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CLONE AUTOMORPHISMS AND HYPERSUBSTITUTIONS[1]

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Abstract. A hypersubstitution maps the operation symbols of a type τ to terms of the same arity and can be uniquely extended to a mapping defined on the set of all terms of this type. In this paper we prove that the group of all clone automorphisms of an algebra $\mathcal{A}$ is isomorphic to a certain group of hypersubstitutions supposed the variety $V(\mathcal{A})$ generated by $\mathcal{A}$ is solid.

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1. Introduction

Let $(f_i)_{i\in I}$ be a sequence of operation symbols, where f_i is n_i-ary and $n_i \geq 1$. We denote by $X_n = \{x_1, \ldots, x_n\}$ a finite and by $X = \{x_1, \ldots\}$ an infinite alphabet. The sequence $\tau = (n_i)_{i\in I}$ is called the type of the language and $W_\tau(X_n)$ denotes the set of all n-ary terms of type τ. Let $W_\tau(X) := \bigcup_{n\geq 1} W_\tau(X_n)$ be the set of all terms of type τ. There are different possibilities to define operations on the set of all terms of type τ (see, e.g., [6]-[8], [10], [12], [13]). Using for every $i \in I$ the operations $\overline{f}_i : W_\tau(X_n)^{n_i} \to W_\tau(X_n)$ with $\overline{f}_i(t_1, \ldots, t_{n_i}) := f_i(t_1, \ldots, t_{n_i})$, we obtain the absolutely free n-generated algebra $\mathcal{F}_\tau(X_n) = (W_\tau(X_n); (\overline{f}_i)_{i\in I})$ of type τ. Let $\mathcal{F}_\tau(X) = (W_\tau(X); (\overline{f}_i)_{i\in I})$ be the absolutely free algebra generated by X.

Another possibility is the operation of composition or superposition of terms which plays an important role in Universal Algebra, Clone Theory and Computer Science. For each $m, n \in \mathbb{N} \setminus \{0\}$ the superposition operation

$$S^n_m : W_\tau(X_n) \times W_\tau(X_m)^n \to W_\tau(X_m)$$

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is a mapping which maps an n-ary term and n m-ary terms into an m-ary term and fulfils the following rules:

(i) $S^n_m(x_j, t_1, \ldots, t_n) := t_j$ if x_j, $1 \le j \le n$, is a variable from X_n,

(ii) $S^n_m(f_i(r_1, \ldots, r_{n_i}), t_1, \ldots, t_n) :=$
$= f_i(S^n_m(r_1, t_1, \ldots, t_n), \ldots, S^n_m(r_{n_i}, t_1, \ldots, t_n))$ for $i \in I$.

We obtain a *multi-based (heterogeneous)* algebra

$$Cl(\tau) := ((W_\tau(X_n))_{n\in\mathbb{N}\setminus\{0\}}; (S^n_m)_{m,n\in\mathbb{N}\setminus\{0\}}, (x_i)_{i\le n\in\mathbb{N}\setminus\{0\}})$$

which is called *unitary Menger system* (or *clone of all terms of type* τ) since it satisfies the following identities:

(C1) $\widetilde{S}^p_m(X_0, \widetilde{S}^n_m(Y_1, X_1, \ldots, X_n), \ldots, \widetilde{S}^n_m(Y_p, X_1, \ldots, X_n)) \approx$
$\widetilde{S}^n_m(\widetilde{S}^p_m(X_0, Y_1, \ldots, Y_p), X_1, \ldots, X_n)$,

(C2) $\widetilde{S}^n_m(\lambda_i, X_1, \ldots, X_n) \approx X_i$, $1 \le i \le n$,

(C3) $\widetilde{S}^n_m(X_1, \lambda_1, \ldots, \lambda_n) \approx X_1$.

Here $\widetilde{S}^n$ is an $(n+1)$-ary operation symbol, $\lambda_1, \ldots, \lambda_n$ are nullary operation symbols and $X_0, X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are variables.
The algebra $Cl(\tau)$ is free with respect to the heterogeneous variety of all unitary Menger systems.

By restriction to an n-ary type τ_n, where all operation symbols have the same arity $n \ge 1$ and to the set $W_\tau(X_n)$ we obtain a *one-based* algebra n-$Cl(\tau_n) := (W_\tau(X_n); S^n, x_1, \ldots, x_n)$ of type $(n+1, 0, \ldots, 0)$ with the $(n+1)$-ary operation $S^n := S^n_n$. This algebra is an example for a *unitary Menger algebra of rank n*. It satisfies axioms which can be derived from (C1), (C2), (C3) if we identify all operation symbols S^n_m with S^n. These identities define the variety of all unitary Menger algebras of rank n. The algebra n-$Cl(\tau_n)$ is free in the variety of all unitary Menger algebras of rank n and is freely generated by $\{f_i(x_1, \ldots, x_n) \mid i \in I\}$ (see, e.g., [3], Theorem 1.2, and [2]).

