family $\{f_{x,y} \mid x, y \in U_0(\mathbf{G}),\ x \neq y\}$ separates elements of $\mathbf{S}$ we conclude that $\mathbf{S}$ is a subdirect power of $\mathbf{M}_2$, thus its quotient $\Phi_0\mathbf{G}$ belongs to $\mathrm{Var}(\mathbf{M}_2)$. □

For a compatible mapping $f : \mathbf{G} \to \mathbf{G}'$ where $\mathbf{G} = (V, E), \mathbf{G}' = (W, D) \in \mathbb{GRA}$ define a mapping $\Phi_0 f : U_1(\mathbf{G}) \to U_1(\mathbf{G}')$ by

$$\Phi_0 f(x) = \begin{cases} (f(v), i) & \text{if } x = (v, i) \text{ for } v \in V \text{ and } i \in 3, \\ x_i & \text{if } x = x_i \in X \text{ for some } i \in 9, \\ \{f(v), f(w)\} & \text{if } x = \{v, w\} \in R(\mathbf{G}),\ f(v) \neq f(w),\ \{f(v), f(w)\} \notin D, \\ u & \text{if } x = u \text{ or } x = \{v, w\} \in R(\mathbf{G}) \text{ with } \{f(v), f(w)\} \in D, \\ 0 & \text{if } x = 0 \text{ or } x = \{v, w\} \in R(\mathbf{G}) \text{ with } f(v) = f(w). \end{cases}$$

Lemma 4.2 *If $f : \mathbf{G} \to \mathbf{G}'$ is a compatible mapping from $\mathbb{GRA}$ then $\Phi_0 f : \Phi_0\mathbf{G} \to \Phi_0\mathbf{G}'$ is a semigroup homomorphism.*

Proof. First consider $x, y \in U_0(\mathbf{G}) \subseteq U_1(\mathbf{G})$. Then $xy = 0$ just when $x = y$ or $\{x, y\} \in P_0(\mathbf{G})$. Since $\Phi_0 f(x_i) = x_i$ for all $i \in 9$ and since $\Phi_0 f(v, i) = (f(v), i)$ for all $v \in V$ and $i \in 3$ we conclude that $\{\Phi_0 f(x), \Phi_0 f(y)\} \in P_0(\mathbf{G}')$ whenever $\{x, y\} \in P_0(\mathbf{G})$. Thus $\Phi_0 f(x)\Phi_0 f(y) = \Phi_0 f(xy)$ for all $x, y \in U_0(\mathbf{G})$ with $xy = 0$ because $\Phi_0 f(0) = 0$. Further $xy = u$ just when $x \neq y$ and $\{x, y\} \in P_1(\mathbf{G}) \cup P_2(\mathbf{G})$. Analogously as for $P_0(\mathbf{G})$ we obtain that $\{\Phi_0 f(x), \Phi_0 f(y)\} \in P_1(\mathbf{G}')$ whenever $\{x, y\} \in P_1(\mathbf{G})$. Since $\Phi_0 f(V \times \{i\}) \subseteq W \times \{i\}$ for $i \in 3$ and since f is a compatible mapping we deduce that $\{\Phi_0 f(x), \Phi_0 f(y)\} \in P_2(\mathbf{G}')$ whenever $\{x, y\} \in P_2(\mathbf{G})$. Thus $\Phi_0 f(x)\Phi_0 f(y) = \Phi_0 f(xy)$ for all $x, y \in U_0(\mathbf{G})$ with $xy = u$ because $\Phi_0 f(u) = u$. By Lemma 2.2, $\{x, y\} \in P_3(\mathbf{G})$ in the remaining case. Then there exist distinct $v, w \in V$ and $i, j \in 3$ such that $\{v, w\} \notin E$, $|i - j| \leq 1$ and $x = (v, i)$, $y = (w, j)$. We have $(v, i)(w, j) = \{v, w\} = p(\{x, y\}) \in R(\mathbf{G})$, $\Phi_0 f(x) = (f(v), i)$ and $\Phi_0 f(y) = (f(w), j)$. If $f(v) = f(w)$ then $\Phi_0 f(\{v, w\}) = 0$ and

$$\Phi_0 f(x)\Phi_0 f(y) = (f(v), i)(f(w), j) = 0 = \Phi_0 f(\{v, w\}) = \Phi_0 f(xy).$$

Assume that $f(v) \neq f(w)$ and $\{f(v), f(w)\} \in D$. Then $\{\Phi_0 f(x), \Phi_0 f(y)\} \in P_2(\mathbf{G}')$ and $\Phi_0 f(\{v, w\}) = u$. Thus

$$\Phi_0 f(x)\Phi_0 f(y) = (f(v), i)(f(w), j) = u = \Phi_0 f(\{v, w\}) = \Phi_0 f(xy).$$

Assume that $f(v) \neq f(w)$ and $\{f(v), f(w)\} \notin D$. Then $\Phi_0 f(\{v, w\}) = \{f(v), f(w)\} \in R(\mathbf{G}') \subseteq U_1(\mathbf{G}')$ and $\{\Phi_0 f(x), \Phi_0 f(y)\} \in P_3(\mathbf{G}')$. Thus

$$\Phi_0 f(x)\Phi_0 f(y) = (f(v), i)(f(w), j) = \{f(v), f(w)\} = \Phi_0 f(\{v, w\}) = \Phi_0 f(xy).$$

We can summarize that $\Phi_0 f(x)\Phi_0 f(y) = \Phi_0 f(xy)$ for all $x, y \in U_0(\mathbf{G})$.

Since $xy = 0$ whenever $x \in R(\mathbf{G}) \cup \{u, 0\}$ or $y \in R(\mathbf{G}) \cup \{u, 0\}$ and since $\Phi_0 f(R(\mathbf{G}) \cup \{u, 0\}) \subseteq R(\mathbf{G}') \cup \{u, 0\}$ we conclude that $\Phi_0 f(x)\Phi_0(y) = 0 = \Phi_0 f(0) = \Phi_0 f(xy)$ whenever $x \in R(\mathbf{G}) \cup \{u, 0\}$ or $y \in R(\mathbf{G}) \cup \{u, 0\}$. Thus $\Phi_0 f$ is a semigroup homomorphism from $\Phi_0\mathbf{G}$ to $\Phi_0\mathbf{G}'$. $\square$

Observe that for $\mathbf{G} \in \mathbb{GRA}$, the subsemigroup of $\Phi_0\mathbf{G}$ generated by the set $\{x_1, x_3\}$ is isomorphic to $\mathbf{M}_2$ (the underlying set is $\{x_1, x_3, u, 0\}$). Thus for every compatible mapping $f \in \mathbb{GRA}$, $\mathbf{M}_2$ is isomorphic to a subsemigroup of $\mathrm{Im}(\Phi_0 f)$. Hence we immediately obtain

Corollary 4.3 *Φ_0 is an embedding of $\mathbb{GRA}$ into Var$(\mathbf{M}_2)$ such that $\Phi_0 f(u) = u$ and $\Phi_0 f \notin \mathcal{Z}_{Var(\mathbf{M}_2')}$ for every compatible mapping $f \in \mathbb{GRA}$.* $\square$

To complete our results about $\mathrm{Var}(\mathbf{M}_2)$, we investigate semigroup homomorphisms from $\Phi_0\mathbf{G}$ to $\Phi_0\mathbf{G}'$.

Lemma 4.4 *If $f : \Phi_0\mathbf{G} \to \Phi_0\mathbf{G}'$ is a semigroup homomorphism for graphs $\mathbf{G} = (V, E)$ and $\mathbf{G}' = (W, D)$ from $\mathbb{GRA}$ with $f(u) = u$, then $f(0) = 0$, $f(x_i) = x_i$ for all $i \in 9$, and there exists a mapping $g : V \to W$ with $f(v, i) = (g(v), i)$ for all $v \in V$ and $i = 0, 1, 2$.*

Proof. From $f(u) = u$ it follows that $f(0) = 0$ because $u^2 = 0$. By Lemma 2.2, either $\{f(x_i) \mid i \in 5\} = \{x_i \mid i \in 5\}$ or $|\{f(x_i) \mid i \in 5\}| < 5$. If $|\{f(x_i) \mid i \in 5\}| < 5$ then, by Lemma 2.3, there exist $i, j \in 5$ such that $i - j \not\equiv -1, 0, 1 \bmod 5$ and either $f(x_i) = f(x_j)$ or $f(x_i) = f(x_k)$ and $f(x_j) = f(x_{k+1 \bmod 5})$ for some $k \in 5$. Thus $\{x_i, x_j\} \in P_1(\mathbf{G})$ and therefore $x_i x_j = u$. If $f(x_i) = f(x_j)$ then $f(u) = f(x_i x_j) = f(x_i)f(x_j) = 0$ – a contradiction. If $f(x_i) = f(x_k)$ and $f(x_j) = f(x_{k+1 \bmod 5})$ then $\{x_k, x_{k+1 \bmod 5}\} \in P_0(\mathbf{G})$ and hence

$$f(u) = f(x_i x_j) = f(x_i)f(x_j) = f(x_k)f(x_{k+1 \bmod 5}) = f(x_k x_{k+1 \bmod 5}) = f(0) = 0$$

again a contradiction. Thus $\{f(x_i) \mid i \in 5\} = \{x_i \mid i \in 5\}$. Since V is a non-empty set we conclude, by Lemma 2.2(1), that $C = \{x_0\} \cup \{x_i \mid i \in \{4, 5, \ldots, 8\}\} \cup \{(v, i) \mid i \in 3\}$ is a cycle in $(U_0(\mathbf{G}), P_0(\mathbf{G}))$ for every $v \in V$. By Lemma 2.3, if $|f(C)| < 9$ then there exist $x, y \in C$ such that $x \neq y$, $\{x, y\} \notin P_0(\mathbf{G})$ and either $f(x) = f(y)$ or $f(x) = f(x')$ and $f(y) = f(y')$ for some $\{x', y'\} \in P_0(\mathbf{G})$. Then $xy = u$ and analogously as above $f(x)f(y) = 0$, this is a contradiction. Thus f is one-to-one on the set C and hence $f(C)$ is a cycle (of length 9) in $(U_0(\mathbf{G}'), P_0(\mathbf{G}'))$. By Lemma 2.2(1), there exists $w_v \in W$ such that

$$f(\{x_0\} \cup \{(v, i) \mid i \in 3\} \cup \{x_j \mid j = 4, \ldots, 8\}) =$$

$$= \{x_0\} \cup \{(w_v, i) \mid i \in 3\} \cup \{x_j \mid j = 4, \ldots, 8\}.$$

Define a mapping $g : V \to W$ such that $g(v) = w_v$ for all $v \in V$. Since the intersection of cycles

$$(x_0, x_1, x_2, x_3, x_4, x_0) \quad \text{and} \quad (x_0, x_4, x_5, x_6, x_7, x_8, (v, 2), (v, 1), (v, 0), x_0)$$

is the set $\{x_0, x_4\}$ we conclude that $f(\{x_0, x_4\}) = \{x_0, x_4\}$. We prove that $f(x_0) = x_0$. If $f(x_0) = x_4$ then $f(v, 0) = x_5$, for all $v \in V$. Since $\emptyset \neq E$, there exists $\{v, w\} \in E$, then $(v, 0)(w, 0) = u$ but $u = f(u) = f(v, 0)f(w, 0) = x_5 x_5 = 0$ – a contradiction. Thus $f(x_0) = x_0$ and $f(x_4) = x_4$. Since f is injective on the sets $\{x_i \mid i \in 5\}$ and $\{x_i \mid i = 0, 4, 5, 6, 7, 8\} \cup \{(v, i) \mid i \in 3\}$ for all $v \in V$ and since $f(0) = 0$ we conclude, by Lemma 2.2(1), that $f(x_i) = x_i$ for all $i \in 9$ and $f(v, i) = (g(v), i)$ for all $v \in V$ and $i \in 3$. □

Lemma 4.5 *If $f : \Phi_0\mathbf{G} \to \Phi_0\mathbf{G}'$ is a semigroup homomorphism for graphs $\mathbf{G} = (V, E)$ and $\mathbf{G}' = (W, D)$ from $\mathbb{GRA}$ with $f(u) = u$, then there exists a compatible mapping $g : \mathbf{G} \to \mathbf{G}'$ with $\Phi_0 g = f$.*

Proof. By Lemma 4.4, there exists a mapping $g : V \to W$ such that $f(x_i) = x_i$ for all $i \in 9$, $f(u) = u$, $f(0) = 0$ and $f(v, i) = (g(v), i)$ for all $v \in V$ and $i \in 3$. Assume that $v, w \in V$ with $\{v, w\} \in E$. Then $(v, 0)(w, 1) = u$ and hence

$$u = f(u) = f((v, 0)(w, 1)) = f(v, 0)f(w, 1) = (g(v), 0)(g(w, 1)).$$

Hence $\{(g(v), 0), (g(w), 1)\} \in P_2(\mathbf{G}')$ and thus $\{g(v), g(w)\} \in D$ and therefore $g : \mathbf{G} \to \mathbf{G}'$ is a compatible mapping. To prove that $f = \Phi_0 g$ consider $\{v, w\} \in R(\mathbf{G})$. Then $\{v, w\} \notin E$ and $(v, 0)(w, 1) = \{v, w\}$. If $g(v) = g(w)$ then $\{f(v, 0), f(w, 1)\} \in P_0(\mathbf{G}')$ and hence

$$f(\{v, w\}) = f((v, 0)(w, 1)) = f(v, 0)f(w, 1) = (g(v), 0)(g(w), 1) = 0 = \Phi_0 g(\{v, w\}).$$

If $g(v) \neq g(w)$ then

$$f(\{v, w\}) = f((v, 0)(w, 1)) = f(v, 0)f(w, 1) = (g(v), 0)(g(w), 1).$$

If $\{g(v), g(w)\} \in D$ then $f(\{v, w\}) = (g(v), 0)(g(w), 1) = u = \Phi_0 g(\{v, w\})$, and if $\{g(v), g(w)\} \notin D$ then $f(\{v, w\}) = (g(v), 0)(g(w), 1) = \{g(v), g(w)\} = \Phi_0 g(\{v, w\})$. Thus $f(x) = \Phi_0 g(x)$ for all $x \in U_1(\mathbf{G})$, and the proof is complete. $\square$

