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HYPERSUBSTITUTIONS AND GROUPS

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Abstract. We consider groups as algebras of type $(2,1,0)$. A hypersubstitution of type $(2,1,0)$ is a mapping σ from the set of the operation symbols $\{\cdot, ^{-1}, e\}$ into the set of terms of type $(2,1,0)$ preserving the arity. For a monoid M of hypersubstitutions of type $(2,1,0)$ a variety V is called M-solid if for each group $(G; \cdot, ^{-1}, e) \in V$ the derived group $(G; \sigma(\cdot), \sigma(^{-1}), \sigma(e))$ also belongs to V for all $\sigma \in M$. The class S_M^{Gr} of all M-solid varieties of groups forms a complete sublattice of the lattice $\mathcal{L}(Gr)$ of all varieties of groups. In this way we get a tool for a better description of the whole lattice $\mathcal{L}(Gr)$ by characterization of complete sublattices S_M^{Gr}.

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1. Introduction

It is of some interest to know what the lattice of all varieties of some type τ looks like, but it has become clear that it is very complicated, even for such special case as the lattice of all varieties of semigroups. In [3] a new method to study these lattices was proposed, using complete sublattices consisting of M-solid varieties, where M is a monoid of hypersubstitutions. M-solid varieties of semigroups are considered in a range of papers (see for example [1], [2], and [7]). Although groups can be considered as semigroups not every variety of groups correponds to a variety of semigroups. Considering groups as algebras of type $(2,1,0)$ we can use the method of M-solid varieties for the description of the lattice of all varieties of groups.

In the next section we introduce the concept of a M-solid variety and collect some basic properties. In the third section we determine the set $\mathcal{H}_{nt}$ of all monoids M of hypersubstitutions of type $(2,1,0)$ such that there is a nontrivial M-solid variety of groups. It turns out that $\mathcal{H}_{nt}$ has infinitely many maximal and one minimal element, and $\mathcal{H}_{nt}$ consists of the submonoids of its maximal elements. The last section is devoted to the main result: For all maximal elements H_p of $\mathcal{H}_{nt}$ we characterize the complete lattice of all H_p-solid varieties

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of groups. An open problem is the characterization of the lattice of all M-solid varieties of groups for arbitrary monoids M. In the commutative case this problem is already solved (see [4]). The present paper will give the answer for another class of varieties of groups, namely for varieties of groups satisfying the identity $x^2y \approx yx^2$.

2. M-solid varieties of groups

Let $W(X)$ be the set of all terms of type $(2,1,0)$ over some fixed alphabet $X := \{x_1, x_2, x_3, ...\}$ where $\{f, g, e\}$ denotes the set of operation symbols (f is binary, g is unary and e is 0-ary). Instead of $x_1, x_2, x_3, ...$ we write also $x, y, z, ...$ Further, $W(X_2)$ ($W(X_1)$, $W(\emptyset)$) denotes the set of all terms of type $(2,1,0)$ over $X_2 := \{x_1, x_2\}$ ($X_1 := \{x_1\}$, $\emptyset$).
We recall that the identities $g(f(y,x)) \approx f(g(x), g(y))$, $g(g(x)) \approx x$, and $g(e) \approx e$ hold in every variety of groups and usually one writes x^{-1} instead of $g(x)$ ([5]). This allows us to write a term $t \in W(X)$ as a semigroup word over the alphabet $X^* := X \cup \{w^{-1} \mid w \in X\} \cup \{e\}$. For example, for $t = f(f(g(x), x), g(f(x, e)))$ one can write $t = x^{-1}xex^{-1}$. (But if necessary we will write terms by using the operation symbols f and g.)
For a variable $w \in X^*$ and a term $t \in W(X)$ we put:
$w^0 := e$, $w^1 := w$, and $w^{m+1} := w^m w$ for $m \geq 1$;
$w^{-m} := (w^{-1})^m$ for any $m \geq 2$;
$c_w(t)$ - the number of occurrences of w in the term t regarded as a semigroup word. For example, for $t = f(f(g(x), x), g(f(x, e)))$ we have $c_x(t) = 1$ and $c_{x^{-1}}(t) = 2$ since the semigroup word $x^{-1}xex^{-1}$ corresponds to this term.

A mapping $\sigma : \{f, g, e\} \longrightarrow W(X_2)$ with $\sigma(g) \in W(X_1)$ and $\sigma(e) \in W(\emptyset)$ is called a hypersubstitution of type $(2,1,0)$ (for short hypersubstitution). Any hypersubstitution σ can be uniquely extended to a map $\widehat{\sigma} : W(X) \longrightarrow W(X)$, this is defined inductively by
(i) $\widehat{\sigma}[w] := w$ for any $w \in X \cup \{e\}$,
(ii) $\widehat{\sigma}[f(t_1, t_2)] := \sigma(f)(\widehat{\sigma}[t_1], \widehat{\sigma}[t_2])$, and $\widehat{\sigma}[g(t)] := \sigma(g)(\widehat{\sigma}[t])$.
Here $\sigma(f)$ and $\sigma(g)$ on the right-hand side of (ii) have to be interpreted as operations induced by the term $\sigma(f)$ and $\sigma(g)$, respectively, on the term algebra induced on $W(X)$.
We denote by Hyp the set of all hypersubstitutions. If we define a product $\circ_h$ of hypersubstitutions by $\sigma_1 \circ_h \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$, where $\circ$ is the usual composition of functions, then $Hyp = (Hyp; \circ_h, \sigma_{id})$ is a monoid. Note that σ_{id} is the identity hypersubstitution, defined by $\sigma_{id}(f) = x_1x_2$, $\sigma_{id}(g) = x_1^{-1}$, and $\sigma_{id}(e) = e$.
Let M be a submonoid of Hyp. Further let V be a variety of type $(2,1,0)$. Then an identity $s \approx t$ of V is called an M-hyperidentity of V if for every $\sigma \in M$ the equation $\widehat{\sigma}[s] \approx \widehat{\sigma}[t]$ is an identity in V. If every identity in V is an M-hyperidentity then V is called M-solid. In the special case that M is all of Hyp, we speak of a hyperidentity and a solid variety. In order to show that any