For every algebra $\mathcal{A} = (A; ({f_i}^{\mathcal{A}})_{i\in I})$ each n-ary term from $W_\tau(X_n)$ induces an n-ary term operation $t^{\mathcal{A}}$ in the usual way.

We denote by $(W_\tau(X_n))^{\mathcal{A}} := \{t^{\mathcal{A}} \mid t \in W_\tau(X_n)\}$ the set of all term operations induced by n-ary terms of type τ. On the set $(W_\tau(X_n))^{\mathcal{A}}$ one may define an $(n+1)$-ary superposition operation $S^{n,A}$ by $S^{n,A}(t^{\mathcal{A}}, {t_1}^{\mathcal{A}}, \ldots, {t_n}^{\mathcal{A}})(a_1, \ldots, a_n) := t^{\mathcal{A}}({t_1}^{\mathcal{A}}, \ldots, {t_n}^{\mathcal{A}})(a_1, \ldots, a_n) := t^{\mathcal{A}}({t_1}^{\mathcal{A}}(a_1, \ldots, a_n), \ldots, {t_n}^{\mathcal{A}}(a_1, \ldots, a_n))$.

Sets of operations defined on A containing all projections and being closed under the application of the superposition operation are called *clones of operations*. We denote by $T^{(n)}(\mathcal{A})$ the clone of all n-ary operations generated by the fundamental operations $\{{f_i}^{\mathcal{A}} \mid i \in I\}$ of the algebra $\mathcal{A}$ (also called *n-clone* of $\mathcal{A}$ in [6]). It is easy to check that $T^{(n)}(\mathcal{A}) = W_\tau(X_n)^{\mathcal{A}}$. Further we have $T(\mathcal{A}) := \bigcup_{n\geq 1} T^{(n)}(\mathcal{A}) = W_\tau(X)^{\mathcal{A}}$. The algebra

$$\mathcal{T}^{(n)}(\mathcal{A}) := (T^{(n)}(\mathcal{A}); S^{n,A}, e_1^{n,A}, \ldots, e_n^{n,A})$$

is of type $(n+1, 0, \ldots, 0)$. It turns out that $\mathcal{T}^{(n)}(\mathcal{A})$ satisfies the defining identities of the variety of unitary Menger algebras of rank n. $T^{(n)}(\mathcal{A})$ is also called the *clone of n-ary term operations* of the algebra $\mathcal{A}$. In a similar way as we did for terms we can define superposition operations $S_m^{n,A}$ and obtain a multi-based algebra

$$\mathcal{T}(\mathcal{A}) := ((T^{(n)}(\mathcal{A}))_{n\in\mathbb{N}\setminus\{0\}}; (S_m^{n,A})_{m,n\in\mathbb{N}}, (e_i^{n,A})_{i\leq n\in\mathbb{N}})$$

which is called *the clone of term operations of* $\mathcal{A}$.

For an algebra $\mathcal{A}$ of type τ we denote by $V(\mathcal{A})$ the variety generated by $\mathcal{A}$ and by $Id\mathcal{A}$ the set of all equations of type τ which are satisfied as identities in $\mathcal{A}$, i.e.

$$Id\mathcal{A} := \{s \approx t \mid s, t \in W_\tau(X) \text{ and } s^{\mathcal{A}} = t^{\mathcal{A}}\}.$$

For the variety $V(\mathcal{A})$ we denote by $IdV(\mathcal{A})$ the set of all identities which are satisfied in every algebra of $V(\mathcal{A})$, i.e.

$$IdV(\mathcal{A}) := \{s \approx t \mid s, t \in W_\tau(X) \text{ and } \forall\, \mathcal{B} \in V(\mathcal{A})\ (s^{\mathcal{B}} = t^{\mathcal{B}})\}.$$

It is well-known that $Id\mathcal{A} = IdV(\mathcal{A})$.

2. Hypersubstitutions

An arbitrary mapping $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$ which preserves the arity, that is, which maps every n_i-ary operation symbol of type τ to an n_i-ary term of the same type, is called a *hypersubstitution* of type τ. Any hypersubstitution σ induces a mapping

$$\widehat{\sigma} : W_\tau(X) \to W_\tau(X)$$

in the following inductive way:

(i) $\hat{\sigma}[x_i] := x_i \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] = S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$.