Lemma 4.6 *If $f : \Phi_0\mathbf{G} \to \Phi_0\mathbf{G}'$ is a semigroup homomorphism for graphs $\mathbf{G} = (V, E)$ and $\mathbf{G}' = (W, D)$ from $\mathbb{GRA}$ such that $f(u) \neq u$, then $f(u) = 0$, $f(R(\mathbf{G})) = \{0\}$, and there exists $\{x, y\} \in P_0(\mathbf{G}')$ with $Im(f) \cap U_0(\mathbf{G}) \subseteq \{x, y\}$. Thus $Im(f)$ is a zero-semigroup.*

Proof. First we recall that $V \neq \emptyset \neq W$. Observe that for every $y \in R(\mathbf{G}) \cup \{u, 0\}$ there exist $v, w \in U_0(\mathbf{G})$ with $vw = y$. On the other hand, any $y \in U_0(\mathbf{G}')$ is irreducible in $\Phi_0\mathbf{G}'$ and hence we obtain that $f(R(\mathbf{G}) \cup \{u, 0\}) \cap U_0(\mathbf{G}') = \emptyset$. Since $y^2 = 0$ for all $y \in U_1(\mathbf{G})$ we conclude that $f(0) = 0$. Assume that $f(u) = \{v, w\} \in R(\mathbf{G}')$. Then $\{x_1, x_5\}, \{x_1, x_7\}, \{x_5, x_7\} \in P_1(\mathbf{G})$ and hence $x_1x_5 = x_1x_7 = x_5x_7 = u$. By (m10), $\{y \mid \exists z,\, yz = \{v, w\}\} = \{v, w\} \times 3 = Z_{v,w}$ and for $z, z' \in Z_{v,w}$ we have $zz' = \{v, w\}$ just when $\{z, z'\} \in P_3(\mathbf{G})$. Observe that $\{y, z\} \cap (\{v\} \times 3)$ and $\{y, z\} \cap (\{w\} \times 3)$ are singletons for all $\{y, z\} \in P_3(\mathbf{G}) \cap \mathfrak{P}_2(Z_{v,w})$. Thus $(Z_{v,w}, P_3(\mathbf{G}) \cap \mathfrak{P}_2(Z_{v,w}))$ is a bipartite graph but $f(x_1)$, $f(x_5)$ and $f(x_7)$ form a cycle of length 3 in $(Z_{v,w}, P_3(\mathbf{G}) \cap \mathfrak{P}_2(Z_{v,w}))$, this is a contradiction. Thus $f(u) \notin R(\mathbf{G}')$ and whence $f(u) = 0$. Assume that $f(\{v, w\}) = \{v', w'\}$ for some $\{v, w\} \in R(\mathbf{G})$ and $\{v', w'\} \in R(\mathbf{G}')$. Then, by (m9) and (m10), $(v, i)(w, j) = \{v, w\} = (w, j)(v, i)$ for $i, j \in 3$ with $|i - j| \leq 1$ and $(v, 0)(w, 2) = (v, 2)(w, 0) = (w, 0)(v, 2) = (w, 2)(v, 0) = u$. Since $(v, 1)(w, j) = \{v, w\}$ for all $j \in 3$, it follows that $f(v, 1)f(w, j) = f(\{v, w\}) = \{v', w'\}$ for all $j \in 3$. Thus we may assume that $f(v, 1) = (v', i)$ and $f(w, j) \in \{w'\} \times 3$ for all $j \in 3$ and, for an analogous reason, $f(v, j) \in \{v'\} \times 3$ for all $j \in 3$. Since $\{(v, i)(w, j) \mid i, j \in 3\} = \{\{v, w\}, u\}$ and $\{(v', i), (w', j) \mid i, j \in 3\} = \{\{v', w'\}, u\}$ we conclude that $f(u) \in \{\{v', w'\}, u\}$ and this is a contradiction with $f(u) = 0$. Thus $f(\{v,w\}) \notin R(\mathbf{G}')$. Assume that $f(\{v,w\}) = u$. By (m8) and (m9),

$$\begin{aligned} f(v,i)f(v,j) = {} & f(w,i)f(w,j) = f(v,0)f(w,2) = f(v,2)f(w,0) = f(w,0)f(v,2) \\ = {} & f(w,2)f(v,0) = 0 \end{aligned}$$

for all $i,j \in 3$. Since for every $i \in 3$ there exists some $j_i \in 3$ with $f(v,i)f(w,j_i) = u = f(\{v,w\}) = f(w,i)f(v,j_i)$, we must have $f(v,i), f(w,i) \in U_0(\mathbf{G}')$ for all $i \in 3$. Hence $(f(v,0), f(v,1), f(v,2), f(v,0))$ and $(f(w,0), f(w,1), f(w,2), f(w,0))$ are cycles of the graph $(U_0(\mathbf{G}'), P_0(\mathbf{G}'))$ and, by Lemma 2.2(1), $|\{f(v,i) \mid i \in 3\}|, |\{f(w,i) \mid i \in 3\}| \le 2$. Thus

$$(f(v,0), f(v,1), f(v,2), f(w,0), f(w,1), f(w,2), f(v,0))$$

is a cycle of length at most 4 in $(U_0(\mathbf{G}'), P_0(\mathbf{G}'))$, and, by Lemma 2.2(1), $|\{f(v,i), f(w,i) \mid i \in 3\}| \le 2$. From $(v,1)(w,j) = \{v,w\}$ for all $j \in 3$ it follows that $f(v,1) \ne f(w,j)$ for all $j \in 3$ and analogously we obtain that $f(w,1) \ne f(v,j)$ for all $j \in 3$. Hence $f(v,i) = f(v,j) \ne f(w,i) = f(w,j)$ for all $i,j \in 3$ and thus

$$f(u) = f((v,0)(w,2)) = f(v,0)f(w,2) = f(v,1)f(w,1) = f((v,1)(w,1)) = f(\{v,w\}),$$

this is a contradiction. Whence we conclude that $f(R(\mathbf{G}) \cup \{u,0\}) = \{0\}$ and $\mathrm{Im}(f)$ is a zero-semigroup. By (m8), if $x,y \in \mathrm{Im}(f) \cap U_0(\mathbf{G}')$ are distinct then $\{x,y\} \in P_0(\mathbf{G}')$. By Lemma 2.2(1), there exists no cycle of length 3 in $(U_0(\mathbf{G}'), P_0(\mathbf{G}'))$ and thus $|\mathrm{Im}(f) \cap U_0(\mathbf{G}')| \le 2$ and if $|\mathrm{Im}(f) \cap U_0(\mathbf{G}')| = 2$ then $\mathrm{Im}(f) \cap U_0(\mathbf{G}') \in P_0(\mathbf{G}')$. The proof is complete. □

It is obvious that any zero-semigroup belongs to $\mathrm{Var}(\mathbf{M}_2')$. This fact is used in the theorem below.

Theorem 4.7 *The variety $\mathrm{Var}(\mathbf{M}_2)$ has an ff-alg-universal 1-expansion and is $\mathrm{Var}(\mathbf{M}_2')$-relatively ff-alg-universal.*

Proof. By Corollary 4.3 and Lemmas 4.5 and 4.6, $\Phi_0 : \mathbb{GRA} \to \mathrm{Var}(\mathbf{M}_2)$ is $\mathrm{Var}(\mathbf{M}_2')$-relatively full embedding from $\mathbb{GRA}$ into $\mathrm{Var}(\mathbf{M}_2)$ preserving finiteness and, by Theorem 2.1, the variety $\mathrm{Var}(\mathbf{M}_2)$ is $\mathrm{Var}(\mathbf{M}_2')$-relatively ff-alg-universal.

To prove that the variety $\mathrm{Var}(\mathbf{M}_2)$ has an ff-alg-universal 1-expansion consider a functor Φ_1 from $\mathbb{GRA}$ into the 1-expansion of the variety $\mathrm{Var}(\mathbf{M}_2)$ such that $\Phi_1\mathbf{G} = (\Phi_0\mathbf{G}, \xi_{\Phi_1\mathbf{G}})$ where $\xi_{\Phi_1\mathbf{G}}(0) = u$ for every $\mathbf{G} \in \mathbb{GRA}$ and $\Phi_1 f = \Phi_0 f$ for every compatible mapping $f \in \mathbb{GRA}$. By Corollary 4.3 and Lemma 4.5, Φ_1 is a full embedding from $\mathbb{GRA}$ into the expansion of the variety $\mathrm{Var}(\mathbf{M}_2)$ by one nullary operation. Since Φ_1 preserves finiteness we conclude, using Theorem 2.1, that the variety $\mathrm{Var}(\mathbf{M}_2)$ has an ff-alg-universal 1-expansion. □

Theorem 4.8 *Var*$(\mathbf{M}_2)$ *is* α*-determined for no cardinal* α*.*

Proof. Consider a graph $\mathbf{G} = (V, E) \in \mathbb{GRA}$. Observe that $I = R(\mathbf{G}) \cup \{u, 0\}$ is an ideal of $\Phi_0(\mathbf{G})$ and the Ree's quotient $\Phi_0(\mathbf{G})/I$ of $\Phi_0(\mathbf{G})$ is a zero semigroup. Further, if $Z \subseteq R(\mathbf{G}) \cup \{x, y, u, 0\}$ for some $\{x, y\} \in P_0(\mathbf{G})$, then Z is a subsemigroup of $\Phi_0(\mathbf{G})$ isomorphic to a zero-semigroup. Thus a mapping $f : U_1(\mathbf{G}) \to U_1(\mathbf{G})$ such that $f(R(\mathbf{G}) \cup \{u, 0\}) = \{0\}$ and $\mathrm{Im}(f) \subseteq R(\mathbf{G}) \cup \{x, y, u, 0\}$ for some $\{x, y\} \in P_0(\mathbf{G})$ is a semigroup endomorphism of $\Phi_0(\mathbf{G})$. Conversely, by Lemma 4.6, if $f \in \mathrm{End}(\Phi_0(\mathbf{G}))$ with $f(u) \neq u$ then $f(R(\mathbf{G}) \cup \{u, 0\}) = \{0\}$ and $\mathrm{Im}(f) \subseteq R(\mathbf{G}) \cup \{x, y, u, 0\}$ for some $\{x, y\} \in P_0(\mathbf{G})$. Hence $E(\mathbf{G}) = \{f \in \mathrm{End}(\Phi_0\mathbf{G}) \mid f(u) \neq u\}$ is a subsemigroup of the monoid $\mathrm{End}(\Phi_0\mathbf{G})$.

Consider graphs $\mathbf{G} = (V, E), \mathbf{G}' = (W, D) \in \mathbb{GRA}$ such that $|V| = |W|$ and $|E| = |D|$. Then also $|R(\mathbf{G})| = |R(\mathbf{G}')|$. Choose bijections $\phi : V \to W$ and $\psi : R(\mathbf{G}) \to R(\mathbf{G}')$ and define a mapping $\kappa : U_1(\mathbf{G})\} \to U_1(\mathbf{G}')$ by

$$\kappa(z) = \begin{cases} z & \text{if } z \in \{x_i \mid i \in 9\} \cup \{u, 0\}, \\ (\phi(v), i) & \text{if } z = (v, i) \text{ for } v \in V \text{ and } i \in 3, \\ \psi(z) & \text{if } z \in R(\mathbf{G}). \end{cases}$$

Then for every $f \in \mathrm{End}(\Phi_0\mathbf{G})$ with $f(u) \neq u$ there exists a unique semigroup endomorphism $\phi_0(f) \in \mathrm{End}(\Phi_0\mathbf{G}')$ with $\phi_0(f)(u) \neq u$ and $\phi_0(f) \circ \kappa = \kappa \circ f$. Whence ϕ_0 is a semigroup isomorphism between $E(\mathbf{G})$ and $E(\mathbf{G}')$. If $\mathbf{G}$ and $\mathbf{G}'$ are rigid graphs then the extension ϕ of ϕ_0 such that ϕ maps the identity mapping of $\Phi_0\mathbf{G}$ to the identity mapping of $\Phi_0\mathbf{G}'$ is a monoid isomorphism between $\mathrm{End}(\Phi_0\mathbf{G})$ and $\mathrm{End}(\Phi_0\mathbf{G}')$. Since for every infinite cardinal α there exists a family $\{\mathbf{G}_i = (V_i, E_i) \mid i \in 2^\alpha\}$ of non-isomorphic rigid graphs from $\mathbb{GRA}$ with $|V_i| = |E_i| = \alpha$ for all $i \in 2^\alpha$ we conclude that for every infinite cardinal α there exists a family $\{\Phi_0\mathbf{G}_i \mid i \in 2^\alpha\}$ of non-isomorphic equimorphic semigroups from the variety $\mathrm{Var}(\mathbf{M}_2)$.

Whence $\mathrm{Var}(\mathbf{M}_2)$ is α-determined for no cardinal α. $\square$

Thus the proof of the second statement of Theorem 1.7 is complete.

5 The variety Var$(\mathbf{M}_3)$

The aim of this section is to investigate the least semigroup variety containing the semigroup $\mathbf{M}_3$. We shall construct a functor $\Gamma_0 : \mathbb{DG}_s \to \mathrm{Var}(\mathbf{M}_3)$ from the category $\mathbb{DG}_s$ defined in Theorem 2.6. For a digraph $\mathbf{G} = (X, R)$ with distinguished nodes $a_\mathbf{G}, b_\mathbf{G} \in X$, let $\Gamma_0\mathbf{G}$ be a groupoid on the set $Z_0(\mathbf{G}) = R \cup X \cup \{a, b, a_1, b_1, u, v\}$ (we assume that a, b, a_1, b_1, u, and v are pairwise distinct vertices with $\{a, b, a_1, b_1, u, v\} \cap (X \cup R) = \emptyset = X \cap R$) such that

(m11) $a(x, y) = x$, $b(x, y) = y$ for all $(x, y) \in R$;
(m12) $ax = u$, $bx = v$ for all $x \in X$;
(m13) $aa = ab = au = av = ab_1 = u$ and $bb = ba = bu = bv = ba_1 = v$;

(m14) $aa_1 = a_{\mathbf{G}}$ and $bb_1 = b_{\mathbf{G}}$ ($a_{\mathbf{G}}$ and $b_{\mathbf{G}}$ are determined by Theorem 2.6(3));
(m15) $st = s$ for all $s \in X \cup R \cup \{a_1, b_1, u, v\}$ and all $t \in Z_0(\mathbf{G})$.