identity is an M-hyperidentity in V we have not to check all $\sigma \in M$, we need only one representative of each equivalence class with respect to the following equivalence relation on Hyp, established by J. Płonka ([6]):
$\sigma_1 \sim_V \sigma_2$ iff $\sigma_1(\mu) \approx \sigma_2(\mu)$ is an identity in V for all operation symbols $\mu \in \{f, g, e\}$.
If $\sigma_1 \sim_V \sigma_2$ we say that σ_1 and σ_2 are V-equivalent. In [6] was shown that if σ_1 and σ_2 are V-equivalent and $\widehat{\sigma}_1[s] \approx \widehat{\sigma}_1[t]$ holds in V then also $\widehat{\sigma}_2[s] \approx \widehat{\sigma}_2[t]$ holds in V.
By definition, to tell if a variety is M-solid, one has to test that application of any hypersubstitution σ to any identity of V results in an identity of V. Denecke and Reichel have developed a reduction in [3]. It suffices to show that every identity of the generating system of V is an M-hyperidentity.
We denote by IdV the set of all identities in V and by $\mathcal{L}(V)$ we mean the subvariety lattice of V. The set $P(V)$ of all hypersubstitutions σ with $\widehat{\sigma}[s] \approx \widehat{\sigma}[t] \in IdV$ for all $s \approx t \in IdV$ forms a submonoid of Hyp [8]. An element of $P(V)$ is called proper hypersubstitution ([6]). The variety $Gr := Mod\{f(f(x,y),z) \approx f(x,f(y,z)),\ f(g(x),x) \approx f(x,g(x)) \approx e,\ f(x,e) \approx f(e,x) \approx x\}$ is the variety of all groups (considered as algebras of type $(2,1,0)$). For a set Σ of equations let $Gr(\Sigma)$ be the variety of groups satisfying Σ. By S_M^{Gr} we denote the class of all M-solid varieties of groups. S_M^{Gr} forms a complete sublattice of $\mathcal{L}(Gr)$. Moreover, if $M_1 \subseteq M_2$ then $S_{M_2}^{Gr} \subseteq S_{M_1}^{Gr}$ (see [3]).

3. Characterization of $\mathcal{H}_{nt}$

For each monoid M of hypersubstitutions the trivial variety $TR := Mod\{x \approx y\}$ belongs to S_M^{Gr}. This is clear, since the application of any $\sigma \in Hyp$ to $x \approx y$ provides again $x \approx y$, i.e. gives an identity of TR. But there are monoids M such that S_M^{Gr} consists only of TR, for example in the case $M = Hyp$. To make this clear we consider a hypersubstitution $\sigma \in Hyp$ with $\sigma(f)$ and $\sigma(e) = e$. If we apply this σ to the group identity $f(e,x) \approx x$ we get $e \approx x$ which holds only in the trivial variety. This shows that TR is the only solid variety of groups. Moreover, this example shows that $S_M^{Gr} = \{TR\}$ for all monoids M containing the previously defined σ. It raises the question: For which monoids M there are nontrivial M-solid varieties of groups. In this section we determine the set $\mathcal{H}_{nt}$ of all such submonoids M of Hyp for which S_M^{Gr} contains not only TR:

$$\mathcal{H}_{nt} := \{M \mid M \subseteq Hyp,\ S_M^{Gr} \neq \{TR\}\}.$$

For $a \geq 1$ let V_a^c be the variety of all commutative groups of order a:
$V_a^c := Gr(\{f(x,y) \approx f(y,x), x^a \approx e\})$.

Note that $V_1^c = TR$. Clearly, $V_i^c \neq V_j^c$ for $i \neq j$.

Definition 1. *Let $a \geq 2$ be a natural number. Let H_a be the set of all hyper-*

substitutions σ satisfying the following properties:

$$\begin{array}{lllll} a) & c_x(\sigma(f)) & - & c_{x-1}(\sigma(f)) & \equiv \quad 1(a); \\ b) & c_y(\sigma(f)) & - & c_{y^{-1}}(\sigma(f)) & \equiv \quad 1(a); \\ c) & c_x(\sigma(g)) & - & c_{x^{-1}}(\sigma(g)) & \equiv \; - \; 1(a). \end{array}$$

Proposition 2. *For all natural numbers $a \geq 2$ we have $H_a = P(V_a^c)$.*

Proof. Let $\sigma \in P(V_a^c)$. We will show that σ satisfies the properties a), b), and c).
Assume that a) does not hold. Then $c_x(\sigma(f)) - c_{x^{-1}}(\sigma(f)) \equiv m(a)$ for some natural number m with $1 < m \leq a$. We apply σ to $f(x, e) \approx x \in IdV_a^c$ and get $x^{c_x(\sigma(f)) - c_{x^{-1}}(\sigma(f))} \approx x \in IdV_a^c$ since σ is a proper hypersubstitution for V_a^c. But $x^{c_x(\sigma(f)) - c_{x^{-1}}(\sigma(f))} \approx x$, $x^a \approx e$, and $c_x(\sigma(f)) - c_{x^{-1}}(\sigma(f)) \equiv m(a)$ imply $x^m \approx x$, i.e. $x^{m-1} \approx e$ with $1 \leq m-1 < a$ is an identity in V_a^c, a contradiction. Dually we get that b) is satisfied.
Assume that c) does not hold. Then $c_x(\sigma(g)) - c_{x^{-1}}(\sigma(g)) \equiv m(a)$ for some natural number m with $0 \leq m < a - 1$. Then $\sigma(g)(x) \approx x^{c_x(\sigma(g)) - c_{x^{-1}}(\sigma(g))} \approx x^m \in IdV_a^c$ because of $x^a \approx e \in IdV_a^c$. By a) and b) we have $c_x(\sigma(f)) - c_{x^{-1}}(\sigma(f)) \equiv 1(a)$ and $c_y(\sigma(f)) - c_{y^{-1}}(\sigma(f)) \equiv 1(a)$, respectively. Thus $x^{m(c_x(\sigma(f)) - c_{x^{-1}}(\sigma(f))) + c_y(\sigma(f)) - c_{y^{-1}}(\sigma(f))} \approx x^{m+1}$ because of $x^a \approx e \in IdV_a^c$.
Further, there holds $\widehat{\sigma}[f(g(x), x)] = \sigma(f)(\widehat{\sigma}[g(x)], \widehat{\sigma}[x]) = \sigma(f)(\sigma(g)(x), x) \approx \sigma(f)(x^m, x) \approx x^{m(c_x(\sigma(f)) - c_{x^{-1}}(\sigma(f))) + c_y(\sigma(f)) - c_{y^{-1}}(\sigma(f))} \approx x^{m+1}$. Since σ is a proper hypersubstitution for V_a^c from $f(g(x), x) \approx e \in IdV_a^c$ follows $\widehat{\sigma}[f(g(x), x)] \approx \widehat{\sigma}[e] \in IdV_a^c$, i.e. $x^{m+1} \approx e$ with $1 \leq m+1 < a$ is an identity in V_a^c, a contradiction.