This extension is uniquely determined and allows us to define a multiplication, denoted by $\circ_h$, on the set $Hyp(\tau)$ of all hypersubstitutions of type τ by

$$\sigma_1 \circ_h \sigma_2 = \hat{\sigma}_1 \circ \sigma_2,$$

where $\circ$ is the usual composition of functions. This multiplication is associative, and if we denote by σ_{id} the identity hypersubstitution which maps each n_i-*ary* operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$, we obtain a monoid

$$\mathcal{H}yp(\tau) := (Hyp(\tau); \circ_h, \sigma_{id}).$$

Hypersubstitutions can be used to define the concept of a *hyperidentity* in a variety V of algebras of type τ. An equation $s \approx t$ consisting of terms of type τ forms a hyperidentity in V if for all $\sigma \in Hyp(\tau)$ the equations $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are satisfied as identities in V. A variety V is called *solid* if every identity in V is a hyperidentity in this variety. For more information on hyperidentities and solidity, we refer to [5].

Not all hypersubstitutions are important if we want to check the hyperidentity property in a variety V. To reduce the complexity of this checking J. Płonka introduced the following relation ([11]):

Definition 2.1 Let $\sigma_1, \sigma_2 \in Hyp(\tau)$ and let V be a variety of type τ. Then

$$\sigma_1 \sim_V \sigma_2 :\Longleftrightarrow (\forall i \in I)\ (\sigma_1(f_i) \approx \sigma_2(f_i) \in IdV).$$

The relation $\sim_V$ is an equivalence relation and has the following properties ([11]):

Proposition 2.2 (i) *If $\sigma_1 \sim_V \sigma_2$, then for any term $t \in W_\tau(X)$ the equation $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$ is an identity in V.*

(ii) *If $\sigma_1 \sim_V \sigma_2$ and $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdV$, then $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV$.*

(iii) *If V is solid, then $\sim_V$ is a congruence relation on $\mathcal{H}yp(\tau)$.*

Endomorphisms of the unitary Menger algebra n-$Cl(\tau_n)$ of rank n and hypersubstitutions are closely related to each other. Indeed, we have

Proposition 2.3 *The endomorphism monoid $End(n\text{-}Cl(\tau_n))$ is isomorphic to the monoid $(Hyp(\tau_n); \circ_h, \sigma_{id})$.*

The set $\{f_i^{\mathcal{A}} \mid i \in I\}$ can be regarded as a sequence $(\{f_i^{n,\mathcal{A}} \mid i \in I_n\})_{n \in \mathbb{N}\setminus\{0\}}$ where $I = \bigcup_{n \in \mathbb{N}\setminus\{0\}} I_n$ and where $f_i^{n,\mathcal{A}}$ is n-ary. We use the following result:

Proposition 2.4 ([1]) *Let $\mathcal{A}$ be an algebra of type τ. Then $V(\mathcal{A})$ is solid iff the clone of all term operations of $\mathcal{A}$, i.e. the heterogeneous algebra $\mathcal{T}(\mathcal{A})$ is free with respect to itself, freely generated by $(\{f_i^{n,\mathcal{A}} \mid i \in I_n\})_{n \in \mathbb{N}\setminus\{0\}}$ that means, every heterogeneous mapping*

$$(\eta_n)_{n \in \mathbb{N}\setminus\{0\}} : (\{f_i^{n,\mathcal{A}} \mid i \in I_n\})_{n \in \mathbb{N}\setminus\{0\}} \to (T^{(n)}(\mathcal{A}))_{n \in \mathbb{N}\setminus\{0\}}$$

can be uniquely extended to an endomorphism

$$\overline{\eta} : \mathcal{T}(\mathcal{A}) \to \mathcal{T}(\mathcal{A})$$

.

3. The Kernel Monoid of Hypersubstitutions

A restricted version of the concept of a hyperidentity is that of an M-*hyperidentity*, where $\mathcal{M} = (M;\ \circ_h, \sigma_{id})$, $M \subseteq \mathcal{H}yp(\tau)$ is a submonoid of the monoid $\mathcal{H}yp(\tau)$ of all hypersubstitutions of type τ. Using M-hyperidentities one can define M*-solid varieties*. All M-solid varieties of type τ form a complete lattice $S_M(\tau)$ which is a complete sublattice of the lattice $\mathcal{L}(\tau)$ of all varieties of type τ with $S_{M_1}(\tau) \supseteq S_{M_2}(\tau)$ whenever $\mathcal{M}_1 \subseteq \mathcal{M}_2$. In fact, there is a Galois connection between submonoids of $\mathcal{H}yp(\tau)$ and complete sublattices of $\mathcal{L}(\tau)$.