The following lemma gives basic properties of our groupoid.

Lemma 5.1 *If $\mathbf{G} = (X, R) \in \mathbb{DG}_s$ then*

1. *the set of all left zeros of $\Gamma_0\mathbf{G}$ is the set $X \cup R \cup \{a_1, b_1, u, v\}$ and, moreover, $X \cup R \cup \{a_1, b_1, u, v\}$ is the greatest subgroupoid of $\Gamma_0\mathbf{G}$ that is a left-zero semigroup;*
2. *there exists no $z \in Z_0(\mathbf{G})$ such that $bz = u$ or $az = v$;*
3. *there exists no $z \in Z_0(\mathbf{G})$ such that $az \in R \cup \{a_1, b_1\}$ or $bz \in R \cup \{a_1, b_1\}$;*
4. *for every pair $x, y \in X$ there exists a sequence $x = x_0, x_1, \ldots, x_{2n} = y$ such that $x_{2i} \in X$, $x_{2i+1} \in R$ and $\{ax_{2i+1}, bx_{2i+1}\} = \{x_{2i}, x_{2i+2}\}$ for all $i = 0, 1, \ldots, n-1$;*
5. *if $az \in X$ then $z \in R \cup \{a_1\}$, if $bz \in X$ then $z \in R \cup \{b_1\}$;*
6. *$X = \{az \mid z \in R\} = \{bz \mid z \in R\}$;*
7. *$\Gamma_0\mathbf{G}$ is a semigroup from the variety $Var(\mathbf{M}_3)$;*
8. *the least congruence $\sim$ of $\Gamma_0\mathbf{G}$ with $u \sim v$ has exactly one non-singleton class $e = \{u, v\}$;*
9. *if T is a subsemigroup of $\Gamma_0\mathbf{G}$ with $T \cap \{a, b\} \neq \emptyset \neq T \cap R$ then T generates the variety $Var(\mathbf{M}_3)$;*
10. *$X \cup \{a, b, u, v\}$ is a subsemigroup of $\Gamma_0\mathbf{G}$ belonging to the variety $Var(\mathbf{M}'_3)$.*

Proof. By (m15), any element from the set $X \cup R \cup \{a_1, b_1, u, v\}$ is a left zero of $\Gamma_0\mathbf{G}$ and, by (m13), a and b are not idempotent elements of $\Gamma_0\mathbf{G}$, hence (1) follows. From (m11)–(14) it follows that $\{az \mid z \in Z_0(\mathbf{G})\} = \{x \in X \mid \exists(x, y) \in R\} \cup \{u, a_{\mathbf{G}}\}$ and $\{bz \mid z \in Z_0(\mathbf{G})\} = \{y \in X \mid \exists(x, y) \in R\} \cup \{v, b_{\mathbf{G}}\}$. The statements (2) and (3) are consequences of these equalities. By (m12) and (m13) $ax = aa = ab = au = av = ab_1 = u$ and $bx = bb = ba = bu = bv = ba_1 = v$ for all $x \in X$. Thus the statements (5) and (6) follow from (m11) and (m14), and the statement (4) is a consequence of (m11) and the fact that $\mathbf{G}$ is strongly connected (indeed, if $x, y \in X$ then there exists a sequence $x = y_0, y_1, \ldots, y_k = y$ of nodes of $\mathbf{G}$ with $(y_i, y_{i+1}) \in R$ for all $i = 0, 1, \ldots, k-1$, set $x_{2i} = y_i$ and $x_{2i+1} = (y_i, y_{i+1})$). To prove (7) consider a groupoid $\mathbf{G}_1$ on the set $Z'(\mathbf{G}) = (R \times 3) \cup \{a, b, u, v, a_1, a_2, b_1, b_2\}$ such that

$a(r, 0) = (r, 1)$, $b(r, 0) = (r, 2)$ for all $r \in R$ and $aa_1 = a_2$, $bb_1 = b_2$;
$a(r, i) = aa_2 = aa = ab = ab_1 = ab_2 = au = av = u$ and $b(r, i) = bb_2 = bb = ba = ba_1 = ba_2 = bu = bv = v$ for all $r \in R$ and $i = 1, 2$;
$zt = z$ for all $z \in Z'(\mathbf{G}) \setminus \{a, b\}$ and $t \in Z'(\mathbf{G})$.

For $r \in R$, let $f_r, g_r : Z'(\mathbf{G}) \to \{a, b, c, d\}$ be mappings such that

$$f_r(z) = \begin{cases} d & \text{if } z = a, \\ c & \text{if } z = (r, 0), \\ b & \text{if } z = (r, 1), \\ a & \text{if } z \neq a, (r, 0), (r, 1), \end{cases} \qquad g_r(z) = \begin{cases} d & \text{if } z = b, \\ c & \text{if } z = (r, 0), \\ b & \text{if } z = (r, 2), \\ a & \text{if } z \neq b, (r, 0), (r, 2), \end{cases}$$

and let $f', g', h' : Z'(\mathbf{G}) \to \{a,b,c,d\}$ be mappings such that

$$f'(z) = \begin{cases} d & \text{if } z = a, \\ c & \text{if } z = a_1, \\ b & \text{if } z = a_2, \\ a & \text{if } z \neq a, a_1, a_2, \end{cases} \qquad g'(z) = \begin{cases} d & \text{if } z = b, \\ c & \text{if } z = b_1, \\ b & \text{if } z = b_2, \\ a & \text{if } z \neq b, b_1, b_2, \end{cases}$$

$$h'(z) = \begin{cases} b & \text{if } z \in \{b, b_2, v\} \cup R \times \{2\}, \\ a & \text{if } z \notin \{b, b_2, v\} \cup R \times \{2\}. \end{cases}$$

By a direct verification, f_r and g_r for all $r \in R$ and f', g' and h' are homomorphisms from $\mathbf{G}_1$ to $\mathbf{M}_3$ separating elements of $\mathbf{G}_1$. Thus $\mathbf{G}_1$ is a subdirect power of $\mathbf{M}_3$. Hence $\mathbf{G}_1 \in \mathrm{Var}(\mathbf{M}_3)$ and $Z'(\mathbf{G}) \setminus \{a,b\}$ is the subsemigroup of $\mathbf{G}_1$ consisting of all left zeros of $\mathbf{G}_1$. Observe that $f_r((R \times \{1,2\}) \cup \{a,b,u,v\}) \subseteq \{a,b,d\}$ and $g_r((R \times \{1,2\}) \cup \{a,b,u,v\}) \subseteq \{a,b,d\}$ for all $r \in R$, and $f'((R \times \{1,2\}) \cup \{a,b,u,v\}) \subseteq \{a,b,d\}$ and $g'((R \times \{1,2\}) \cup \{a,b,u,v\}) \subseteq \{a,b,d\}$. For $r \in R$, let $C_r = \{(r,0)\}$, for $x \in X \setminus \{a_{\mathbf{G}}, b_{\mathbf{G}}\}$, let us define $C_x = \{(r,1) \mid r = (x,y) \in R\} \cup \{(r,2) \mid r = (z,x) \in R\}$, $C_{a_{\mathbf{G}}} = \{(r,1) \mid r = (a_{\mathbf{G}},y) \in R\} \cup \{(r,2) \mid r = (z,a_{\mathbf{G}}) \in R\} \cup \{a_2\}$, $C_{b_{\mathbf{G}}} = \{(r,1) \mid r = (b_{\mathbf{G}},y) \in R\} \cup \{(r,2) \mid r = (z,b_{\mathbf{G}}) \in R\} \cup \{b_2\}$, and $C_z = \{z\}$ for each $z \in \{a,b,a_1,b_1,u,v\}$. It is easy to verify that $\{C_z \mid z \in Z_0(\mathbf{G})\}$ is a decomposition of the set $Z'(\mathbf{G})$. Consider the equivalence $\sim$ on the set $Z'(\mathbf{G})$ corresponding to the decomposition $\{C_z \mid z \in Z_0(\mathbf{G})\}$. Observe that any non-singleton class is a subset of $(R \times \{1,2\}) \cup \{u,v,a_2,b_2\}$ and therefore it is a congruence of $\mathbf{G}_1$ and if we identify a class C_z with z for all $z \in Z_0(\mathbf{G})$ then we obtain that the groupoid $\Gamma_0\mathbf{G}$ is a quotient of $\mathbf{G}_1$. Hence $\Gamma_0\mathbf{G} \in \mathrm{Var}(\mathbf{M}_3)$ and (7) is proved. Since $(R \times \{1,2\}) \cup \{a,b,a_2,b_2,u,v\}$ is saturated by $\sim$ we conclude that $X \cup \{a,b,u,v\}$ is a subsemigroup of $\Gamma_0\mathbf{G}$ belonging to the variety $\mathrm{Var}(\mathbf{M}'_3)$ and (10) is proved. The statement (8) is an easy consequence of (1) and (7). For $(x,y) \in R$, a direct calculation shows that the subsemigroup $\{(x,y), x, u, a\}$ of $\Gamma_0\mathbf{G}$ is generated by $\{(x,y), a\}$, and that the subsemigroup $\{(x,y), y, v, b\}$ of $\Gamma_0\mathbf{G}$ is generated by $\{(x,y), b\}$. It is easy to see that both these subsemigroups of $\Gamma_0\mathbf{G}$ are isomorphic to $\mathbf{M}_3$ and the proof of (9) is complete. □

For a compatible mapping $f : \mathbf{G} \to \mathbf{G}' \in \mathbb{DG}_s$ define $\Gamma_0 f : Z_0(\mathbf{G}) \to Z_0(\mathbf{G}')$ so that

(z1) $\Gamma_0 f(s) = s$ for all $s \in \{a,b,a_1,b_1,u,v\}$;
(z2) $\Gamma_0 f(x) = f(x)$ for all $x \in X$;
(z3) $\Gamma_0 f((x,y)) = (f(x), f(y))$ for all $(x,y) \in R$.

Lemma 5.2 $\Gamma_0 f : \Gamma_0\mathbf{G} \to \Gamma_0\mathbf{G}'$ *is a homomorphism for every compatible mapping* $f : \mathbf{G} \to \mathbf{G}' \in \mathbb{DG}_s$.

Proof. By Lemma 5.1(1), $X \cup R \cup \{a_1,b_1,u,v\}$ (or $X' \cup R' \cup \{a_1,b_1,u,v\}$) is a left-zero subsemigroup of $\Gamma_0\mathbf{G}$ (or $\Gamma_0\mathbf{G}'$). Thus $\Gamma_0 f(xy) = \Gamma_0 f(x) =$

$\Gamma_0 f(x)\Gamma_0 f(y)$ for all $x \in X \cup R \cup \{a_1, b_1, u, v\}$ and all $y \in Z_0(\mathbf{G})$ because $\Gamma_0 f(X \cup R \cup \{a_1, b_1, u, v\}) \subseteq X' \cup R' \cup \{a_1, b_1, u, v\}$. Since

$$\begin{aligned} &ax = u \text{ and } bx = v \text{ for all } x \in X \cup \{u, v\},\\ &a(x, y) = x \text{ and } b(x, y) = y \text{ for all } (x, y) \in R,\\ &aa_1 = a_{\mathbf{G}},\ bb_1 = b_{\mathbf{G}} \text{ and } ab_1 = ba_1 = u \end{aligned}$$

and since

$$\begin{aligned} &\Gamma_0 f(x, y) = (f(x), f(y)) \text{ for all } (x, y) \in R,\\ &\Gamma_0 f(x) = f(x) \text{ for all } x \in X,\\ &\Gamma_0 f(s) = s \text{ for all } s \in \{a, b, u, v\} \end{aligned}$$

we conclude that $\Gamma_0 f(st) = \Gamma_0 f(s)\Gamma_0 f(t)$ for all $s \in \{a, b\}$ and all $t \in Z_0(\mathbf{G})$. Thus $\Gamma_0 f : \Gamma_0 \mathbf{G} \to \Gamma_0 \mathbf{G}'$ is a homomorphism. □

As a consequence we obtain the following corollary.

Corollary 5.3 Γ_0 *is an embedding of* $\mathbb{DG}_s$ *into the variety* $Var(\mathbf{M}_3)$ *which preserves finiteness.* □

Let $\mathbb{V}$ denote the variety that is the expansion of $\mathrm{Var}(\mathbf{M}_3)$ by two nullary operations ξ and ζ. Define a functor $\Gamma_1 : \mathbb{DG}_s \to \mathbb{V}$ such that $\Gamma_1 \mathbf{G} = (\hat{\Gamma}_0 \mathbf{G}, \xi_{\Gamma_1 \mathbf{G}}, \zeta_{\Gamma_1 \mathbf{G}})$ for every $\mathbf{G} \in \mathbb{DG}_s$ where $\hat{\Gamma}_0 \mathbf{G}$ is the subsemigroup of $\Gamma_0 \mathbf{G}$ on the set $Z_0(\mathbf{G}) \setminus \{a_1, b_1\}$, see Lemma 5.1(1) and (3), $\xi_{\Gamma_1 \mathbf{G}}(0) = a$, $\zeta_{\Gamma_1 \mathbf{G}}(0) = b$ and $\Gamma_1 f$ is the domain-range restriction of $\Gamma_0 f$ to $\hat{\Gamma}_0 \mathbf{G}$ and $\hat{\Gamma}_0 \mathbf{G}'$ (this is correct because $\Gamma_0 f^{-1}(a_1) = \{a_1\}$, $\Gamma_0 f^{-1}(b_1) = \{b_1\}$) for every compatible mapping $f : \mathbf{G} \to \mathbf{G}'$. We immediately obtain the following corollary.