Conversely, let $\sigma \in H_a$. We will show that σ is a proper hypersubstitution for H_a. For this we show that σ is V_a^c-equivalent to the identity hypersubstitution σ_{id}. There are natural numbers k, l, m, n such that $c_x(\sigma(f)) = k$, $c_{x^{-1}}(\sigma(f)) = l$, $c_y(\sigma(f)) = m$, and $c_{y^{-1}}(\sigma(f)) = n$. Then $\sigma(f) \approx x^{k-l} y^{m-n}$ because of the commutative law. Because of a) and b) we have $k - l \equiv 1(a)$ and $m - n \equiv 1(a)$, respectively. Thus $\sigma(f) \approx xy$ (because of $x^a \approx e$).
Further, there are natural numbers i, j such that $c_x(\sigma(g)) = i, c_{x^{-1}}(\sigma(g)) = j$. Then $\sigma(g) \approx x^{i-j}$ because of the commutative law. Because of c) we have $i - j \equiv -1(a)$. Thus $\sigma(g) \approx x^{-1}$ (because of $x^a \approx e$).
Obviously, we have $\widehat{\sigma}[e] \approx e$. □

Notation 3 *For a monoid M of hypersubstitutions of type $(2, 1, 0)$ we define $gcd(M)$ as be the greatest common divisor of the following integers: $c_x(\sigma(f)) - c_{x^{-1}}(\sigma(f)) - 1$, $c_y(\sigma(f)) - c_{y^{-1}}(\sigma(f)) - 1$, and $c_x(\sigma(g)) - c_{x^{-1}}(\sigma(g)) + 1$ for all $\sigma \in M$.*

Theorem 4. *Let M be a monoid of hypersubstitutions of type $(2, 1, 0)$. Then $S_M^{Gr} \neq \{TR\}$ iff there is a prime number p with $M \subseteq H_p$.*

Proof. Let $S_M^{Gr} \neq \{TR\}$. Then there is an M-solid variety V of groups with $V \neq TR$.
Assume that $M \nsubseteq H_p$ for all prime numbers p. Then for each prime number p there is a $\sigma \in M$ with $c_x(\sigma(f)) - c_{x^{-1}}(\sigma(f)) \not\equiv 1(p)$ or $c_y(\sigma(f)) - c_{y^{-1}}(\sigma(f)) \not\equiv 1(p)$ or $c_x(\sigma(g)) - c_{x^{-1}}(\sigma(g)) \not\equiv -1(p)$. This means $gcd(M) = 1$.
On the other hand we have
$\{\widehat{\sigma}[f(x,e)] \approx \widehat{\sigma}[x] \mid \sigma \in M\} \cup \{\widehat{\sigma}[f(e,x)] \approx \widehat{\sigma}[x] \mid \sigma \in M\} \cup \{\widehat{\sigma}[f(g(x),x)] \approx \widehat{\sigma}[e] \mid \sigma \in M\} \subseteq IdV$. This provides $\{x^{c_x(\sigma(f))-c_{x^{-1}}(\sigma(f))} \approx x \mid \sigma \in M\} \cup \{x^{c_y(\sigma(f))-c_{y^{-1}}(\sigma(f))} \approx x \mid \sigma \in M\} \cup \{x^{[c_x(\sigma(f))-c_{x^{-1}}(\sigma(f))][c_x(\sigma(g))-c_{x^{-1}}(\sigma(g))]+c_y(\sigma(f))-c_{y^{-1}}(\sigma(f))} \approx e \mid \sigma \in M\} \subseteq IdV$. For $\sigma \in M$, using $x^{c_x(\sigma(f))-c_{x^{-1}}(\sigma(f))} \approx x$ and $x^{c_y(\sigma(f))-c_{y^{-1}}(\sigma(f))} \approx x$ from $x^{[c_x(\sigma(f))-c_{x^{-1}}(\sigma(f))][c_x(\sigma(g))-c_{x^{-1}}(\sigma(g))]+c_y(\sigma(f))-c_{y^{-1}}(\sigma(f))} \approx e$ it follows $x^{c_x(\sigma(g))-c_{x^{-1}}(\sigma(g))+1} \approx e$ and thus $x^{c_x(\sigma(g))-c_{x^{-1}}(\sigma(g))+2} \approx x$. This shows that $\{x^{c_x(\sigma(f))-c_{x^{-1}}(\sigma(f))} \approx x \mid \sigma \in M\} \cup \{x^{c_y(\sigma(f))-c_{y^{-1}}(\sigma(f))} \approx x \mid \sigma \in M\} \cup \{x^{c_x(\sigma(g))-c_{x^{-1}}(\sigma(g))+2} \approx x \mid \sigma \in M\} \subseteq IdV$. From these identities we can derive $x^{gcd(M)+1} \approx x \in IdV$. Since $gcd(M) = 1$, we have $x^2 \approx x \in IdV$, i.e. $x \approx e \in IdV$ and $x \approx y \in IdV$. Thus $V = TR$, a contradiction.

Conversely, let $M \subseteq H_p$ for some prime number p. Then $S_{H_p}^{Gr} \subseteq S_M^{Gr}$. Since $P(V_p^c) = H_p$ (Proposition 2) we have $V_p^c \in S_{H_p}^{Gr} \subseteq S_M^{Gr}$ and thus $S_M^{Gr} \neq \{TR\}$.
□

Remark 5. *The previous theorem shows that the monoids H_p are maximal elements in $\mathcal{H}_{nt}$, where for two different prime numbers p_1 and p_2 the monoids H_{p_1} and H_{p_2} are different.*
Moreover, it is easy to check that

$$M_1 := \{\sigma_{id}\}$$

forms a monoid. M_1 is the least element in $\mathcal{H}_{nt}$.

The following set D of hypersubstitutions of type $(2,1,0)$ is the set of all proper hypersubstitutions of the variety of all commutative groups ([4]).

Definition 6. *Let D be the set of all hypersubstitutions σ satisfying the following properties:*

$$\begin{array}{llcll} a) & c_x(\sigma(f)) & - & c_{x^{-1}}(\sigma(f)) & = \quad 1; \\ b) & c_y(\sigma(f)) & - & c_{y^{-1}}(\sigma(f)) & = \quad 1; \\ c) & c_x(\sigma(g)) & - & c_{x^{-1}}(\sigma(g)) & = \; -1. \end{array}$$

Obviously, we have $D \subseteq H_n$ for all natural numbers $n \geq 2$. We will determine such monoids M with $M \subseteq H_n$ for all natural numbers $n \geq 2$.

Definition 7. *For any submonoid $M \subseteq Hyp$ we denote by $\mathcal{L}(M)$ the submonoid lattice of M.*

Proposition 8. *There holds* $\bigcap_{2\leq i\in\mathbb{N}} \mathcal{L}(H_i) = \mathcal{L}(D)$.