In this section, we will introduce a new monoid of hypersubstitutions which is defined by using the *kernel of a hypersubstitution.* If σ is a hypersubstitution of type τ, it is very natural to ask for its kernel,

$$ker(\sigma) := \{(s,t) \mid s,t \in W_\tau(X) \text{ and } \hat{\sigma}[s] = \hat{\sigma}[t]\}.$$

The kernel of a hypersubstitution is a fully invariant congruence on the absolutely free algebra $\mathcal{F}_\tau(X)$. Clearly, σ is injective iff $ker(\sigma) = \Delta_{W_\tau(X)}$. The diagonal $\Delta_{W_\tau(X)}$ is the set of all identities satisfied in the variety $Alg(\tau)$ of all algebras of type τ. Therefore, it is quite natural to generalize the concept of a kernel of a hypersubstitution in the following way:

Definition 3.1 ([4]) The set

$$\ker_V(\sigma) := \{(s,t) \mid s,t \in W_\tau(X) \text{ and } \hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV\}$$

will be called the *kernel of* σ *with respect to* V or the *semantical kernel* of σ. The kernel $\ker(\sigma)$ of a hypersubstitution will be called the syntactical kernel.

In [4] was proved

Proposition 3.2 *Let σ be a hypersubstitution of type $\tau = (n_i)_{i \in I}$ with $n_i \geq 1$ for all $i \in I$. Then $\ker_V(\sigma)$ is a fully invariant congruence relation on the absolutely free algebra $\mathcal{F}_\tau(X)$.*

If $\mathcal{A} = (A; ({f_i}^{\mathcal{A}})_{i \in I})$ is an algebra of type τ, then we consider the following set $M^{\mathcal{A}}_{\ker}$ of hypersubstitutions:

$$M^{\mathcal{A}}_{\ker} := \{\sigma \mid \sigma \in Hyp(\tau) \text{ and } \ker_{V(\mathcal{A})}(\sigma) = IdV(\mathcal{A}) \text{ and } \hat{\sigma}[W_\tau(X)]^{\mathcal{A}} = T(\mathcal{A})\}.$$

Then we have:

Lemma 3.3 *For every algebra $\mathcal{A}$ of type τ the set $M^{\mathcal{A}}_{\ker}$ forms a submonoid of the monoid $\mathcal{H}yp(\tau)$ of all hypersubstitutions of type τ.*

Proof. Consider two hypersubstitutions $\sigma_1, \sigma_2 \in M^{\mathcal{A}}_{\ker}$. Then we have
$(s,t) \in \ker_{V(\mathcal{A})}(\sigma_1 \circ_h \sigma_2)$

$$\begin{aligned} &\iff (\hat{\sigma}_1 \circ \hat{\sigma}_2)[s] \approx (\hat{\sigma}_1 \circ \hat{\sigma}_2)[t] \in IdV(\mathcal{A}) \\ &\iff \hat{\sigma}_1[\hat{\sigma}_2[s]] \approx \hat{\sigma}_1[\hat{\sigma}_2[t]] \in IdV(\mathcal{A}) \\ &\iff (\hat{\sigma}_2[s], \hat{\sigma}_2[t]) \in \ker_{V(\mathcal{A})}(\sigma_1) \end{aligned}$$

by definition of the semantical kernel. Since $\ker_{V(\mathcal{A})}(\sigma_1) = IdV(\mathcal{A})$, we obtain $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV(\mathcal{A})$ and again, by definition of the kernel, we have $(s,t) \in \ker_{V(\mathcal{A})}(\sigma_2)$. Since $\ker_{V(\mathcal{A})}(\sigma_2) = IdV(\mathcal{A})$, we obtain $s \approx t \in IdV(\mathcal{A})$ and then $\ker_{V(\mathcal{A})}(\sigma_1 \circ_h \sigma_2) = IdV(\mathcal{A})$.

Further, if $(\hat{\sigma}_i[W_\tau(X)])^{\mathcal{A}} = T(\mathcal{A})$, $i = 1, 2$, then from $(\hat{\sigma}_1[W_\tau(X)])^{\mathcal{A}} = T(\mathcal{A})$ we obtain that for every $t^{\mathcal{A}} \in T(\mathcal{A})$ there is a term $s \in W_\tau(X)$ such that $(\hat{\sigma}_1[s])^{\mathcal{A}} = t^{\mathcal{A}}$ and then $\hat{\sigma}_1[s] \approx t \in IdV(\mathcal{A})$. For $s^{\mathcal{A}} \in T(\mathcal{A})$ there is a term $s' \in W_\tau(X)$ such that $(\hat{\sigma}_2[s'])^{\mathcal{A}} = s^{\mathcal{A}}$ and $\hat{\sigma}_2[s'] \approx s \in IdV(\mathcal{A})$. Applying $\hat{\sigma}_1$ on both sides and using that $IdV(\mathcal{A}) = \ker_{V(\mathcal{A})}(\sigma_1)$, we have $\hat{\sigma}_1[\hat{\sigma}_2[s']] \approx \hat{\sigma}_1[s] \approx t \in IdV(\mathcal{A})$, thus $(\hat{\sigma}_1[\hat{\sigma}_2[s']])^{\mathcal{A}} = t^{\mathcal{A}}$ and this means that for $t^{\mathcal{A}} \in T(\mathcal{A})$ there is a term $s' \in W_\tau(X)$ such that $((\hat{\sigma}_1 \circ \hat{\sigma}_2)[s'])^{\mathcal{A}} = t^{\mathcal{A}}$ and thus $((\sigma_1 \circ_h \sigma_2)\ \hat{}\ [W_\tau(X)])^{\mathcal{A}} = T(\mathcal{A})$.