Corollary 5.4 Γ_1 *is an embedding of* $\mathbb{DG}_s$ *into the variety* $\mathbb{V}$ *which preserves finiteness.* □

Next we prove that Γ_1 is a full embedding.

Theorem 5.5 Γ_1 *is a full embedding of* $\mathbb{DG}_s$ *into the variety* $\mathbb{V}$*. The variety* $Var(\mathbf{M}_3)$ *has an ff-alg-universal* 2*-expansion.*

Proof. Let $\mathbf{G} = (X, R)$ and $\mathbf{G}' = (X', R')$ be digraphs from $\mathbb{DG}_s$ and let $f : \Gamma_1 \mathbf{G} \to \Gamma_1 \mathbf{G}'$ be a homomorphism of $\mathbb{V}$. Since $\xi_{\Gamma_1 \mathbf{G}}(0) = \xi_{\Gamma_1 \mathbf{G}'}(0) = a$ and $\zeta_{\Gamma_1 \mathbf{G}}(0) = \zeta_{\Gamma_1 \mathbf{G}'}(0) = b$ we have $f(a) = a$ and $f(b) = b$. From $a^2 = u$ and $b^2 = v$ it follows that $f(u) = u$ and $f(v) = v$. By Lemma 5.1(1), we conclude that $f(X \cup R \cup \{u, v\}) \subseteq X' \cup R' \cup \{u, v\}$. First we prove that $f(X) \subseteq X'$ and $f(R) \subseteq R'$. Since $\mathbf{G}$ is strongly connected we conclude that for every $x \in X$ there exist $y, z \in X$ with $(x, y), (z, x) \in R$. Then $a(x, y) = x = b(z, x)$, by (m11). From Lemma 5.1(3) it follows that $f(x) \notin R'$ and from Lemma 5.1(2) it follows $f(x) \notin \{u, v\}$. Thus $f(x) \in X'$ and since $x \in X$ is an arbitrary element we conclude $f(X) \subseteq X'$. Lemma 5.1(5) implies that $f(R) \subseteq R'$ because

$a(x, y) = x \in X$ and $b(x, y) = y \in X$ for all $(x, y) \in R$. Let g be the domain-range restriction of f to X and X'. Since $a(x, y) = x$, $b(x, y) = y$ for all $(x, y) \in R$ we conclude that $g(x) = f(x) = f(a(x, y)) = f(a)f(x, y) = af(x, y)$ and $g(y) = f(y) = f(b(x, y)) = f(b)f(x, y) = bf(x, y)$. Whence $f(x, y) = (g(x), g(y))$ and hence $g : \mathbf{G} \to \mathbf{G}'$ is a compatible mapping and by a direct calculation we obtain that $\Gamma_1 g = f$. Thus Γ_1 is a full embedding and Theorem 2.6 completes the proof. $\square$

Next, we prove that the 1-expansion of $\mathrm{Var}(\mathbf{M}_3)$ is not alg-universal.

Lemma 5.6 *Any rigid algebra in the* 1*-expansion of Var*$(\mathbf{M}_3)$ *has at most two elements.*

Proof. Let $\mathbf{S} = (S, \cdot, \xi)$ be a rigid algebra in the 1-expansion of $\mathrm{Var}(\mathbf{M}_3)$, by the nullary operation ξ. Let us assume that $\xi(0) = s \in S$. By a direct inspection, we obtain that every element of $\mathbf{M}_3$ is either irreducible or idempotent. This property is preserved by products, subalgebras and homomorphic images, and thus $\mathbf{S}$ satisfies this property. If s is idempotent then the constant mapping with value s is an endomorphism of $\mathbf{S}$ and because $\mathbf{S}$ is rigid we obtain $S = \{s\}$. If s is irreducible then the mapping

$$g(t) = \begin{cases} s & \text{if } t = s, \\ s^2 & \text{if } t \in S \setminus \{s\} \end{cases}$$

is an endomorphism of $\mathbf{S}$. Whence $S = \{s, s^2\}$ and the proof is complete. $\square$

By Theorem 5.5 and Lemma 5.6 we obtain the following corollary.

Corollary 5.7 *The variety Var*$(\mathbf{M}_3)$ *has an alg-universal α-expansion for a cardinal α if and only if $\alpha \geq 2$.* $\square$

Next we modify the functor Γ_0 to obtain a $\mathcal{Z}_{\mathrm{Var}(\mathbf{M}_3')}$-full embedding $\Gamma_2 : \mathbb{DG}_s \to \mathrm{Var}(\mathbf{M}_3)$. For a digraph $\mathbf{G} \in \mathbb{DG}_s$, let $\Gamma_2\mathbf{G}$ be the quotient semigroup $\Gamma_0\mathbf{G}/\sim$ where $\sim$ is the least congruence of $\Gamma_0\mathbf{G}$ with $u \sim v$. By Lemma 5.1(8), $e = \{u, v\}$ is the unique non-singleton class of $\sim$. Let us denote $Z_1(\mathbf{G}) = R \cup X \cup \{a, b, a_1, b_1, e\} = Z_0(\mathbf{G})/\sim$. For a compatible mapping $f : \mathbf{G} \to \mathbf{G}'$, let us define $\Gamma_2 f$ so that $\Gamma_2 f(z) = \Gamma_0 f(z)$ for all $z \in R \cup X \cup \{a, b, a_1, b_1\}$ and $\Gamma_2 f(e) = e$. Then we can write $\Gamma_2 f = \Gamma_0 f/\sim$ because $(\Gamma_0 f)^{-1}(\{u, v\}) = \{u, v\}$. Hence we obtain the following proposition.

Proposition 5.8 $\Gamma_2 : \mathbb{DG}_s \to$ *Var*$(\mathbf{M}_3)$ *is an embedding such that the subsemigroup Im*$(\Gamma_2 f)$ *of* $\Gamma_2\mathbf{G}'$ *generates the variety Var*$(\mathbf{M}_3)$ *for every compatible mapping* $f : \mathbf{G} \to \mathbf{G}' \in \mathbb{DG}_s$.

Proof. From Lemma 5.1(7), (8) and Corollary 5.3 it follows that $\Gamma_2 : \mathbb{DG}_s \to \mathrm{Var}(\mathbf{M}_3)$ is an embedding. For a compatible mapping $f : \mathbf{G} \to \mathbf{G}' \in \mathbb{DG}_s$, we have $\mathrm{Im}(\Gamma_2 f) \cap \{a,b\} \neq \emptyset \neq \mathrm{Im}(\Gamma_2 f) \cap R$ and Lemma 5.1(9) completes the proof. $\square$

To prove that Γ_2 is $\mathrm{Var}(\mathbf{M}_3')$-full we describe semigroup homomorphisms between $\Gamma_2\mathbf{G}$ and $\Gamma_2\mathbf{G}'$ for $\mathbf{G}, \mathbf{G}' \in \mathbb{DG}_s$.

Proposition 5.9 *Let $\mathbf{G} = (X,R)$ and $\mathbf{G}' = (X',R')$ be digraphs from $\mathbb{DG}_s$. Then a mapping $f : Z_1(\mathbf{G}) \to Z_1(\mathbf{G}')$ is a semigroup homomorphism from $\Gamma_2\mathbf{G}$ to $\Gamma_2\mathbf{G}'$ if and only if it satisfies the following conditions:*

1. *$f(R \cup X \cup \{a_1,b_1,e\}) \subseteq R' \cup X' \cup \{a_1,b_1,e\}$ and $\{f(a),f(b)\} \subseteq \{a,b,f(e)\}$;*
2. *if $\{a,b\} \neq \{f(a),f(b)\}$ then $f(X \cup \{e\})$ is a singleton;*
3. *if $\{f(a),f(b)\} \cap \{a,b\}$ is a singleton then $f(e) = e$ and $Im(f) \subseteq X \cup \{a,b,e,d\}$ where*

$$d = \begin{cases} b_1 & \text{if } a \in \{f(a),f(b)\}, \\ a_1 & \text{if } b \in \{f(a),f(b)\}; \end{cases}$$

4. *if $\{f(a),f(b)\} = \{a,b\}$ then either there exists a compatible mapping $g : \mathbf{G} \to \mathbf{G}'$ with $f = \Gamma_2 g$ or $f(X \cup \{e\}) = \{e\}$ and $Im(f) \subseteq X \cup \{a,b,e\}$.*

Proof. Let $f : \Gamma_2\mathbf{G} \to \Gamma_2\mathbf{G}'$ be a semigroup homomorphism. By Lemma 5.1(1), $f(R \cup X \cup \{a_1,b_1,e\}) \subseteq R' \cup X' \cup \{a_1,b_1,e\}$. Since $a^2 = b^2 = e$, by Lemma 5.1(1) we obtain $f(a) = f(e)$ or $f(a) \in \{a,b\}$ and analogously $f(b) = f(e)$ or $f(b) \in \{a,b\}$ and the statement (1) is true. Observe that for every $x \in X$ and $c \in \{a,b\}$ there exists $z \in R$ with $x = cz$. Thus from $f(c) = f(e)$ it follows that $f(x) = f(c)f(z) = f(e)f(z) = f(e)$. Hence if $f(a) \in R' \cup X' \cup \{a_1,b_1,e\}$ or $f(b) \in R' \cup X' \cup \{a_1,b_1,e\}$ then $f(X \cup \{e\})$ is a singleton. If $\{f(a),f(b)\} \cap \{a,b\} = \emptyset$ then the statements (3) and (4) hold (because the hypothesis of these statements are not satisfied) and therefore the statements (1), (2), (3), and (4) are true.

Next assume that there exists $c \in \{a,b\}$ with $f(c) \in \{a,b\}$. From $a^2 = b^2 = e$ it follows that $f(e) = e$. Observe that, by (m12), (m13) and (m14), $\{z \in Z_1(\mathbf{G}') \mid f(c)z = e\} = X' \cup \{e,d\}$ where

$$d = \begin{cases} b_1 & \text{if } f(c) = a, \\ a_1 & \text{if } f(c) = b. \end{cases}$$

If, moreover, $f(X) = \{e\}$ then, by (1), (m11) and (m14), $f(z) \in X' \cup \{e,d\}$ for all $z \in R \cup \{a_1,b_1\}$ because $cz \in X \cup \{e\}$. Therefore if $\{f(a),f(b)\} \cap \{a,b\} \neq \emptyset \neq \{f(a),f(b)\} \setminus \{a,b\}$ then the statements (1), (2), (3), (4) are true (the hypothesis of the statement (4) is not satisfied).

Assume that $\{f(a),f(b)\} \subseteq \{a,b\}$. First we prove that either $f(X) = \{e\}$ or $f(X) \subseteq X'$. Assume that $f(x) \neq e$ for some $x \in X$. Since for $c \in \{a,b\}$ there exists $z \in R$ with $x = cz$ we have $f(x) = f(c)f(z)$ and, by Lemma 5.1(3), $f(x) \notin R' \cup \{a_1,b_1\}$ because $f(c) \in \{a,b\}$. Whence $f(x) \in X'$. By Lemma

5.1(4), for every $y \in X$ there exists a sequence $x = x_0, x_1, \ldots, x_{2n} = y$ such that $x_{2i} \in X$, $x_{2i+1} \in R$, $\{ax_{2i+1}, bx_{2i+1}\} = \{x_{2i}, x_{2i+2}\}$ for all $i = 0, 1, \ldots, n-1$. We prove that if $f(x_{2i}) \in X'$ for some $i = 0, 1, \ldots, n-1$, then $f(x_{2i+2}) \in X'$. Assume that $cx_{2i+1} = x_{2i}$ for $c \in \{a, b\}$. Then $f(c)f(x_{2i+1}) = f(cx_{2i+1}) = f(x_{2i}) \in X'$ and, by Lemma 5.1(5), if $f(c) = a$ then $f(x_{2i+1}) \in R' \cup \{a_1\}$, if $f(c) = b$ then $f(x_{2i+1}) \in R' \cup \{b_1\}$. Since $c'x_{2i+1} = x_{2i+2}$ for $c' \in \{a, b\} \setminus \{c\}$ we deduce that $f(x_{2i+2}) = f(c')f(x_{2i+1}) \in X'$ whenever $f(x_{2i+1}) \in R'$ because $f(c') \in \{a, b\}$. If $f(x_{2i+1}) \notin R'$ and $f(c) = f(c')$ then $f(x_{2i+2}) = f(c)f(x_{2i+1}) = f(x_{2i}) \in X'$. It remains to consider the case of $\{f(a), f(b)\} = \{a, b\}$ and $f(x_{2i+1}) \notin R'$. Then $f(c) \neq f(c')$ implies $f(x_{2i+2}) = f(c')f(x_{2i+1}) = e$, by (m13) and (m14), and if $f(c) = a$ then $f(x_{2i+1}) = a_1$ and $f(x_{2i}) = a_{\mathbf{G}'}$, if $f(c) = b$ then $f(x_{2i+1}) = b_1$ and $f(x_{2i}) = b_{\mathbf{G}'}$. By the definition of $\Gamma_0\mathbf{G}$, either $(x_{2i}, x_{2i+2}) \in R$ or $(x_{2i+2}, x_{2i}) \in R$, and, by Theorem 2.6(2), there exists a node $z \in X$ such that $\{x_{2i}, x_{2i+2}, z\}$ is a cycle of length 3 in $\mathbf{G}$. Thus there exist $z_1, z_2 \in R$ such that $\{az_1, bz_1\} = \{x_{2i+2}, z\}$, $\{az_2, bz_2\} = \{x_{2i}, z\}$. Since $cx_{2i+1} = x_{2i}$, $c'x_{2i+1} = x_{2i+2}$ we deduce that $cz_1 = x_{2i+2}$, $c'z_1 = z$, $cz_2 = z$, $c'z_2 = x_{2i}$. If $f(c) = a$ then, by (m12), (m13) and (m14), $f(z_1) \in X \cup \{e, b_1\}$ and $f(z) = f(c')f(z_1) \in \{e, b_{\mathbf{G}'}\}$, if $f(c) = b$ then $f(z_1) \in X \cup \{e, a_1\}$ and $f(z) = f(c')f(z_1) \in \{e, a_{\mathbf{G}'}\}$. If $f(z_1) \in X \cup \{e\}$, then, by (m13), $f(z) = e$ and, by the same argument, if $f(c) = a$ then $f(x_{2i}) \in \{e, b_{\mathbf{G}'}\}$, if $f(c) = b$ then $f(x_{2i}) \in \{e, a_{\mathbf{G}'}\}$ – this is a contradiction with the value of $f(x_{2i})$. Thus, if $f(c) = a$ then $f(z_1) = b_1$ and $f(z) = b_{\mathbf{G}'}$ and if $f(c) = b$ then $f(z_1) = a_1$ and $f(z) = a_{\mathbf{G}'}$. From $\{f(c)f(z_2), f(c')f(z_2)\} = \{f(x_{2i}), f(z)\} = \{a_{\mathbf{G}'}, b_{\mathbf{G}'}\}$ it follows that either $(a_{\mathbf{G}'}, b_{\mathbf{G}'}) \in R'$ or $(b_{\mathbf{G}'}, a_{\mathbf{G}'}) \in R'$ – this is a contradiction with Theorem 2.6(4). Therefore $f(x_{2i+2}) \in X$ and, by an easy induction, we obtain that $f(y) \in X'$. From Lemma 5.1(4) it follows that if $f(x) \neq e$ for some $x \in X$ then $f(X) \subseteq X'$. Thus either $f(X) \subseteq X'$ or $f(X) = \{e\}$.