Proof. " $\supseteq$ " : Clearly, for $2 \leq i \in \mathbb{N}$ we have $D \subseteq H_i$, i.e. $D \in \mathcal{L}(H_i)$. Thus $\mathcal{L}(D) \subseteq \mathcal{L}(H_i)$ for $2 \leq i \in \mathbb{N}$, i.e. $\mathcal{L}(D) \subseteq \bigcap_{2\leq i\in\mathbb{N}} \mathcal{L}(H_i)$.

" $\subseteq$ " : Let $M \in \bigcap_{2\leq i\in\mathbb{N}} \mathcal{L}(H_i)$ and let $\sigma \in M$. Then there is a natural number $n \geq 1$ with $c_x(\sigma(f)) - c_{x^{-1}}(\sigma(f)) = n$. Assume that $n \neq 1$. Then $n \not\equiv 1(n)$, i.e. $\sigma \notin H_n$ and $M \notin \mathcal{L}(H_n)$, contradicts $M \in \bigcap_{2\leq i\in\mathbb{N}} \mathcal{L}(H_i)$. Thus $c_x(\sigma(f)) - c_{x^{-1}}(\sigma(f)) = 1$. Similarly, one can show that $c_y(\sigma(f)) - c_{y^{-1}}(\sigma(f)) = 1$ and $c_x(\sigma(g)) - c_{x^{-1}}(\sigma(g)) = -1$.
Consequently, $\sigma \in D$ and thus $M \subseteq D$, i.e. $M \in \mathcal{L}(D)$. □

4. All H_p-solid varieties of groups

The monoids H_p, for prime numbers p, are the maximal elements in $\mathcal{H}_{nt}$. In particular, for any $M \in \mathcal{H}_{nt}$ there is a prime number p with $M \subseteq H_p$, i.e. $S^{Gr}_{H_p} \subseteq S^{Gr}_M$. If we have a characterization of the lattice $S^{Gr}_{H_p}$ for all prime numbers p then we have some knowledge about a complete sublattice of S^{Gr}_M for any monoid $M \in \mathcal{H}_{nt}$. The main theorem of the present paper, the characterization of $S^{Gr}_{H_p}$ for all prime numbers p, is the topic of this section. We start with some properties of H_p-solid varieties of groups.

Lemma 9. *Let $n \geq 2$ be a natural number. Then in each H_n-solid variety V of groups there holds $xyx^{-1}zxy^{-1}x^{-1} \approx yzy^{-1}$.*

Proof. We consider the following hypersubstitution σ:

$$\sigma(f) := x^2yx^{-1}$$
$$\sigma(g) := x^{-1}$$
$$\sigma(e) := e.$$

We have $c_x(\sigma(f)) - c_{x^{-1}}(\sigma(f)) = 2 - 1 = 1 \equiv 1(n)$, $c_y(\sigma(f)) - c_{y^{-1}}(\sigma(f)) = 1 - 0 = 1 \equiv 1(n)$, and $c_x(\sigma(g)) - c_{x^{-1}}(\sigma(g)) = 0 - 1 = -1 \equiv -1(n)$, i.e. $\sigma \in H_n$. Since V is H_n-solid, the application of σ to the associative law provides the identities $x^2yx^{-1}x^2yx^{-1}z(x^2yx^{-1})^{-1} \approx x^2y^2zy^{-1}x^{-1}$, $x^2yxyx^{-1}zxy^{-1}x^{-2} \approx x^2y^2zy^{-1}x^{-1}$, $xyx^{-1}zxy^{-1}x^{-1} \approx yzy^{-1}$ in V. □

For a group $\mathcal{A} = (A; \cdot, ^{-1}, e)$, by $\mathcal{C}(\mathcal{A}) := \{a \in A \mid xa = ax \text{ for all } x \in A\}$ we denote the centre of $\mathcal{A}$. In particular, $\mathcal{C}(\mathcal{A})$ forms a subgroup of $\mathcal{A}$ (see [5]). For $a, b \in \mathcal{A}$ let $[a, b] := aba^{-1}b^{-1}$ be the commutator of a and b. The commutator group of $\mathcal{A}$, i.e. the group generated by the set $\{[a, b] \mid a, b \in \mathcal{A}\}$, is denoted by $[\mathcal{A}, \mathcal{A}]$.

Proposition 10. *Let $n \geq 2$ be a natural number, V be an H_n-solid variety of groups and $\mathcal{A} \in V$. Then*

$$[\mathcal{A}, \mathcal{A}] \subseteq \mathcal{C}(\mathcal{A}),$$

i.e. the commutator group is a subgroup of the centre.

Proof. We will show that for any $a, b \in A$ the commutator $[a, b]$ belongs to the centre of $\mathcal{A}$, i.e. $\{[a, b] \mid a, b \in \mathcal{A}\} \subseteq \mathcal{C}(\mathcal{A})$.
Let $a, b \in A$. Then for any $x \in A$ holds $ba^{-1}b^{-1}xbab^{-1} \approx a^{-1}xa$ by Lemma 9. This implies $\underline{a}ba^{-1}b^{-1}xbab^{-1}\underline{ba^{-1}b^{-1}} \approx \underline{a}a^{-1}xa\underline{ba^{-1}b^{-1}}$, i.e. $aba^{-1}b^{-1}x \approx xaba^{-1}b^{-1}$ and thus the commutator $[a, b] = aba^{-1}b^{-1}$ belongs to the centre of $\mathcal{A}$. Since $\{[a, b] \mid a, b \in \mathcal{A}\} \subseteq \mathcal{C}(\mathcal{A})$ and $\mathcal{C}(\mathcal{A})$ is a subgroup of $\mathcal{A}$ the group generated by the set $\{[a, b] \mid a, b \in \mathcal{A}\}$, i.e. the commutator group $[\mathcal{A}, \mathcal{A}]$, is a subgroup of $\mathcal{C}(\mathcal{A})$. □

Lemma 11. *Let $n \geq 2$ be a natural number. Then in each H_n-solid variety V of groups there holds $x^n \approx e$.*

Proof. We consider the following hypersubstitution σ:

$$\begin{aligned}\sigma(f) &:= x^{n+1}y\\ \sigma(g) &:= x^{-1}\\ \sigma(e) &:= e.\end{aligned}$$

We have $c_x(\sigma(f)) - c_{x^{-1}}(\sigma(f)) = n+1-0 = n+1 \equiv 1(n)$, $c_y(\sigma(f)) - c_{y^{-1}}(\sigma(f)) = 1-0 = 1 \equiv 1(n)$, and $c_x(\sigma(g)) - c_{x^{-1}}(\sigma(g)) = 0-1 = -1 \equiv -1(n)$, i.e. $\sigma \in H_n$. Since V is H_n-solid, the application of σ to the group identity $f(x, e) \approx x$ provides an identity in V, namely $x^{n+1} \approx x$, i.e. $x^n \approx e$. □

By Proposition 10 and Lemma 11, respectively, it becomes clear that an H_n-solid variety of groups consists of solvable groups.