This proves that $M_{\ker}^{\mathcal{A}}$ is closed under the multiplication of hypersubstitutions. The set $M_{\ker}^{\mathcal{A}}$ contains the identity hypersubstitution since

$$\begin{aligned} (s,t) \in \ker_{V(\mathcal{A})}(\sigma_{id}) &\iff \hat{\sigma}_{id}[s] \approx \hat{\sigma}_{id}[t] \in IdV(\mathcal{A}) \\ &\iff s \approx t \in IdV(\mathcal{A}) \text{ and } (\hat{\sigma}_{id}[W_\tau(X)])^{\mathcal{A}} = W_\tau(X)^{\mathcal{A}} = T(\mathcal{A}). \end{aligned}$$

□

We call $\mathcal{M}_{\ker}^{\mathcal{A}}$ the *kernel monoid of hypersubstitutions with respect to* $\mathcal{A}$.

An interesting property of the kernel monoid $\mathcal{M}_{\ker}^{\mathcal{A}}$ is that it consists of full blocks of the equivalence relation $\sim_{V(\mathcal{A})}$, i.e. this relation sautrates the kernel monoid $\mathcal{M}_{\ker}^{\mathcal{A}}$.

Proposition 3.4 *The kernel monoid with respect to an algebra $\mathcal{A}$ of type τ is a union of equivalence classes of the relation $\sim_{V(\mathcal{A})}$.*

Proof. We show that for $\sigma_1 \sim_{V(\mathcal{A})} \sigma_2$ we have $\ker_{V(\mathcal{A})}(\sigma_1) = \ker_{V(\mathcal{A})}(\sigma_2)$. By Proposition 2.2 (ii) for $\sigma_1 \sim_{V(\mathcal{A})} \sigma_2$, we get $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdV(\mathcal{A})$ iff $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV(\mathcal{A})$ and therefore the two kernels are equal.

Further, if $(\hat{\sigma}_1[W_\tau(X)])^{\mathcal{A}} = T(\mathcal{A})$, then for every $t^{\mathcal{A}} \in T(\mathcal{A})$ there is a term $s \in W_\tau(X)$ such that $(\hat{\sigma}_1[s])^{\mathcal{A}} = t^{\mathcal{A}}$. If $\sigma_1 \sim_{V(\mathcal{A})} \sigma_2$ then by Proposition 2.2 (i), $\hat{\sigma}_2[s] \approx \hat{\sigma}_1[s] \in IdV(\mathcal{A})$ and thus $(\hat{\sigma}_2[s])^{\mathcal{A}} = (\hat{\sigma}_1[s])^{\mathcal{A}} = t^{\mathcal{A}}$ and then for every $t^{\mathcal{A}} \in T(\mathcal{A})$ there is a term $s \in W_\tau(X)$ such that $(\hat{\sigma}_2[s])^{\mathcal{A}} = t^{\mathcal{A}}$ and this means $(\hat{\sigma}_2[W_\tau(X)])^{\mathcal{A}} = T(\mathcal{A})$.

If now $\sigma_1 \in M_{\ker}^{\mathcal{A}}$ and $\sigma_2 \sim_{V(\mathcal{A})} \sigma_1$, then $\ker_{V(\mathcal{A})}(\sigma_1) = IdV(\mathcal{A}) = \ker_{V(\mathcal{A})}(\sigma_2)$ and $(\hat{\sigma}_2[W_\tau(X)])^{\mathcal{A}} = T(\mathcal{A})$ and thus $\sigma_2 \in M_{\ker}^{\mathcal{A}}$ □

Another consequence of Proposition 2.2 is the following corollary:

Corollary 3.5 *If $V(\mathcal{A})$ is a solid variety, then $M_{\ker}^{\mathcal{A}}/\sim_{V(\mathcal{A})}$ is a monoid.*

This is clear, since for solid varieties the relation $\sim_V$ is a congruence.