Next assume that $f(a) = f(b) = c \in \{a, b\}$. Then

$$f(a_{\mathbf{G}}) = f(aa_1) = f(a)f(a_1) = cf(a_1) = f(b)f(a_1) = f(ba_1) = f(e) = e$$

and hence $f(X) = \{e\}$. Thus if $|\{f(a), f(b)\} \cap \{a, b\}| \leq 1$ then the statements (1), (2), (3), and (4) are true (because the hypothesis of the statement (4) is not satisfied).

If $\{f(a), f(b)\} = \{a, b\}$ and $f(X) = \{e\}$ then, by (m11), (m13) and (m14), $f(R \cup \{a_1, b_1\}) \subseteq \big(X \cup \{e, a_1\}\big) \cap \big(X \cup \{e, b_1\}\big) = X \cup \{e\}$. Thus $\operatorname{Im}(f) \subseteq X \cup \{a, b, e\}$.

If $\{f(a), f(b)\} = \{a, b\}$ and $f(X) \subseteq X'$ then, by Lemma 5.1(5), $f(R) \subseteq R'$. Let $g : X \to X'$ be the domain-range restriction of f to X and X'. If $(x, y) \in R$ then $a(x, y) = x$ and $b(x, y) = y$. If $f(a) = a$ then $f(b) = b$ and

$$\begin{aligned} g(x) = f(x) = f(a(x, y)) = f(a)f(x, y) = af(x, y) \text{ and} \\ g(y) = f(y) = f(b(x, y)) = f(b)f(x, y) = bf(x, y), \end{aligned}$$

so that $f(x, y) = (g(x), g(y)) \in R'$ and $g : \mathbf{G} \to \mathbf{G}'$ is a compatible mapping. Since $aa_1 = a_{\mathbf{G}}$, $ba_1 = ab_1 = e$ and $bb_1 = b_{\mathbf{G}}$ and since $g(a_{\mathbf{G}}) = a_{\mathbf{G}'}$ and

$g(b_{\mathbf{G}}) = b_{\mathbf{G}'}$ (by Theorem 2.6(3)), we conclude that $f(a_1) = a_1$, $f(b_1) = b_1$ and $f = \Gamma_2 g$. If $f(a) = b$ then $f(b) = a$ and

$$g(x) = f(x) = f(a(x,y)) = f(a)f(x,y) = bf(x,y) \text{ and}$$
$$g(y) = f(y) = f(b(x,y)) = f(b)f(x,y) = af(x,y),$$

so that $f(x,y) = (g(y), g(x)) \in R'$ for all $(x,y) \in R$. This is a contradiction with Theorem 2.6(5) and hence if $\{f(a), f(b)\} = \{a,b\}$, then the statements (1), (2), (3), and (4) hold.

Therefore any semigroup homomorphism $f : \Gamma_2\mathbf{G} \to \Gamma_2\mathbf{G}'$ satisfies the statements (1), (2), (3), and (4).

Conversely, assume that a mapping $f : Z_1(\mathbf{G}) \to Z_1(\mathbf{G}')$ satisfies the statements (1), (2), (3) and (4). We prove that f is a semigroup homomorphism from $\Gamma_2\mathbf{G}$ to $\Gamma_2\mathbf{G}'$. First assume that $f(a), f(b) \notin \{a,b\}$. Then $f(X \cup \{a,b,e\})$ is a singleton and $\mathrm{Im}(f) \subseteq R' \cup X' \cup \{a_1, b_1, e\}$. By Lemma 5.1(1), $f(z) = f(z)f(t) = f(zt) = f(z)$ for all $z, t \in R \cup X \cup \{a_1, b_1, e\}$. Since $f(X \cup \{a,b,e\})$ is a singleton we conclude that $f(e) = f(c)f(z) = f(cz)$ for all $z \in Z_1(\mathbf{G})$ and $c \in \{a,b\}$ and $f : \Gamma_2\mathbf{G} \to \Gamma_2\mathbf{G}'$ is a semigroup homomorphism. If $\{f(a), f(b)\} \cap \{a,b\}$ is a singleton then $f(X \cup \{e\}) = \{e\}$, $\{f(a), f(b)\} \subseteq \{a,b,e\}$ and $f(R \cup X \cup \{a_1, b_1, e\}) \subseteq X' \cup \{e, d\}$ where $d = b_1$ if $a \in \{f(a), f(b)\}$ and $d = a_1$ if $b \in \{f(a), f(b)\}$. By Lemma 5.1(1), $f(z) = f(z)f(t) = f(zt) = f(z)$ for all $z \in R \cup X \cup \{a_1, b_1, e\}$ and $t \in Z_1(\mathbf{G})$. By Lemma 5.1(6), $\{az \mid z \in Z_1(\mathbf{G})\} = X \cup \{e\} = \{bz \mid z \in Z_1(\mathbf{G})\}$ and thus $e = f(c)f(z) = f(cz)$ for all $c \in \{a,b\}$ and $z \in Z_1(\mathbf{G})$. Hence $f : \Gamma_2\mathbf{G} \to \Gamma_2\mathbf{G}'$ is a semigroup homomorphism. If $\{f(a), f(b)\} = \{a,b\}$ then either $f = \Gamma_2 g$ for a compatible mapping $g : \mathbf{G} \to \mathbf{G}'$ or $f(X \cup \{e\}) = \{e\}$ and $f(R \cup X \cup \{a_1, b_1, e\}) \subseteq X' \cup \{e\}$. In the first case, f is a homomorphism because Γ_2 is a functor into $\mathrm{Var}(\mathbf{M}_3)$, in the second case, we obtain, by the argument as above, that $f : \Gamma_2\mathbf{G} \to \Gamma_2\mathbf{G}'$ is a semigroup homomorphism, and the proof is complete. □

As a consequence we obtain the following corollary.

Corollary 5.10 *Let $\mathbf{G} = (X, R)$ and $\mathbf{G}' = (X', R')$ be digraphs from $\mathbb{DG}_s$. If $f : \Gamma_2\mathbf{G} \to \Gamma_2\mathbf{G}'$ is a semigroup homomorphism then either $f = \Gamma_2 g$ for a compatible mapping $g : \mathbf{G} \to \mathbf{G}'$ or $f(X \cup \{e\})$ is a singleton and Im(f) is a subsemigroup of $\Gamma_2\mathbf{G}'$ belonging to the variety Var$(\mathbf{M}_3')$. Let $E(\mathbf{G})$ be the set of all semigroup endomorphisms f of $\Gamma_2\mathbf{G}$ such that $f(X \cup \{e\})$ is a singleton. Then $E(\mathbf{G})$ is a subsemigroup of End$(\Gamma_2\mathbf{G})$. Semigroups $E(\mathbf{G})$ and $E(\mathbf{G}')$ are isomorphic if and only if $|X| = |X'|$ and $|R| = |R'|$.*

Proof. If $\{f(a), f(b)\} \neq \{a,b\}$ then, by Proposition 5.9(2) and (3), $f(X \cup \{e\})$ is a singleton and either $\mathrm{Im}(f) \subseteq R' \cup X' \cup \{a_1, b_1, e\}$ or $\mathrm{Im}(f) \subseteq X' \cup \{a, b, e, d\}$ where $d = b_1$ if $a \in \mathrm{Im}(f)$ and $d = a_1$ if $b \in \mathrm{Im}(f)$. In the first case, by Lemma 5.1(1), $\mathrm{Im}(f) \in \mathrm{Var}(\mathbf{M}_3')$ in the second case, by Lemma 5.1.(10), $\mathrm{Im}(f) \in \mathrm{Var}(\mathbf{M}_3')$. If $\{f(a), f(b)\} = \{a,b\}$ then Proposition 5.9(4) and Lemma 5.1(10) complete the proof of the first statement.

To prove the second and the third statement, observe that, by Proposition 5.9, a mapping $f : Z_1(\mathbf{G}) \to Z_1(\mathbf{G})$ belongs to $E(\mathbf{G})$ if and only if it satisfies one of the following conditions

1. $f(X \cup \{a, b, e\})$ is a singleton and $\mathrm{Im}(f) \subseteq R \cup X \cup \{a_1, b_1, e\}$;
2. $a \in \{f(a), f(b)\} \subseteq \{a, e\}$, $f(X \cup \{e\}) = \{e\}$ and $f(R \cup \{a_1, b_1\}) \subseteq X \cup \{b_1, e\}$;
3. $b \in \{f(a), f(b)\} \subseteq \{b, e\}$, $f(X \cup \{e\}) = \{e\}$ and $f(R \cup \{a_1, b_1\}) \subseteq X \cup \{a_1, e\}$;
4. $\{f(a), f(b)\} = \{a, b\}$, $f(X \cup \{e\}) = \{e\}$ and $f(R \cup \{a_1, b_1\}) \subseteq X \cup \{e\}$.

Now it is easy to see that $E(\mathbf{G})$ is closed under the composition, and that $E(\mathbf{G})$ and $E(\mathbf{G}')$ are isomorphic if and only if $|X| = |X'|$ and $|R| = |R'|$. Indeed, if $\phi : X \to X'$ and $\psi : R \to R'$ are bijections then define a bijection $\mu : Z_1(\mathbf{G}) \to Z_1(\mathbf{G})$ by setting

$$\mu(z) = \begin{cases} \phi(z) & \text{if } z \in X, \\ \psi(z) & \text{if } z \in R, \\ z & \text{if } z \in \{e, a, b, e_1, b_1\}. \end{cases}$$

For $f \in E(\mathbf{G})$, let us denote $\nu(f) = \mu \circ f \circ \mu^{-1}$. Then ν is an isomorphism between $E(\mathbf{G})$ and $E(\mathbf{G}')$. If $|X| \neq |X'|$ or $|R| \neq |R'|$ then $|E(\mathbf{G})| \neq |E(\mathbf{G}')|$ and thus $E(\mathbf{G})$ and $E(\mathbf{G}')$ are not isomorphic. The proof is complete. □

The two theorems below follow from Proposition 5.8 and Corollary 5.10.

Theorem 5.11 *The variety Var*$(\mathbf{M}_3)$ *is Var*$(\mathbf{M}_3')$*-relatively* ff*-alg-universal.*

Proof. From Proposition 5.8 and the first statement of Corollary 5.10, Γ_2 is $\mathrm{Var}(\mathbf{M}_3')$-full embedding of $\mathbb{DG}_s$ into $\mathrm{Var}(\mathbf{M}_3)$. It is easy to see that Γ_2 preserves finiteness. Thus $\mathrm{Var}(\mathbf{M}_3)$ is $\mathrm{Var}(\mathbf{M}_3')$-relatively ff-alg-universal. □

Theorem 5.12 *The variety Var*$(\mathbf{M}_3)$ *is* α*-determined for no cardinal* α*.*

Proof. If $\mathbf{G}$ is a rigid graph from $\mathbb{DG}_s$ then $\mathrm{End}(\Gamma_2\mathbf{G})$ is a semigroup $E(\mathbf{G})$ with the outer identity. Thus if $\mathbf{G} = (X, R)$ and $\mathbf{G}' = (X', R')$ are rigid graphs from the category $\mathbb{DG}_s$ with $|X| = |X'|$ and $|R| = |R'|$, then, by the third statement of Corollary 5.10, $\Gamma_2\mathbf{G}$ and $\Gamma_2\mathbf{G}'$ are equimorphic. Since for any infinite cardinal α there exist 2^α non-isomorphic connected rigid graphs in $\mathbb{GRA}$, see [31] and since for $(V, E) \in \mathbb{GRA}$ with an infinite set V we have $|V| = |E| = |X| = |R|$ where $(X, R) = \Lambda(\Omega(V, E))$, we conclude that for every infinite cardinal α there exist 2^α non-isomorphic rigid digraphs in $\mathbb{DG}_s$ with an underlying set of cardinality α. Hence it follows that the variety $\mathrm{Var}(\mathbf{M}_3)$ is α-determined for no cardinal α. □

The third statement of Theorem 1.7 is a consequence of Theorems 5.5, 5.11 and 5.12.