Definition 12. *We define a hypersubstitution σ_d by*

$$\begin{aligned}\sigma_d(f) &:= yx\\ \sigma_d(g) &:= x^{-1}\\ \sigma_d(e) &:= e.\end{aligned}$$

A variety V of groups is called self-dual if the application of σ_d to any identity of V gives again an identity in V:

$$\{\widehat{\sigma}_d u] \approx \widehat{\sigma}_d[v] \mid u \approx v \in IdV\} \subseteq IdV.$$

Lemma 13. *Let $n \geq 2$ be a natural number. Any H_n-solid variety V of groups is self-dual.*

Proof. We have $c_x(\sigma_d(f)) - c_{x^{-1}}(\sigma_d(f)) = c_y(\sigma_d(f)) - c_{y^{-1}}(\sigma_d(f)) = 1 - 0 = 1 \equiv 1(n)$, and $c_x(\sigma_d(g)) - c_{x^{-1}}(\sigma_d(g)) = 0 - 1 = -1 \equiv -1(n)$, i.e. $\sigma_d \in H_n$. Since V is H_n-solid, the application of σ_d to an identity of V gives again an identity of V. □

Lemma 14. *Let V be a variety of groups satisfying $xyx^{-1}zxy^{-1}x^{-1} \approx yzy^{-1}$. Then for any integer a there holds*

$$xyx^{-1}y^a \approx y^a xyx^{-1} \in IdV.$$

Proof. All is clear for $a = 0$. Let $a \neq 0$ be an integer. Then we have $xyx^{-1}y^a \approx y^a xy^{-a}yy^a x^{-1}y^{-a}y^a \approx y^a xyx^{-1}$ (using $xyx^{-1}zxy^{-1}x^{-1} \approx yzy^{-1}$). □

Lemma 15. *Let V be a variety of groups satisfying $xyx^{-1}zxy^{-1}x^{-1} \approx yzy^{-1}$. Then for integers $r, s, t, u \neq 0$ the following identities (i)-(iv) are satisfied in V:*

(i) $x^r y^s x^{-r} y^t x^u \approx y^t x^r y^s x^{u-r}$

(ii) $x^r y^s x^{-t} y^u x^t \approx x^{r-t} y^u x^t y^s$

(iii) $x^r y^s x^r y^t x^u \approx y^{-t} x^r y^{s+2t} x^{r+u}$

(iv) $x^r y^s x^t y^u x^t \approx x^{r+t} y^{u+2s} x^t y^{-s}$.

Proof. The identities (i) and (ii) are immediate consequences of Lemma 14.
We show (iii). The identity (iv) can be checked dually. Using Lemma 14 we have $x^r y^s x^r y^t x^u \approx x^r y^s \underline{x^r y^t x^{-r}} x^{u+r}$
$\approx x^r x^r y^t x^{-r} y^s x^{u+r}$
$\approx x^{2r} \underline{y^t x^{-r} y^{-t}} y^{s+t} x^{u+r}$
$\approx y^t x^{-r} y^{-t} x^{2r} y^{s+t} x^{u+r}$
$\approx y^t x^{-r} y^{-t} \underline{x^{2r} y^{s+t} x^{-2r}} x^{u+3r}$
$\approx y^t x^{-r} x^{2r} y^{s+t} x^{-2r} y^{-t} x^{u+3r}$
$\approx y^t \underline{x^r y^{s+t} x^{-r}} x^{-r} y^{-t} x^{u+3r}$
$\approx x^r y^{s+t} x^{-r} \underline{y^t x^{-r} y^{-t}} x^{u+3r}$
$\approx x^r y^{s+t} y^t x^{-r} y^{-t} x^{-r} x^{u+3r}$
$\approx \underline{x^r y^{s+2t} x^{-r} y^{-t}} x^{u+2r}$
$\approx y^{-t} x^r y^{s+2t} y^t x^{-r} x^{u+2r}$
$\approx y^{-t} x^r y^{s+2t} y^t x^{u+r}$. □

Theorem 16. *Let $r \geq 2$ be a natural number and let V be a variety of groups. V is H_r-solid iff V is self-dual and satisfies both identities $x^r \approx e$ and $xyx^{-1}zxy^{-1}x^{-1} \approx yzy^{-1}$.*

Proof. Suppose that V is H_r-solid. Then V is self-dual by Lemma 13, satisfies $x^r \approx e$ (i.e. it is a variety of r-group) by Lemma 11 and satisfies $xyx^{-1}zxy^{-1}x^{-1} \approx yzy^{-1}$ by Lemma 9.

Suppose now that V is a self-dual variety of r-groups satisfying

$$xyx^{-1}zxy^{-1}x^{-1} \approx yzy^{-1}(\text{i}).$$

Let $\sigma \in H_r$. We will show that $\sigma(f) \approx x^a y^b x^c y^d$ or $\sigma(f) \approx y^d x^c y^b x^a$ for some natural numbers a, b, c, d with $a + c \equiv b + d \equiv 1(r)$.
For this we check that for natural numbers a, n_2, n_3, n_4, n_5 we have

$$x^{an_3} y^{n_2} x^{n_3} y^{n_4} x^{n_5} \approx y^{(-a+1)n_2 - an_4} x^{n_3} y^{an_2+(a+1)n_4} x^{n_5+an_3} \in IdV \text{ (ii)}.$$

We show by induction on k that $x^{kn_3} y^{n_2} x^{n_3} y^{n_4} x^{n_5} \approx y^{(-k+1)n_2 - kn_4} x^{n_3} y^{kn_2+(k+1)n_4} x^{n_5+kn_3} \in IdV$.
For $k = 1$ we have $x^{1n_3} y^{n_2} x^{n_3} y^{n_4} x^{n_5} \approx y^{(-1+1)n_2 - 1n_4} x^{n_3} y^{1n_2+(1+1)n_4} x^{n_5+1n_3} \in IdV$ by Lemma 15(iii).
Suppose now that the statement is true for $k = m$, i.e. $x^{mn_3} y^{n_2} x^{n_3} y^{n_4} x^{n_5} \approx y^{(-m+1)n_2 - mn_4} x^{n_3} y^{mn_2+(m+1)n_4} x^{n_5+mn_3} \in IdV$ (hypothesis).
Then for $k = m + 1$ holds $x^{(m+1)n_3} y^{n_2} x^{n_3} y^{n_4} x^{n_5}$
$\approx x^{n_3} x^{mn_3} y^{n_2} x^{n_3} y^{n_4} x^{n_5}$
$\approx x^{n_3} y^{(-m+1)n_2 - mn_4} x^{n_3} y^{mn_2+(m+1)n_4} x^{n_5+mn_3}$ (by hypothesis)
$\approx y^{-mn_2-(m+1)n_4} x^{n_3} y^{(-m+1)n_2 - mn_4 + 2mn_2 + 2(m+1)n_4} x^{n_5+mn_3+n_3}$ (by Lemma 15(iii))
$\approx y^{(-(m+1)+1)n_2 - (m+1)n_4} x^{n_3} y^{(m+1)n_2+((m+1)+1)n_4} x^{n_5+(m+1)n_3}$.
This shows that (ii) holds.
We show now that the following statement (iii) holds:

For any natural numbers n_1, n_2, n_3, n_4, n_5 there are natural numbers a, b, c, d such that $x^{n_1} y^{n_2} x^{n_3} y^{n_4} x^{n_5} \approx y^a x^b y^c x^d$, $n_1 + n_3 + n_5 \equiv b + d(r)$, and $n_2 + n_4 \equiv a + c(r)$.

Let a_1, b_1, c_1, d_1, e_1 be natural numbers. Then there are natural numbers k_1 and r_1 with $r_1 < c_1$ such that $a_1 = k_1 c_1 + r_1$. Then we have $x^{a_1} y^{b_1} x^{c_1} y^{d_1} x^{e_1}$
$\approx x^{r_1} x^{k_1 c_1} y^{b_1} x^{c_1} y^{d_1} x^{e_1}$
$\approx x^{r_1} y^{(-k_1+1)b_1 - k_1 d_1} x^{c_1} y^{k_1 b_1 + (k_1+1)d_1} x^{e_1+k_1 c_1}$ (by (ii))
$\approx x^{r_1} y^{(-k_1+1)b_1 - k_1 d_1} x^{c_1} y^{(k_1-1)b_1 + k_1 d_1} y^{b_1+d_1} x^{e_1+k_1 c_1}$
$\approx y^{(-k_1+1)b_1 - k_1 d_1} x^{c_1} y^{(k_1-1)b_1 + k_1 d_1} x^{r_1} y^{b_1+d_1} x^{e_1+k_1 c_1}$ (by Lemma 14)
$\approx y^{f_2} x^{a_2} y^{b_2} x^{c_2} y^{d_2} x^{e_2}$ with $a_2 := c_1$, $b_2 := (k_1 - 1)b_1 + k_1 d_1$, $c_2 := r_1$, $d_2 := b_1 + d_1$, $e_2 := e_1 + k_1 c_1$ and $f_2 := (-k_1+1)b_1 - k_1 d_1$ where $b_2 + d_2 + f_2 = b_1 + d_1$ and $a_2 + c_2 + e_2 = a_1 + c_1 + e_1$. In $n \geq 1$ such steps we can derive from $x^{a_1} y^{b_1} x^{c_1} y^{d_1} x^{e_1}$ a term
$y^{f_2} \ldots y^{f_{n+1}} x^{a_{n+1}} y^{b_{n+1}} x^{c_{n+1}} y^{d_{n+1}} x^{e_{n+1}}$ with integers $a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}, e_{n+1}$, $f_2, \ldots, f_{n+1}$ such that $c_{n+1} = 0$ and $b_{n+1} + d_{n+1} + \sum_{i=1}^{n} f_{i+1} = b_1 + d_1$ and $a_{n+1} + c_{n+1} + e_{n+1} = a_1 + c_1 + e_1$. Because of $x^r \approx e$ there are natural numbers a, b, c, d such that $\sum_{i=1}^{n} f_{i+1} \equiv a(r)$, $a_{n+1} \equiv b(r)$, $b_{n+1} + d_{n+1} \equiv c(r)$ and $e_{n+1} \equiv d(r)$, i.e., $y^{f_2} \ldots y^{f_{n+1}} x^{a_{n+1}} y^{b_{n+1}} x^{c_{n+1}} y^{d_{n+1}} x^{e_{n+1}} \approx y^a x^b y^c x^d \in IdV$ and altogether we have $x^{n_1} y^{n_2} x^{n_3} y^{n_4} x^{n_5} \approx y^a x^b y^c x^d \in IdV$. This shows the

statement (iii).

On the other hand there are natural numbers $n \geq 1$ and $a_1, ..., a_{2n}$ such that $\sigma(f) \approx x^{a_1}y^{a_2}...x^{a_{2n-1}}y^{a_{2n}} \in IdV$ with $\sum_{i=0}^{n-1} a_{2i+1} \equiv \sum_{i=1}^{n} a_{2i} \equiv 1(r)$. Using (iii) we get $\sigma(f) \approx x^a y^b x^c y^d \in IdV$ or $\sigma(f) \approx y^d x^c y^b x^a \in IdV$ for some natural numbers a, b, c, d with $a + c \equiv \sum_{i=0}^{n-1} a_{2i+1}$ and $\sum_{i=1}^{n} a_{2i} \equiv b + d$, i.e. $a + c \equiv b + d \equiv 1(r)$.

Now we check that the application of σ to the group identities gives again identities in V. We note that $\sigma(e) \approx e$ and $\sigma(g) \approx x^{-1}$ since $c_x(\sigma(g)) - c_{x^{-1}}(\sigma(g)) \equiv -1(r)$ and $x^r \approx e \in IdV$. Thus we have
$\widehat{\sigma}[f(x,e)] \approx x^{a+c} \approx x = \widehat{\sigma}[x]$ and
$\widehat{\sigma}[f(x,g(x))] \approx x^{a+c}(x^{-1})^{b+d} \approx xx^{-1} = e = \widehat{\sigma}[e]$ since $a + c \equiv b + d \equiv 1(r)$ and $x^r \approx e \in IdV$. Dually we get $\widehat{\sigma}[f(e,x)] \approx \widehat{\sigma}[x] \in IdV$ and $\widehat{\sigma}[f(g(x),x)] \approx \widehat{\sigma}[e] \in IdV$.