4. Clone Automorphisms

We mentioned already that extended hypersubstitutions correspond to endomorphisms of $n\text{-}Cl(\tau_n)$. Since $n\text{-}Cl(\tau_n)$ is free in the variety of all unitary Menger algebras of rank n, the Menger algebra $(T^{(n)}(\mathcal{A}); S^{n,A}, e_1^{n,A}, \ldots, e_n^{n,A})$ is a homomorphic image of $n\text{-}Cl(\tau_n)$.

Here we ask whether the group of all automorphisms of the multi-based algebra $\mathcal{T}(\mathcal{A})$ can be described by hypersubstitutions. Indeed, we make the following observations:

(*) To every clone autmomorphism $\varphi \in Aut(\mathcal{T}(\mathcal{A}))$ there corresponds a class of $M_{\text{ker}}^{\mathcal{A}}/\sim_{V(\mathcal{A})}$.

In fact, if φ maps the fundamental operation $f_i^{\mathcal{A}}$ to a term operation $t_i^{\mathcal{A}}$, then we assign to φ the class $A_\varphi := \{\sigma \mid \sigma \in Hyp(\tau) \text{ and } \sigma(f_i)^{\mathcal{A}} = t_i^{\mathcal{A}}\}$. Therefore A_φ is an equivalence class with respect to $\sim_{V(\mathcal{A})}$. Indeed, if $\sigma, \sigma' \in A_\varphi$, then $\sigma(f_i)^{\mathcal{A}} = t_i^{\mathcal{A}} = \sigma'(f_i)^{\mathcal{A}}$, and then $\sigma(f_i) \approx \sigma'(f_i) \in IdV(\mathcal{A})$, which means $\sigma \sim_{V(\mathcal{A})} \sigma'$. A_φ is a full equivalence class, since from $\sigma \in A_\varphi$ and $\sigma' \sim_{V(\mathcal{A})} \sigma$ there follows $\sigma'(f_i)^{\mathcal{A}} = \sigma(f_i)^{\mathcal{A}} = t_i^{\mathcal{A}}$ and thus $\sigma' \in A_\varphi$.

Therefore, φ maps $f_i^{\mathcal{A}}$ to $[\sigma]_{\sim_{V(\mathcal{A})}}$ with $\sigma(f_i)^{\mathcal{A}} = \varphi(f_i^{\mathcal{A}})$.
If we can show that $\sigma \in M_{\text{ker}}^{\mathcal{A}}/\sim_{V(\mathcal{A})}$, then, by Proposition 3.4, $[\sigma]_{\sim_{V(\mathcal{A})}} \subseteq M_{\text{ker}}^{\mathcal{A}}/\sim_{V(\mathcal{A})}$. First, we show that from $\varphi(f_i^{\mathcal{A}}) = \sigma(f_i)^{\mathcal{A}}$ for every term t there follows $\varphi(t^{\mathcal{A}}) = \hat{\sigma}[t]^{\mathcal{A}}$. If $t = x_i$ is a variable, then $\varphi(x_i^{\mathcal{A}}) = \varphi(e_i^{n,A}) = e_i^{n,A} = x_i^{\mathcal{A}} = \hat{\sigma}[x_i]^{\mathcal{A}}$. If $t = f(t_1, \ldots, t_{n_i})$ is a composite term and assume that $\varphi(t_i^{\mathcal{A}}) = \hat{\sigma}[t_i]^{\mathcal{A}}$ for $i = 1, \ldots, n_i$, then from $\varphi(f_i^{\mathcal{A}}) = \sigma(f_i)^{\mathcal{A}}$ we get by superposition $\varphi(f_i^{\mathcal{A}})(\varphi(t_1^{\mathcal{A}}), \ldots, \varphi(t_{n_i}^{\mathcal{A}})) = \sigma(f_i)^{\mathcal{A}}(\hat{\sigma}[t_1]^{\mathcal{A}}, \ldots, \hat{\sigma}[t_{n_i}]^{\mathcal{A}}) = (\hat{\sigma}[f(t_1, \ldots, t_{n_i})])^{\mathcal{A}}$.

Now, using the property of $\varphi \in Aut(T(\mathcal{A}))$ as an automorphism of $\mathcal{T}(\mathcal{A})$, we have

$$\begin{aligned} s \approx t \in IdV(\mathcal{A}) \quad &\Longleftrightarrow \quad s^{\mathcal{A}} = t^{\mathcal{A}} \\ &\Longleftrightarrow \quad \varphi(s^{\mathcal{A}}) = \varphi(t^{\mathcal{A}}) \\ &\Longleftrightarrow \quad (\hat{\sigma}[s])^{\mathcal{A}} = (\hat{\sigma}[t])^{\mathcal{A}} \\ &\Longleftrightarrow \quad \hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV(\mathcal{A}) \\ &\Longleftrightarrow \quad (s,t) \in \ker_{V(\mathcal{A})}(\sigma). \end{aligned}$$

This implies $IdV(\mathcal{A}) = \ker_{V(\mathcal{A})}(\sigma)$.