6 Semigroup varieties

The aim of this section is to characterize the varieties $\mathrm{Var}(\mathbf{M}_1)$, $\mathrm{Var}(\mathbf{M}_2)$ and $\mathrm{Var}(\mathbf{M}_3)$ and to describe their positions in the lattice $\mathcal{L}(\mathbb{S})$ of all semigroup varieties. We derive some consequences of Theorem 1.7 for other varieties. First we recall several results about band varieties that are analogous to Theorem 1.7. For this purpose we list the varieties near the zero of $\mathcal{L}(\mathbb{S})$ together with their defining identities. We denote

$\mathbb{T}$ – the variety of trivial semigroups, $x = y$;

$\mathbb{SL}$ – the variety of semilattices, $xy = yx$ and $x^2 = x$;

$\mathbb{ZS}$ – the variety of zero semigroups, $xy = uv$; observe that $\mathrm{Var}(\mathbf{M}_2') = \mathbb{ZS}$;

$\mathbb{LZS}$ – the variety of left-zero semigroups, $xy = x$;

$\mathbb{RZS}$ – the variety of right-zero semigroups, $yx = x$;

$\mathbb{AB}_p$ – the variety of commutative groups of order p for a natural number $p > 1$, $xy = yx$ and $x^p y = y$;

$\mathbb{RCB}$ – the variety of rectangular bands, $x^2 = x$ and $xyx = x$;

$\mathbb{LNB}$ – the variety of left normal bands, $x^2 = x$ and $xyz = xzy$;

$\mathbb{RNB}$ – the variety of right normal bands, $x^2 = x$ and $yzx = zyx$;

$\mathbb{NB}$ – the variety of normal bands, $x^2 = x$ and $xyzx = xzyx$;

$\mathbb{SLZ}$ – the variety of semilattices of left zero semigroups, $x^2 = x$ and $xyx = xy$;

$\mathbb{SRZ}$ – the variety of semilattices of right zero semigroups, $x^2 = x$ and $xyx = yx$;

$\mathbb{LQN}$ – the variety of left quasi-normal bands, $x^2 = x$ and $xyz = xyxz$;

$\mathbb{RQN}$ – the variety of right quasi-normal bands, $x^2 = x$ and $yzx = yxzx$;

$\mathbb{RB}$ – the variety of regular bands, $x^2 = x$ and $xyzx = xyxzx$;

$\mathbb{LSN}$ – the variety of left semi-normal bands, $x^2 = x$ and $xyz = xyzxz$;

$\mathbb{RSN}$ – the variety of right semi-normal bands, $x^2 = x$ and $xyz = xzxyz$.

It is well known that $\mathbb{T}$ is the zero in $\mathcal{L}(\mathbb{S})$ and that the varieties $\mathbb{SL}$, $\mathbb{ZS}$, $\mathbb{LZS}$, $\mathbb{RZS}$ and $\mathbb{AB}_p$ for a prime p are atoms of the lattice $\mathcal{L}(\mathbb{S})$. The lattice $\mathcal{L}(\mathbb{B})$ of all varieties of bands was described independently by A. P. Birjukov [8], Ch. Fennemore [13] and J. Gerhard [14]. The lattice $\mathcal{L}(\mathbb{B})$ is a sublattice of $\mathcal{L}(\mathbb{S})$. The inclusions between varieties of bands near the zero of $\mathcal{L}(\mathbb{S})$ are drawn in Fig. 2.

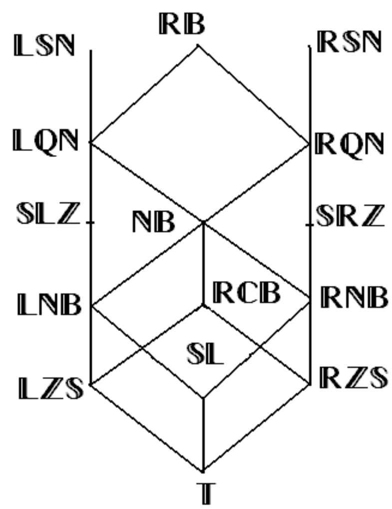

Figure 2. The inclusion of varieties of bands..

B. M. Schein proved that semilattices are 3-determined [34] and normal bands are 5-determined [35]. The theorem below summarizes other known results concerning varieties of bands.

Theorem 6.1 *The band varieties satisfy:*

1. *the varieties* $\mathbb{SLZ}$ *and* $\mathbb{SRZ}$ *are* 3*-determined and the varieties* $\mathbb{LQN}$ *and* $\mathbb{RQN}$ *are* 5*-determined, [10];*
2. *the variety* $\mathbb{V}$ *of bands has an alg-universal* α*-expansion for some cardinal* α *if and only if* $\mathbb{LNB} \subseteq \mathbb{V}$ *or* $\mathbb{RNB} \subseteq \mathbb{V}$*, in this case* $\mathbb{V}$ *has an ff-alg-universal* 3*-expansion, [11];*
3. *the variety* $\mathbb{V}$ *of bands has an alg-universal* 2*-expansion if and only if* $\mathbb{SLZ} \subseteq \mathbb{V}$ *or* $\mathbb{SRZ} \subseteq \mathbb{V}$*, in this case* $\mathbb{V}$ *has an ff-alg-universal* 2*-expansion; no variety of bands has an alg-universal* 1*-expansion, [11];*
4. *the variety* $\mathbb{V}$ *of bands is var-relatively alg-universal if and only if* $\mathbb{LSN} \subseteq \mathbb{V}$ *or* $\mathbb{RSN} \subseteq \mathbb{V}$*, in this case* $\mathbb{V}$ *is* $\mathbb{LQN}$*-relatively ff-alg-universal or* $\mathbb{RQN}$*-relatively ff-alg-universal, [12].* □

It is an open question whether a variety $\mathbb{V}$ of bands properly containing $\mathbb{LQN}$ or $\mathbb{RQN}$ is α-determined for some cardinal α. In [4], M. E. Adams and W. Dziobiak used the techniques from [12] to prove that the varieties $\mathbb{LSN}$ and $\mathbb{RSN}$ are Q-universal. Sapir generalized this result (cf. [4]) by showing that the varieties $\mathbb{LQN}$ and $\mathbb{RQN}$ are Q-universal. J. Gerhard and A. Shafaat [15], and M. Petrich [30] independently proved that the variety $\mathbb{NB}$ has only finitely many subquasivarieties and thus it is not Q-universal. The question of whether the varieties $\mathbb{SLZ}$ and $\mathbb{SRZ}$ are Q-universal remains open.

Next we describe semigroup varieties $\mathrm{Var}(\mathbf{M}_1)$, $\mathrm{Var}(\mathbf{M}_2)$ and $\mathrm{Var}(\mathbf{M}_3)$. First, for a semigroup $\mathbf{S} = (S, \cdot)$ define its dual $\mathcal{D}(\mathbf{S}) = (S, \odot)$ by $s \odot t = ts$ for all $s, t \in S$. It is easily seen that $\mathcal{D}(\mathbf{M}_2)$ is isomorphic to $\mathbf{M}_2$ and that $\mathcal{D}(\mathbf{M}_1)$ or $\mathcal{D}(\mathbf{M}_3)$ is not isomorphic to $\mathbf{M}_1$ or $\mathbf{M}_3$, respectively. Since for semigroups $\mathbf{S} = (S, \cdot)$ and $\mathbf{T} = (T, \cdot)$ a mapping $f : S \to T$ is a semigroup homomorphism from $\mathbf{S}$ to $\mathbf{T}$ if and only if f is a semigroup homomorphism from $\mathcal{D}(\mathbf{S})$ to $\mathcal{D}(\mathbf{T})$ we obtain by Theorem 1.7

Corollary 6.2 *The variety $\mathrm{Var}(\mathcal{D}(\mathbf{M}_1))$ has an ff-alg-universal 3-expansion, it is 3-determined and it is not var-relatively universal.*
The variety $\mathrm{Var}(\mathcal{D}(\mathbf{M}_3))$ has an ff-alg-universal 2-expansion, it is α-determined for no cardinal α and it is $\mathrm{Var}(\mathcal{D}(\mathbf{M}_3'))$-relatively ff-alg-universal. □

Below we describe algebraic properties of semigroup varieties $\mathrm{Var}(\mathbf{M}_1)$, $\mathrm{Var}(\mathcal{D}(\mathbf{M}_1))$, $\mathrm{Var}(\mathbf{M}_2)$, $\mathrm{Var}(\mathbf{M}_3)$, and $\mathrm{Var}(\mathcal{D}(\mathbf{M}_3))$.

Proposition 6.3 *The variety $\mathrm{Var}(\mathbf{M}_1)$ is determined by the identities $x^2y = xy$, $x^2y^2 = y^2x^2$ and $x^2y^2 = (xy)^2$. The variety $\mathrm{Var}(\mathcal{D}(\mathbf{M}_1))$ is determined by the identities $xy^2 = xy$, $x^2y^2 = y^2x^2$ and $x^2y^2 = (xy)^2$. The varieties $\mathrm{Var}(\mathbf{M}_1)$ and $\mathrm{Var}(\mathcal{D}(\mathbf{M}_1))$ are join-irreducible in the lattice $\mathcal{L}(\mathbb{S})$ and cover the variety $\mathbb{ZS} \vee \mathbb{SL}$ in $\mathcal{L}(\mathbb{S})$.*

Proof. By Lemma 3.1, any semigroup $\mathbf{S} \in \mathrm{Var}(\mathbf{M}_1)$ satisfies the identities $x^2y = xy$, $x^2y^2 = y^2x^2$ and $x^2y^2 = (xy)^2$. If a semigroup $\mathbf{S}$ satisfies the identities $x^2y = xy$, $x^2y^2 = y^2x^2$ and $x^2y^2 = (xy)^2$ then, by Proposition 3.2(7), (8) and (9), the family of homomorphisms from $\mathbf{S}$ to $\mathbf{M}_1$ separates elements of $\mathbf{S}$ and thus $\mathbf{S} \in \mathrm{Var}(\mathbf{M}_1)$.

If a semigroup $\mathbf{S} = (S, \cdot) \in \mathrm{Var}(\mathbf{M}_1)$ contains a reducible element $s \in S \setminus r(\mathbf{S})$, then, by Proposition 3.2(9) and (10), $\mathbf{M}_1$ is a homomorphic image of $\mathbf{S}$ and therefore $\mathrm{Var}(\mathbf{S}) = \mathrm{Var}(\mathbf{M}_1)$. If any element of $S \setminus r(\mathbf{S})$ is irreducible then, by Proposition 3.2(7) and (8), the family of homomorphisms from $\mathbf{S}$ to either the two-element semilattice or to the two-element zero-semigroup separates elements of $\mathbf{S}$. Whence $\mathbf{S} \in \mathbb{ZS} \vee \mathbb{SL}$ and thus $\mathrm{Var}(\mathbf{M}_1)$ covers $\mathbb{ZS} \vee \mathbb{SL}$ in $\mathcal{L}(\mathbb{S})$ and $\mathrm{Var}(\mathbf{M}_1)$ is join-irreducible in $\mathcal{L}(\mathbb{S})$.

The statements for $\mathrm{Var}(\mathcal{D}(\mathbf{M}_1))$ follow dually. □

Remark. Observe that the identities $x^2y = xy$ and $xy^2 = xy$ fail in the semigroup $\mathbf{M}_2$ and the identity $x^2y^2 = y^2x^2$ fails in the semigroups $\mathbf{M}_3$ and $\mathcal{D}(\mathbf{M}_3)$. Further, the identity $xy^2 = xy$ fails in the semigroup $\mathbf{M}_1$. Whence $\mathrm{Var}(\mathbf{M}_1)$ and $\mathrm{Var}(\mathcal{D}(\mathbf{M}_1))$ are incomparable varieties in $\mathcal{L}(\mathbb{S})$ and the varieties $\mathrm{Var}(\mathbf{M}_1)$ and $\mathrm{Var}(\mathcal{D}(\mathbf{M}_1))$ do not contain the semigroups $\mathbf{M}_2$, $\mathbf{M}_3$ and $\mathcal{D}(\mathbf{M}_3)$.

In [2], M. E. Adams and W. Dziobiak introduced the notion of a critical algebra. A finite algebra $\mathbf{A}$ is <u>critical</u> if it is not a subdirect product of its proper subalgebras. M. E. Adams and W. Dziobiak proved [2] that a quasivariety with only finitely many critical algebras cannot be Q-universal. Hence we obtain the following corollary.

Corollary 6.4 *The varieties Var*$(\mathbf{M}_1)$ *and Var*$(\mathcal{D}(\mathbf{M}_1))$ *have only finitely many critical algebras and therefore are not Q-universal.*

Proof. The proof of Proposition 6.3 implies the claim about $\mathrm{Var}(\mathbf{M}_1)$ because any critical algebra in $\mathrm{Var}(\mathbf{M}_1)$ is a subalgebra of $\mathbf{M}_1$. The claim for $\mathrm{Var}(\mathcal{D}(\mathbf{M}_1))$ follows dually. □

Proposition 6.5 *The semigroup variety Var*$(\mathbf{M}_2)$ *is determined by the identities $xy = yx$ and $x^2 = uvw$, covers the variety $\mathbb{ZS}$ in $\mathcal{L}(\mathbb{S})$ and is join-irreducible in $\mathcal{L}(\mathbb{S})$.*

Proof. It is easy to see that $\mathbf{M}_2$ satisfies both identities, and hence any semigroup in $\mathrm{Var}(\mathbf{M}_2)$ satisfies them. Conversely, assume that the identities $xy = yx$ and $x^2 = uvw$ hold in a semigroup $\mathbf{S} = (S, \cdot)$. From the identity $x^2 = uvw$ it follows that $\mathbf{S}$ satisfies the identity $x^2 = x^2y = yx^2$ and thus s^2 is zero of $\mathbf{S}$ for every $s \in S$. Let 0 denote the zero of $\mathbf{S}$ (thus $s^2 = 0$ for all $s \in S$). Since $rst = 0$ for all $r, s, t \in S$, either S is a singleton or $\mathbf{S}$ is generated by the set S' of all its irreducible elements. If S is a singleton then $\mathbf{S} \in \mathrm{Var}(\mathbf{M}_2)$. Let $S' \neq \emptyset$. Let $\mathbf{T} = \mathcal{C}(S')/I = (S' \cup \mathfrak{P}_2(S') \cup \{0\}, \cdot)$ be the Ree's quotient of the free commutative semigroup $\mathcal{C}(S')$ over the set S' by the ideal I generated by the set $\{x^2 \mid x \in S'\} \cup \{xyz \mid x, y, z \in S'\}$. Analogously to Proposition 4.1, $\mathbf{S}$ is a quotient of $\mathbf{T}$ and $\mathbf{T}$ is a subdirect power of $\mathbf{M}_2$. Thus $\mathbf{S} \in \mathrm{Var}(\mathbf{M}_2)$.