Now we show that the application of σ to the associative law gives an identity in V. For this we check by induction on k that

$$(x^a y^b x^c y^d)^k z (y^{-d} x^{-c} y^{-b} x^{-a})^k \approx x^k y^k z y^{-k} x^{-k} \in IdV \text{ (iv)}$$

For $k = 1$ we have $x^a y^b x^c y^d z y^{-d} x^{-c} y^{-b} x^{-a}$
$\approx x^a x^c y^b x^{-c} x^c y^d z y^{-d} x^{-c} x^c y^{-b} x^{-c} x^{-a}$ (by (i))
$\approx x^{a+c} y^{b+d} z y^{-(b+d)} x^{-(a+c)}$
$\approx xyzy^{-1}x^{-1}$ since $a + c \equiv b + d \equiv 1(r)$ and $x^r \approx e \in IdV$.
Suppose now that (iv) is true for $k = m$, i.e. it holds $(x^a y^b x^c y^d)^m z (y^{-d} x^{-c} y^{-b} x^{-a})^m \approx x^m y^m z y^{-m} x^{-m} \in IdV$ (hypothesis).
Then for $k = m + 1$ we have $(x^a y^b x^c y^d)^{m+1} z (y^{-d} x^{-c} y^{-b} x^{-a})^{m+1}$
$\approx (x^a y^b x^c y^d) x^m y^m z y^{-m} x^{-m} (y^{-d} x^{-c} y^{-b} x^{-a})$ (by hypothesis)
$\approx x^a x^c y^b x^{-c} x^c y^d x^m y^m z y^{-m} x^{-m} y^{-d} x^{-c} x^c y^{-b} x^{-c} x^{-a}$ (by (i))
$\approx x^{a+c} y^{b+d} x^m y^m z y^{-m} x^{-m} y^{-(b+d)} x^{-(a+c)}$
$\approx x^{a+c} x^m y^{b+d} x^{-m} x^m y^m z y^{-m} x^{-m} x^m y^{-(b+d)} x^{-m} x^{-(a+c)}$ (by (i))
$\approx x^{a+c+m} y^{b+d+m} z y^{-(b+d+m)} x^{-(a+c+m)}$
$\approx x^{1+m} y^{1+m} z y^{-(1+m)} x^{-(1+m)}$ since $a + c \equiv b + d \equiv 1(r)$ and $x^r \approx e \in IdV$.

Now we have $\widehat{\sigma}[f(f(x,y),z)] \approx (x^a y^b x^c y^d)^a z^b (x^a y^b x^c y^d)^c z^d$
$\approx (x^a y^b x^c y^d)^a z^b (x^a y^b x^c y^d)^{-a} (x^a y^b x^c y^d) z^d$ (since $a + c \equiv 1(r)$ and $x^r \approx e \in IdV$)
$\approx (x^a y^b x^c y^d)^a z^b (y^{-d} x^{-c} y^{-b} x^{-a})^a (x^a y^b x^c y^d) z^d$
$\approx x^a y^a z^b y^{-a} x^{-a} (x^a y^b x^c y^d) z^d$ (by (iv))
$\approx x^a y^a z^b y^{-a+b} x^c y^d z^d$
$\approx x^a y^a z^b y^{-a+b} x^c y^{a+c-b} z^d$ (since $a + c \equiv b + d \equiv 1(r)$ and $x^r \approx e \in IdV$)
$\approx x^a y^b y^{a-b} z^b y^{-a+b} x^c y^{a-b} z^{-b} y^{b-a} y^{-b} y^a z^b y^c z^d$
$\approx x^a y^b z^b x^c z^{-b} y^{-b} y^a z^b y^c z^d$ (by (i))
$\approx x^a (y^a z^b y^c z^d)^b x^c (z^{-d} y^{-c} z^{-b} y^{-a})^b (y^a z^b y^c z^d)$ (by (iv))

$\approx x^a(y^az^by^cz^d)^bx^c(y^az^by^cz^d)^{-b+1}$
$\approx x^a(y^az^by^cz^d)^bx^c(y^az^by^cz^d)^d$ (since $b+d\equiv 1(r)$ and $x^r\approx e\in IdV$)
$=\widehat{\sigma}[f(x,f(y,z))]$.
We will show that the application of σ to any identity in V gives again an identity in V. Let $s\approx t\in IdV$. Since we have already checked that the application of σ to the group identities gives again identities in V we can consider the terms s and t as semigroup words over the alphabet X^*. So there are natural numbers $j,m\geq 1$ and $s_1,...,s_j,t_1,...,t_m\in X^*$ such that $s=s_1...s_j$ and $t=t_1...t_m$.
We will show by induction on j that

$$\widehat{\sigma}[s_1...s_j]\approx s_1^a...s_j^as_j^{b-a}...s_1^{b-a}s_1^d...s_j^d\in IdV.$$

First, we remark that from (i) it follows $x^{n_1}y^mx^{n_2}zx^{-n_2}y^{-m}x^{-n_1}\approx x^{n_1}x^{n_2}y^mx^{-n_2}x^{n_2}zx^{-n_2}x^{n_2}y^mx^{-n_2}x^{-n_1}$, i.e.

$$x^{n_1}y^mx^{n_2}zx^{-n_2}y^{-m}x^{-n_1}\approx x^{n_1+n_2}y^mzy^{-m}x^{-(n_2+n_2)} \text{ (v)}$$