Since φ is surjective, for every $t^{\mathcal{A}} \in T^{(n)}(\mathcal{A})$ there is a term operation $s^{\mathcal{A}} \in T^{(n)}(\mathcal{A})$ such that $\varphi(s^{\mathcal{A}}) = t^{\mathcal{A}}$. But this means that for every $t^{\mathcal{A}} \in T^{(n)}(\mathcal{A})$ there is a term $s \in W_\tau(X_n)$ such that $\hat{\sigma}[t]^{\mathcal{A}} = t^{\mathcal{A}}$ and then $\hat{\sigma}[W_\tau(X_n)]^{\mathcal{A}} = T^{(n)}(\mathcal{A})$. Since this can be done for every $n \geq 1$, we have $\hat{\sigma}[W_\tau(X)]^{\mathcal{A}} = T(\mathcal{A})$.
Altogether, this means, $\sigma \in M_{\text{ker}}^{\mathcal{A}}$.

But we also have a mapping in the opposite direction:

(**) If $V(\mathcal{A})$ is solid, then a clone automorphism corresponds to every class of $M_{\text{ker}}^{\mathcal{A}}/\sim_{V(\mathcal{A})}$

Let $[\sigma]_{\sim_{V(\mathcal{A})}}$ be a class from $M^{\mathcal{A}}_{\ker}/\sim_{V(\mathcal{A})}$. For this class we define a mapping φ by $\varphi(f_i^{\mathcal{A}}) := (\hat{\sigma}[f_i(x_1,\ldots,x_n)])^{\mathcal{A}}$. Clearly, if $\sigma \sim_{V(\mathcal{A})} \sigma'$, we have $(\hat{\sigma}[f_i(x_1,\ldots,x_n)])^{\mathcal{A}} = (\hat{\sigma}'[f_i(x_1,\ldots,x_n)])^{\mathcal{A}}$. So, the whole class is mapped to the same φ. Indeed, φ is well-defined since

$$f_i^{\mathcal{A}} = f_j^{\mathcal{A}} \Rightarrow i = j \Rightarrow f_i = f_j \Rightarrow \sigma(f_i) = \sigma(f_j) \Rightarrow (\sigma(f_i))^{\mathcal{A}} = (\sigma(f_j))^{\mathcal{A}}.$$

Here we used by Proposition 2.4 that $\mathcal{T}(\mathcal{A})$ is free with respect to itself, and that $\{f_i^{\mathcal{A}} \mid i \in I\}$ is an independent set of generators. The mapping φ is one-to-one since

$$\varphi(f_i^{\mathcal{A}}) = \varphi(f_j^{\mathcal{A}}) \Rightarrow \sigma(f_i)^{\mathcal{A}} = \sigma(f_j)^{\mathcal{A}} \Rightarrow \sigma(f_i) \approx \sigma(f_j) \in IdV(\mathcal{A}) \Rightarrow$$

$$\Rightarrow f_i(x_1,\ldots,x_{n_i}) \approx f_j(x_1,\ldots,x_{n_j}) \in IdV(\mathcal{A}).$$

For the last step, we used $\ker_{V(\mathcal{A})}(\sigma) = IdV(\mathcal{A})$.

The surjectivity of φ follows from $(\hat{\sigma}[W_\tau(X)])^{\mathcal{A}} = T(\mathcal{A})$.

We show that $\varphi|_{T^{(n)}(\mathcal{A})}$ is an automorphism of

$$\mathcal{T}^{(n)}(\mathcal{A}) = (T^{(n)}(\mathcal{A}); S^{n,A}, e_1^{n,A}, \ldots, e_n^{n,A})$$

for every n.

Indeed,

$$\begin{array}{rlcl} & \varphi(S^{n,A}(t^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}})) & = & \varphi(S^n(t, t_1, \ldots, t_n)^{\mathcal{A}}) \\ = & (\hat{\sigma}[S^n(t, t_1, \ldots, t_n)])^{\mathcal{A}} & = & (S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]))^{\mathcal{A}} \\ = & S^{n,A}((\hat{\sigma}[t])^{\mathcal{A}}, (\hat{\sigma}[t_1])^{\mathcal{A}}, \ldots, (\hat{\sigma}[t_n])^{\mathcal{A}}) & = & S^{n,A}(\varphi(t^{\mathcal{A}}), \varphi(t_1^{\mathcal{A}}), \ldots, \varphi(t_n^{\mathcal{A}})). \end{array}$$

This works in the same way if we apply the more general operators $S_m^{n,A}$ to sets of term operations of different arities.