In [17] it was proved that $\mathrm{Var}(\mathbf{M}_2)$ covers only $\mathbb{ZS}$ in $\mathcal{L}(\mathbb{S})$, therefore it is join-irreducible in $\mathcal{L}(\mathbb{S})$. □

Remark. Observe that the identity $x^2y = x^2$ fails in semigroups $\mathbf{M}_1$, $\mathcal{D}(\mathbf{M}_1)$, $\mathbf{M}_3$ and $\mathcal{D}(\mathbf{M}_3)$. Therefore the variety $\mathrm{Var}(\mathbf{M}_2)$ does not contain the semigroups $\mathbf{M}_1$, $\mathcal{D}(\mathbf{M}_1)$, $\mathbf{M}_3$ and $\mathcal{D}(\mathbf{M}_3)$.

Proposition 6.6 *The variety Var*$(\mathbf{M}_3)$ *is determined by the identity $xyz = xy$. The variety Var*$(\mathcal{D}(\mathbf{M}_3))$ *is determined by the identity $xyz = yz$. The variety Var*$(\mathbf{M}_3)$ *consists of all semigroups $\mathbf{S} = (S, \cdot)$ such that the subsemigroup $S^2 = \{st \mid s, t \in S\}$ of $\mathbf{S}$ is a left zero semigroup. The variety Var*$(\mathcal{D}(\mathbf{M}_3))$ *consists of all semigroups $\mathbf{S} = (S, \cdot)$ such that the subsemigroup $S^2 = \{st \mid s, t \in S\}$ of $\mathbf{S}$ is a right zero semigroup. The variety Var*$(\mathbf{M}_3)$ *covers the variety $\mathbb{ZS} \vee \mathbb{LZS}$ in $\mathcal{L}(\mathbb{S})$ and it is join-irreducible in $\mathcal{L}(\mathbb{S})$, the variety Var*$(\mathcal{D}(\mathbf{M}_3))$ *covers the variety $\mathbb{ZS} \vee \mathbb{RZS}$ in $\mathcal{L}(\mathbb{S})$ and it is join-irreducible in $\mathcal{L}(\mathbb{S})$. Furthermore, Var*$(\mathbf{M}'_3) = \mathbb{ZS} \vee \mathbb{LZS}$ *and Var*$(\mathcal{D}(\mathbf{M}'_3)) = \mathbb{ZS} \vee \mathbb{RZS}$.

Proof. These statements were proved already in [16]. Statement 9 in [16] says that the semigroup variety determined by the identity $xyz = xy$ consists of all semigroups $\mathbf{S} = (S, \cdot)$ such that the subsemigroup S^2 of $\mathbf{S}$ consists of all left zeros of $\mathbf{S}$ and this variety covers only the variety $\mathbb{ZS} \vee \mathbb{LZS}$. Moreover, in the proof it is shown that this variety is generated by the semigroup $\mathbf{M}_3$ (in this paper, $\mathbf{M}_3$ is denoted as Q). From this it follows the statements for $\mathrm{Var}(\mathbf{M}_3)$ and the statements for $\mathrm{Var}(\mathcal{D}(\mathbf{M}_3))$ follow dually. The characterization of $\mathrm{Var}(\mathbf{M}_3')$ and $\mathrm{Var}(\mathcal{D}(\mathbf{M}_3'))$ is folklore. □

As a consequence we obtain the following corollary.

Corollary 6.7 *The varieties Var*$(\mathbf{M}_1)$*, Var*$(\mathcal{D}(\mathbf{M}_1))$*, Var*$(\mathbf{M}_2)$*, Var*$(\mathbf{M}_3)$ *and Var*$(\mathcal{D}(\mathbf{M}_3))$ *are incomparable.* □

The final part of this section is devoted to sufficient conditions under which a semigroup variety contains one of the semigroups $\mathbf{M}_1$, $\mathcal{D}(\mathbf{M}_1)$, $\mathbf{M}_2$, $\mathbf{M}_3$, and $\mathcal{D}(\mathbf{M}_3)$. First we recall that the semigroup structure determines a partial order $\leq$ on the set of Green's $\mathcal{J}$-classes: for Green's $\mathcal{J}$-classes J_1 and J_2 of a semigroup $\mathbf{S}$ we have $J_1 \leq J_2$ if and only if the ideal of $\mathbf{S}$ generated by J_1 contains J_2. We prove some auxiliary statements.

Statement 6.8 *Let* $\mathbf{S} = (S, \cdot)$ *be a semigroup such that* $\mathbf{S}$ *satisfies the identity* $x^2 = x^{2+n}$ *for some positive integer* n *and* $r(\mathbf{S})$ *is a left ideal. Then*

1. $r(\mathbf{S})$ *is the union of all regular* $\mathcal{J}$*-classes;*
2. *a Green's* $\mathcal{H}$*-class of* $\mathbf{S}$ *contains an idempotent element if and only if it belongs to a regular Green's* $\mathcal{J}$*-class of* $\mathbf{S}$*;*
3. *if* $st, ts \in r(\mathbf{S})$ *for some* $s, t \in S$ *then* st *and* ts *belong to the same Green's* $\mathcal{J}$*-class of* $\mathbf{S}$*;*
4. *the Green's relation* $\mathcal{J}$ *of* $\mathbf{S}$ *is a congruence of the subsemigroup* $r(\mathbf{S})$*;*
5. *if* J *is the greatest Green's* $\mathcal{J}$*-class of* $\mathbf{S}$ *then the least equivalence on* S *coinciding with the Green's relation* $\mathcal{H}$ *on* J *is a congruence of* $\mathbf{S}$*.*

Proof. Since $\mathbf{S}$ satisfies the identity $x^2 = x^{2+n}$, Green's relations $\mathcal{J}$ and $\mathcal{D}$ of $\mathbf{S}$ coincide. For every $s \in r(\mathbf{S})$ the Green's $\mathcal{L}$-class of $\mathbf{S}$ containing s is a subset of $r(\mathbf{S})$ because $r(\mathbf{S})$ is a left ideal. For every $s \in r(\mathbf{S})$, the Green's $\mathcal{H}$-class of $\mathbf{S}$ containing s is a group and thus it contains an idempotent element. Hence the Green's $\mathcal{J}$-class J of $\mathbf{S}$ containing s is regular and therefore any Green's $\mathcal{L}$-class that is a subset of J contains an idempotent element, see [9], and thus $J \subseteq r(\mathbf{S})$. The proof of (1) is complete.

By (1), $r(\mathbf{S})$ is a union of regular $\mathcal{J}$-classes of $\mathbf{S}$. Since for every $s \in r(\mathbf{S})$ the Green's $\mathcal{H}$-class of $\mathbf{S}$ containing s is a subsemigroup of $\mathbf{S}$ that is a group, we obtain (2).

Consider $s, t \in S$. Then the Green's class J_{ts} of $\mathbf{S}$ containing ts is less than the Green's class J_{stst} of $\mathbf{S}$ containing $stst = (st)^2$ and the Green's class J_{st} of $\mathbf{S}$ containing st is less than the Green's class J_{tsts} of $\mathbf{S}$ containing $tsts = (ts)^2$. If $st, ts \in r(\mathbf{S})$ then the Green's $\mathcal{H}$-classes H_{st} and H_{ts} of $\mathbf{S}$ containing st and ts, respectively, are groups and thus $(st)^2 \in H_{st}$ and $(ts)^2 \in H_{ts}$. Therefore

$J_{st} = J_{stst}$ and $J_{ts} = J_{tsts}$. Whence $J_{st} \geq J_{ts} \geq J_{st}$ and thus $J_{st} = J_{ts}$, and (3) is proved.

To prove (4), assume that $s, t \in r(\mathbf{S})$ belong to the same Green's $\mathcal{J}$-class of $\mathbf{S}$. Then there exists $u \in S$ such that $s\mathcal{L}u$ and $u\mathcal{R}t$. From this for every $v \in S$ we have $sv\mathcal{L}uv$ and $vu\mathcal{R}vt$, see [9]. Thus sv and uv and/or vu and vt belong to the same Green's $\mathcal{J}$-class of $\mathbf{S}$, and (3) completes the proof of (4).

By the hypothesis, J is an ideal and the description of semigroup structure, see [9], completes the proof of (5). □

Statement 6.9 *Let $\mathbb{V}$ be a semigroup variety.*

1. *If there exist a semigroup $\mathbf{S} = (S, \cdot) \in \mathbb{V}$ and $s \in S$ such that $s^2 \neq s^i$ for all $i \neq 2$ then $\mathbf{M}_2 \in \mathbb{V}$.*
2. *If the identity $x^2 = x^{2+n}$ fails in $\mathbb{V}$ for all $n > 1$ then $\mathbf{M}_2 \in \mathbb{V}$.*
3. *If $\mathbf{M}_1, \mathcal{D}(\mathbf{M}_1) \in \mathbb{V}$ then $\mathbf{M}_2 \in \mathbb{V}$.*
4. *If $\mathbf{M}_2 \notin \mathbb{V}$ then $r(\mathbf{S})$ is a union of regular Green's $\mathcal{J}$-classes for every semigroup $\mathbf{S} \in \mathbb{V}$.*
5. *If $\mathbf{M}_2 \notin \mathbb{V}$ then $r(\mathbf{S})$ is a subsemigroup for every $\mathbf{S} \in \mathbb{V}$.*
6. *If $\mathbf{M}_2 \notin \mathbb{V}$ and there exists a semigroup $\mathbf{S} \in \mathbb{V}$ such that $r(\mathbf{S})$ is not a right ideal, then $\mathbf{M}_1 \in \mathbb{V}$.*
7. *If $\mathbf{M}_2 \notin \mathbb{V}$ and there exists a semigroup $\mathbf{S} \in \mathbb{V}$ such that $r(\mathbf{S})$ is not a left ideal, then $\mathcal{D}(\mathbf{M}_1) \in \mathbb{V}$.*
8. *If $\mathbf{M}_2 \notin \mathbb{V}$ and if there exist a semigroup $\mathbf{S} = (S, \cdot) \in \mathbb{V}$ and $s, t \in S$ such that $s \in S \setminus r(\mathbf{S})$, $t \in r(\mathbf{S})$, t is contained in the ideal of $\mathbf{S}$ generated by s and $st \neq s^{1+n}t$ for all $n > 0$ then $\mathbf{M}_3 \in \mathbb{V}$.*
9. *If $\mathbf{M}_2 \notin \mathbb{V}$ and if there exist a semigroup $\mathbf{S} = (S, \cdot) \in \mathbb{V}$ and $s, t \in S$ such that $s \in S \setminus r(\mathbf{S})$, $t \in r(\mathbf{S})$, t is contained in the ideal of $\mathbf{S}$ generated by s and $ts \neq ts^{1+n}$ for all $n > 0$ then $\mathcal{D}(\mathbf{M}_3) \in \mathbb{V}$.*
10. *Let $\tilde{\mathbf{M}}_2$ be a semigroup obtained from the semigroup $\mathbf{M}_2$ by setting $ba = 0$ instead of $ba = c$. If $\tilde{\mathbf{M}}_2 \in \mathbb{V}$ then $\mathbf{M}_2 \in \mathbb{V}$.*
11. *If there exist a semigroup $\mathbf{S} = (S, \cdot) \in \mathbb{V}$ and $x, y, z \in S \setminus r(\mathbf{S})$ such that $xy = z$ and $x^2y \neq z \neq xy^2$ then $\mathbf{M}_2 \in \mathbb{V}$.*

Proof. We prove (1). By the hypothesis, $\mathbb{V}$ contains a semigroup $\mathbf{T} = (T, \cdot)$ where $T = \{x, x^2, x^3 = x^4\}$ and x, x^2 and x^3 are pairwise distinct. It is easy to see that $\mathbf{M}_2$ is isomorphic to a quotient of the subsemigroup

$$U = \{(x, x, x^3), (x, x^3, x), (x^2, x^2, x^3), (x^2, x^3, x^2), (x^2, x^3, x^3), (x^3, x^3, x^3)\}$$

of $\mathbf{T} \times \mathbf{T} \times \mathbf{T}$. Hence (1) follows.

(2) is a consequence of (1) – indeed, by (1), if $\mathbf{M}_2 \notin \mathbb{V}$ then for every $\mathbf{S} = (S, \cdot) \in \mathbb{V}$ and for every $s \in S$ there exist $n > 1$ with $s^2 = s^{2+n}$. If there exists a sequence $\{(\mathbf{S}_i, s_i)\}_{i=1}^{\infty}$ and an increasing sequence $\{n_i\}_{i=1}^{\infty}$ such that $\mathbf{S}_i = (S_i, \cdot) \in \mathbb{V}$ is a semigroup and $s_i \in S_i$ is an element such that n_i is the least number with $s_i^m = s_i^{m+n_i}$ for some natural number m, then for

$s = (s_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} \mathbf{S}_i \subseteq \mathbf{S}$ we have $s^2 \neq s^{2+n}$ for all positive integers n. Since $\mathbf{S} \in \mathbb{V}$, by (1), $\mathbf{M}_2 \in \mathbb{V}$. Thus, if $\mathbf{M}_2 \notin \mathbb{V}$ then $\mathbb{V}$ satisfies the identity $x^2 = x^{2+n}$ for some positive integer n.

To prove (3), observe that $\mathbf{M}_2$ is isomorphic to a quotient of the subsemigroup

$$V = \{(1,a),(a,1),(a,a),(1,0),(0,1),(a,0),(0,a),(0,0)\}$$

of $\mathbf{M}_1 \times \mathcal{D}(\mathbf{M}_1)$.