Here we used that $\hat{\sigma}$ is an endomorphism of n-$Cl(\tau_n)$ (Proposition 2.3). Finally we have $\varphi(e_i^{n,A}) = \varphi(x_i^{\mathcal{A}}) = \hat{\sigma}[x_i]^{\mathcal{A}} = x_i^{\mathcal{A}} = e_i^{n,A}$ for all $1 \leq i \leq n$. Therefore, φ is an automorphism of $\mathcal{T}(\mathcal{A})$.

Using (*) and (**) we can prove our main result:

Theorem 4.1 *If $V(\mathcal{A})$ is a solid variety, then the group $Aut(\mathcal{T}(A))$ of all clone automorphisms of $\mathcal{T}(\mathcal{A})$ is isomorphic to $\mathcal{M}^{\mathcal{A}}_{\ker}/_{\sim V(\mathcal{A})}$*

Proof. By (*) we may define $\Psi : Aut(\mathcal{T}(A)) \to \mathcal{M}^{\mathcal{A}}_{\ker}/_{\sim V(\mathcal{A})}$, in the following way

$$\Psi(\varphi) = [\hat{\sigma}]_{\sim_{V(\mathcal{A})}} \text{ with } (\hat{\sigma}[t])^{\mathcal{A}} = \varphi(t^{\mathcal{A}}).$$

We show that Ψ is a bijection. In fact, we have

$$\begin{array}{ll} \varphi_1 = \varphi_2 & \\ \Leftrightarrow & \forall\, t^{\mathcal{A}} \in T^{(n)}(\mathcal{A}) \;\; (\varphi_1(t^{\mathcal{A}}) = \varphi_2(t^{\mathcal{A}})) \\ \Leftrightarrow & \exists [\sigma_1]_{\sim_{V(\mathcal{A})}}, [\sigma_2]_{\sim_{V(\mathcal{A})}} \in M^{\mathcal{A}}_{\ker}/_{\sim_{V(\mathcal{A})}} \;\; \forall\, t \in W_\tau(X_n) \;\; ((\sigma_1[t])^{\mathcal{A}} = (\sigma_2[t])^{\mathcal{A}}) \\ \Leftrightarrow & \forall\, t \in W_\tau(X_n) \;\; (\sigma_1[t] \approx \sigma_2[t] \in IdV(\mathcal{A})) \\ \Leftrightarrow & \sigma_1 \sim_{V(\mathcal{A})} \sigma_2 \\ \Leftrightarrow & [\sigma_1]_{\sim_{V(\mathcal{A})}} = [\sigma_2]_{\sim_{V(\mathcal{A})}}. \end{array}$$

The surjectivity of Φ follows from (**).

We show the compatibility of Φ with the operations.

Let us note that

$$\Phi(\varphi_1 \circ \varphi_2) = [\hat{\sigma}_1 \circ \hat{\sigma}_2]_{\sim_{V(\mathcal{A})}} = [\hat{\sigma}_1]_{\sim_{V(\mathcal{A})}} \circ [\hat{\sigma}_2]_{\sim_{V(\mathcal{A})}} = \Psi(\varphi_1) \circ \Psi(\varphi_2),$$

since

$$(\varphi_1 \circ \varphi_2)(t^{\mathcal{A}}) = \varphi_1(\varphi_2(t^{\mathcal{A}})) = \varphi_1((\hat{\sigma}_2[t])^{\mathcal{A}}) = (\hat{\sigma}_1[\hat{\sigma}_2[t]])^{\mathcal{A}} = ((\hat{\sigma}_1 \circ \hat{\sigma}_2)[t])^{\mathcal{A}}.$$

For the identical automorphism we have

$$\Psi(\varphi_{id}) = [\sigma_{id}]_{\sim_{V(\mathcal{A})}},$$

since

$$\varphi_{id}(t^{\mathcal{A}}) = t^{\mathcal{A}} = (\sigma_{id}[t])^{\mathcal{A}}$$

This completes the proof. □

Finally, we formulate two interesting problems for future research in this area.

Problem 1.] Inner clone automorphisms are induced by weak automorphisms of the algebra $\mathcal{A}$ and form a subgroup of $Aut(\mathcal{T}(\mathcal{A}))$. Describe the corresponding subgroup of $\mathcal{M}^{\mathcal{A}}_{\text{ker}}/_{\sim V(\mathcal{A})}$!

The reader can find some useful hints in [9] (p. 85), and [14].

Problem 2.] For selected semigroups $\mathcal{A}$ determine all $M^{\mathcal{A}}_{\text{ker}}$-solid varieties of semigroups!

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