We prove (4). Consider a semigroup $\mathbf{S} = (S.\cdot) \in \mathbb{V}$ such that there exists a regular Green's class J of $\mathbf{S}$ and $x, y \in J$ with $xy \notin J$. By (2), we can assume that $\mathbb{V}$ satisfies the identity $x^2 = x^{2+n}$ for some positive integer n, and thus the Green's $\mathcal{J}$-classes and the Green's $\mathcal{D}$-classes of $\mathbf{S}$ coincide. Since J is regular, any Green's $\mathcal{L}$-class $L \subseteq J$ and any Green's class $R \subseteq J$ contains an idempotent element, see [9]. For $z \in J$, let R_z be the Green's $\mathcal{R}$-class containing z and L_z be the Green's $\mathcal{L}$-class containing z. From $xy \notin J$ it follows that the Green's $\mathcal{H}$-class $H = L_x \cap R_y$ does not contain an idempotent element, thus $uv \notin H$ for all $u, v \in H$, see [9]. If $u \in L_x$ is an idempotent element then for $w \in H$ we conclude that $wu \in H$ because $u \in L_w \cap R_u$ and $uw, (wu)^2 \notin J$. Whence $\mathcal{D}(\mathbf{M}_1)$ is isomorphic to a quotient of the subsemigroup of $\mathbf{S}$ generated by $\{u, w\}$. If v is an idempotent element of R_y then $\mathbf{M}_1$ is isomorphic to a quotient of the subsemigroup of $\mathbf{S}$ generated by $\{v, w\}$, and (3) completes the proof of (4).

Consider a semigroup $\mathbf{S} = (S, \cdot)$ satisfying the identity $x^2 = x^{2+n}$ for some positive integer n and assume that there exist $s, t \in r(\mathbf{S})$ such that $st \notin r(\mathbf{S})$. From $s, t \in r(\mathbf{S})$ it follows that $s = s^{1+n}$ and $t = t^{1+n}$ and $st \notin r(\mathbf{S})$ implies that $st \neq (st)^{1+n}$. Let W be the subsemigroup of $\mathbf{S}$ generated by $\{s, t\}$. Let I be the greatest ideal from W such that $st \notin W$. Then $s, t, st \notin I$ but $(st)^2 \in I$. Let V be the Ree's quotient semigroup W/I. Then $s^i t$ and st belong to the same Green's $\mathcal{L}$-class of W and st^i and st belong to the same Green's $\mathcal{R}$-class of W for all natural numbers i. Hence $s^i t^j \notin I$ for all positive integers i and j and st and sts (or tst) do not belong to the same Green's $\mathcal{D}$-class and therefore $sts, tst, ts \in I$. Since $s^{2n} = s^n$, $t^{2n} = t^n$, the set $U = \{s^n, t^n, s^n t^n, 0 = I\}$ is a subsemigroup of V. It is easy to see that $\mathbf{M}_1$ is isomorphic to the subsemigroup $\{s^n, s^n t^n, 0\}$ of U and $\mathcal{D}(\mathbf{M}_1)$ is isomorphic to the subsemigroup $\{t^n, s^n t^n, 0\}$ of U. Therefore $\mathbf{M}_1, \mathcal{D}(\mathbf{M}_1) \in \mathrm{Var}(\mathbf{S})$, and (3) completes the proof of (5).

Let $\mathbb{V}$ be a semigroup variety with $\mathbf{M}_2 \notin \mathbb{V}$ such that $r(\mathbf{S})$ is not a right ideal for some semigroup $\mathbf{S} = (S, \cdot) \in \mathbb{V}$. Thus there exist $s \in r(\mathbf{S})$ and $t \in S \setminus r(\mathbf{S})$ with $st \notin r(\mathbf{S})$. From $\mathbf{M}_2 \notin \mathbb{V}$ it follows the existence of a positive integer n such that the identity $x^2 = x^{2+n}$ is satisfied in $\mathbb{V}$, from $s \in r(\mathbf{S})$ we obtain that $s = s^{1+n}$ and therefore s^n is the idempotent element. Then $st = s^{n+1}t = s(s^n t)$ and thus $s^n t \notin r(\mathbf{S})$. Whence $\mathbf{M}_1$ is isomorphic to a Ree's quotient of the subsemigroup of $\mathbf{S}$ generated by $\{s^n, st\}$ and (6) is proved.

The statement (7) is dual to (6).

If $\mathbf{M}_2 \notin \mathbb{V}$ then, by (3), (4), (6) and (7) either $r(\mathbf{S})$ is a left ideal that is the union of all regular Green's $\mathcal{J}$-classes for every semigroup $\mathbf{S} \in \mathbb{V}$ or $r(\mathbf{S})$ is a right ideal that is the union of all regular Green's $\mathcal{J}$-classes for every semigroup $\mathbf{S} \in \mathbb{V}$. Assume that $\mathbf{S} \in \mathbb{V}$ is a semigroup satisfying the hypotheses of the statement (8). Let U be the subsemigroup of $\mathbf{S}$ generated by $\{s, t\}$ and let J be the Green's $\mathcal{J}$-class of $\mathbf{S}$ containing t. According to Statement 6.8(2) the subsemigroup of $\mathbf{S}$ generated by $\{t\}$ is a group and thus $U \cap J$ is the greatest $\mathcal{J}$-class of the semigroup U and, by Statement 6.8(5), the least equivalence $\sim$ on U coinciding with the Green's relation $\mathcal{H}$ of U on $U \cap J$ is a congruence of U. If $st \neq s^{1+n}t$ then st and $s^{1+n}t$ do not belong to the same Green's $\mathcal{L}$-class, because the left inner translation of s must be injective on the Green's $\mathcal{L}$-class containing st. Therefore $st \not\sim s^i t$ for all integers $i \geq 2$. Then $\mathbf{M}_3$ is isomorphic to the quotient of $U/\sim$ by the least congruence $\approx$ such that $s^2 \approx st \approx s^i$ for all integers $i > 2$. Hence (8) follows.

The statement (9) is dual to (8).

It is easy to see that $\mathbf{M}_2$ is isomorphic to a quotient of the subsemigroup of $\tilde{\mathbf{M}}_2 \times \tilde{\mathbf{M}}_2$ generated by $\{(a, b), (b, a)\}$, and (10) follows.

Assume that the hypothesis of (11) holds. It is easy to see that $\mathbf{M}_2$ or $\tilde{\mathbf{M}}_2$ is isomorphic to a quotient of the subsemigroup of $\mathbf{S}$ generated by $\{x, y\}$. Then (10) completes the proof of (11). □

Corollary 6.10 *A semigroup variety $\mathbb{V}$ is var-relatively ff-alg-universal and it is α-determined for no cardinal α whenever it satisfies one of the following conditions:*

1. *$\mathbb{V}$ fails the identity $x^2 = x^{2+n}$ for every positive integer n;*
2. *there exists a semigroup $\mathbf{S} \in \mathbb{V}$ such that $r(\mathbf{S})$ is not the union of all regular Green's $\mathcal{J}$-classes of $\mathbf{S}$;*
3. *there exist semigroups $\mathbf{S}_1, \mathbf{S}_2 \in \mathbb{V}$ such that $r(\mathbf{S}_1)$ is not a left ideal of $\mathbf{S}_1$ and $r(\mathbf{S}_2)$ is not a right ideal of $\mathbf{S}_2$;*
4. *there exist a semigroup $\mathbf{S} \in \mathbb{V}$ and $x, y, z \in S \setminus r(\mathbf{S})$ such that $xy = z$ and $x^2y \neq z \neq xy^2$;*
5. *there exist a semigroup $\mathbf{S} \in \mathbb{V}$, $s \in S \setminus r(\mathbf{S})$ and $t \in r(\mathbf{S})$ such that t belongs to the least ideal generated by s, $s^2 = s^{2+n} \in r(\mathbf{S})$ for some positive integer n and $st \neq s^{1+n}t$ or $ts \neq ts^{1+n}$.*

Proof. The statement is a combination of Theorem 1.7, Corollary 6.2 and Statement 6.9. □

If $\mathbb{V}$ is a semigroup variety satisfying that $x^2 = x^{2+n}$ for some positive integer n and $r(\mathbf{S})$ is a left ideal (or a right ideal) for any semigroup $\mathbf{S} \in \mathbb{V}$, then observe that the existence of a semigroup $\mathbf{S} \in \mathbb{V}$ such that $r(\mathbf{S}) \in \mathbb{LSN}$ or $r(\mathbf{S}) \in \mathbb{RSN}$ implies, by Theorem 6.1, that $\mathbb{V}$ is var-relatively *ff*-alg-universal but it is an open question whether $\mathbb{V}$ is α-determined for some cardinal $\alpha > 1$.

Theorem 6.11 *A semigroup variety $\mathbb{V}$ has an ff-alg-universal* 3*-expansion whenever there exists a semigroup $\mathbf{S} \in \mathbb{V}$ such that $\mathbf{S}$ is neither an inflation of a completely simple semigroup nor an inflation of a semilattice of groups.*

Remark. If $\mathbb{W}$ is a variety consisting of completely simple semigroups (or semilattices of groups) then the variety $\mathbb{ZS} \vee \mathbb{V}$ consists of the inflations of $\mathbb{W}$.

Proof. Let $\mathbb{V}$ be a semigroup variety. By Theorem 1.7, Corollary 6.2 and Statement 6.9, we conclude that $\mathbb{V}$ has an ff-alg-universal 3-expansion if one of the following condition is fulfilled:

(i) $\mathbb{V}$ fails the identity $x^2 = x^{2+n}$ for every positive integer n;
(ii) there exists a semigroup $\mathbf{S} \in \mathbb{V}$ such that $r(\mathbf{S})$ is not an ideal;
(iii) there exist a semigroup $\mathbf{S} = (S, \cdot) \in \mathbb{V}$ and a reducible element of $S \setminus r(\mathbf{S})$;
(iv) there exist a semigroup $\mathbf{S} = (S, \cdot) \in \mathbb{V}$, $s \in S \setminus r(\mathbf{S})$ and $t \in r(\mathbf{S})$ such that $s^2 = s^{2+n}$ for some positive integer n, t belongs to an ideal of $\mathbf{S}$ generated by s and $st \neq s^{1+n}t$ or $ts \neq ts^{1+n}$.

By Proposition 6.8(4), the Green's relation $\mathcal{J}$ of $\mathbf{S}$ is a congruence on $r(\mathbf{S})$ and, by Proposition 6.8(1) and Statement 6.9(4), any regular Green's class of $\mathbf{S}$ is a subsemigroup that is a completely simple semigroup. If there exists a semigroup $\mathbf{S} \in \mathbb{V}$ such that $r(\mathbf{S})$ is neither a completely simple semigroup nor a semilattice of groups then $\mathbb{LNB} \subseteq \mathbb{V}$ or $\mathbb{RNB} \subseteq \mathbb{V}$ and, by Theorem 6.1, $\mathbb{V}$ has an ff-alg-universal 3-expansion. If $r(\mathbf{S})$ is a completely simple semigroup for a semigroup $\mathbf{S} \in \mathbb{V}$, then, by (iv), $\mathbf{S}$ is an inflation of $r(\mathbf{S})$. Thus we can assume that $r(\mathbf{S})$ is a semilattice of groups for a semigroup $\mathbf{S} = (S, \cdot) \in \mathbb{V}$. Then the Green's $\mathcal{J}$-classes of $r(\mathbf{S})$ coincide with the Green's $\mathcal{H}$-classes of $r(\mathbf{S})$, and, by (iii), we have $\{st \mid s, t \in S\} = r(\mathbf{S})$. We prove that $\mathbf{S}$ is an inflation of $r(\mathbf{S})$. Assume that $\mathbb{V}$ satisfies the identity $x^2 = x^{2+n}$ for some positive integer n. To prove that $\mathbf{S}$ is an inflation of $r(\mathbf{S})$ it suffices to prove that $ts = ts^{1+n}$ or $st = s^{1+n}t$ for all $s \in S \setminus r(\mathbf{S})$ and $t \in S$. For any $v \in S$, let $H(v)$ be the Green's $\mathcal{H}$-class containing v. Observe that $ts, st \in r(\mathbf{S})$ and hence $ts = (ts)^{n+1} \in H(ts)$ and $st = (st)^{n+1} \in H(st)$. Therefore $H(ts) = H(st)$ and $ts^2t, st^2s \in H(ts)$. Thus $ts^2, s^2t \in H(ts) = H(ts^2) = H(s^2t)$. From this it follows that $ts^{1+n}, s^{1+n}t \in H(ts)$. Since the left and/or the right inner translation of s in $\mathbf{S}$ maps $H(st)$ into itself, therefore both translations are injective on $H(ts)$. From this it follows that $st = s^{1+n}t$ and $ts = ts^{1+n}$ because $s^2 = s^{2+n}$. We conclude that $\mathbf{S}$ is an inflation of $r(\mathbf{S})$ and the proof is complete. □

Remark. The Q-universality in the semigroup varieties and quasivarieties generated by semigroups $\mathbf{M}_2$, $\mathbf{M}_3$ and $\mathcal{D}(\mathbf{M}_3)$ is studied in the new preprint "Weak alg-universality and Q-universality of semigroup quasivarieties" due to the authors of this paper. It is shown that every variety $\mathrm{Var}(\mathbf{M}_2)$, $\mathrm{Var}(\mathbf{M}_3)$ and $\mathrm{Var}(\mathcal{D}(\mathbf{M}_3))$ contains a finite semigroup which generates the Q-universal and relatively ff-alg-universal quasivariety. On the other hand, the quasivarieties generated by $\mathbf{M}_2$, $\mathbf{M}_3$ and $\mathcal{D}(\mathbf{M}_3)$ respectively are neither Q-universal nor relatively ff-alg-universal. The analogous result was proved by M. V. Sapir [33]. He proved that there exists a finite commutative three-nilpotent semigroup $\mathbf{S} \in \mathrm{Var}(\mathbf{M}_2)$ such that the quasivariety generated by $\mathbf{S}$ is Q-universal.

The semigroup presented in our preprint is substantially lesser than the Sapir's semigroup.

After submission of this paper we have obtained a remark from J. Sichler. He has proved that Conjecture 1.6 is true. In fact, he proved that if $\mathbb{V}$ is a weakly var-relatively alg-universal variety (or a weakly var-relatively ff-alg-universal variety) then $\mathbb{V}$ has an alg-universal α-expansion (or an ff-alg-universal α-expansion, respectively) for some cardinal $\alpha > 1$.

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