for any integers n_1,n_2,m.
If $j=1$ then we have $\widehat{\sigma}[s_1]=s_1\approx s_1^{b+d}$ (since $b+d\equiv 1(r)$ and $x^r\approx e\in IdV$)
$\approx s_1^bs_1^d$
$\approx s_1^as_1^{b-a}\ s_1^d$.
Suppose that the statement is true for $j=k$, i.e. $\widehat{\sigma}[s_1...s_k]\approx s_1^a...s_k^as_k^{b-a}...s_1^{b-a}s_1^d...s_k^d\in IdV$ (hypothesis). We put $r:=s_1...s_k$.
Then for $j=k+1$ holds $\widehat{\sigma}[f(r,s_{k+1})]\approx\widehat{\sigma}[r]^as_{k+1}^b\widehat{\sigma}[r]^cs_{k+1}^d$
$\approx\widehat{\sigma}[r]^as_{k+1}^b\widehat{\sigma}[r]^{-a+1}s_{k+1}^d$ (since $a+c\equiv 1(r)$ and $x^r\approx e\in IdV$)
$\approx\widehat{\sigma}[r]^as_{k+1}^b\widehat{\sigma}[r]^{-a}\widehat{\sigma}[r]s_{k+1}^d$
$\approx (s_1^a...s_k^as_k^{b-a}...s_1^{b-a}s_1^d...s_k^d)^as_{k+1}^b(s_k^{-d}...s_1^{-d}s_1^{-b+a}...s_k^{-b+a}s_k^{-a}...s_1^{-a})^a(s_1^a...s_k^as_k^{b-a}...s_1^{b-a}s_1^d...s_k^d)s_{k+1}^d$ (by hypothesis)
$\approx (s_1^as_2^{a+b-a+d}...s_{k-1}^{a+b-a+d}s_k^as_k^{b-a}s_1^{b-a}s_1^ds_k^d)^as_{k+1}^b(s_k^{-d}s_1^{-d}s_1^{-b+a}s_k^{-b+a}s_k^{-a}s_{k-1}^{-a-b+a-d}...s_2^{-a-b+a-d}s_1^{-a})^a(s_1^a...s_k^as_k^{b-a}...s_1^{b-a}s_1^d...s_k^d)s_{k+1}^d$ (by (v))
$\approx (s_1^as_2...s_{k-1}s_k^bs_1^cs_k^d)^as_{k+1}^b(s_k^{-d}s_1^{-c}s_k^{-b}s_{k-1}^{-1}...s_2^{-1}s_1^{-a})^a(s_1^a...s_k^as_k^{b-a}...s_1^{b-a}s_1^d...s_k^d)s_{k+1}^d$ (since $a+c\equiv b+d\equiv 1(r)$ and $x^r\approx e\in IdV$)
$\approx (s_1^{a+c}s_2...s_{k-1}s_k^{b+d})^as_{k+1}^b(s_k^{-(b+d)}s_{k-1}^{-1}...s_2^{-1}s_1^{-(a+c)})^a(s_1^a...s_k^as_k^{b-a}...s_1^{b-a}s_1^d...s_k^d)s_{k+1}^d$ (by (v))
$\approx (s_1s_2...s_{k-1}s_k)^as_{k+1}^b(s_k^{-1}s_{k-1}^{-1}...s_2^{-1}s_1^{-1})^a(s_1^a...s_k^as_k^{b-a}...s_1^{b-a}s_1^d...s_k^d)s_{k+1}^d$ (since $a+c\equiv b+d\equiv 1(r)$ and $x^r\approx e\in IdV$)
$\approx (s_1^a...s_k^a)s_{k+1}^b(s_k^{-a}...s_1^{-a})(s_1^a...s_k^as_k^{b-a}...s_1^{b-a}s_1^d...s_k^d)s_{k+1}^d$ (it follows from (iv))
$\approx s_1^a...s_k^as_{k+1}^as_{k+1}^{b-a}s_k^{b-a}...s_1^{b-a}s_1^d...s_k^ds_{k+1}^d$.

Similarly, one can show that $\widehat{\sigma}[t]\approx t_1^a...t_m^at_m^{b-a}...t_1^{b-a}t_1^d...t_m^d\in IdV$. Now we substitute in $s\approx t$ each $w\in X^*$ by w^a,w^{b-a}, and w^d, respectively. So we get the following identities satisfied in V:

$$s_1^a...s_j^a \approx t_1^a...t_m^a,$$
$$s_1^{b-a}...s_j^{b-a} \approx t_1^{b-a}...t_m^{b-a},$$
$$s_1^d...s_j^d \approx t_1^d...t_m^d.$$

Moreover, since V is self-dual we have $s_j^{b-a}...s_1^{b-a} \approx t_m^{b-a}...t_1^{b-a} \in IdV$. These three identities provide $s_1^a...s_j^a s_j^{b-a}...s_1^{b-a} s_1^d...s_j^d \approx t_1^a...t_m^a t_m^{b-a}...t_1^{b-a} t_1^d...t_m^d \in IdV$, i.e. $\widehat{\sigma}[s] \approx \widehat{\sigma}[t] \in IdV$.
Consequently, the application of any $\sigma \in H_r$ to any identity of V gives again an identity in V, i.e. V is H_r-solid. □

An open problem is the characterization of the lattice S_M^{Gr} for any given monoid M. For the commutative case we have given the answer in [4]:

Proposition 17. *Let M be a monoid of hypersubstitutions and V be a variety of commutative groups. Then V is M-solid iff $V \subseteq V^c_{\gcd(M)}$.*

In this paper we give the answer for a generalization of the commutative case.

Remark 18. *Let $\underline{Q} := (\{\pm e, \pm i, \pm j, \pm k\}; \cdot, ^{-1}, e)$ be the quaternion group. The commutative law is not valid in $\underline{Q}$. On the other hand we have $a^2 = \pm e$ and $\pm e \cdot b = \pm b = b \cdot \pm e$ for all $a, b \in \{\pm e, \pm i, \pm j, \pm k\}$. Thus $x^2 y \approx y x^2$ is an identity in $\underline{Q}$. This motivates us to consider varieties of groups satisfying $x^2 y \approx y x^2$.*

The next theorem characterizes the lattice of all M-solid varieties of groups satisfying $x^2 y \approx y x^2$ for any given monoid $M \in \mathcal{H}_{nt}$.

Definition 19. *Let $\sigma \in Hyp$ with $\sigma(f) \approx x^{a_1} y^{a_2} ... x^{a_{2n-1}} y^{a_{2n}}$ where $1 \leq n \in \mathbb{N}, a_2, ..., a_{2n-1} \in \mathbb{Z} \setminus \{0\}$ and $a_1, a_{2n} \in \mathbb{Z}$.*
σ is said to be y-odd if there is an $i \in \{1, ...n\}$ such that a_{2i} is odd and $a_1, ..., a_{2i-1}$ are even.

For example, any $\sigma \in Hyp$ with $\sigma(f) = x x^{-1} y y x x y^{-1}$ is y-odd.

Theorem 20. *Let p be a prime number and M be a submonoid of Hyp with $M \subseteq H_p$. A variety V of groups satisfying $x^2 y \approx y x^2$ is M-solid iff $x^p \approx e \in IdV$ and V is self-dual if there exists some y-odd hypersubstitution $\sigma \in M$.*

Proof. Suppose that $x^p \approx e \in IdV$ and V is self-dual if there exists some y-odd hypersubstitution $\sigma \in M$. We show that for any $\sigma \in M$ holds

$$\sigma \sim_V \sigma_d \text{ if } \sigma \text{ is } y\text{-odd and}$$
$$\sigma \sim_V \sigma_{id} \text{ otherwise.}$$

Let $\sigma \in M$ with $\sigma(f) \approx x^{a_1}y^{a_2}...x^{a_{2r-1}}y^{a_{2r}}$ where $1 \leq r \in \mathbb{N}, a_2, ..., a_{2r-1} \in \mathbb{Z} \setminus \{0\}$ and $a_1, a_{2r} \in \mathbb{Z}$. Using $x^2y \approx yx^2$ it is easy to calculate that from $\sigma(f) \approx x^{a_1}y^{a_2}...x^{a_{2r-1}}y^{a_{2r}}$ it follows $\sigma(f) \approx yxy^ax^b$ or $\sigma(f) \approx y^{a+1}x^{b+1}$ if σ is $y - odd$ and $\sigma(f) \approx xyx^ay^b$ or $\sigma(f) \approx x^{a+1}y^{b+1}$ otherwise for some integer a, b. Because of $x^p \approx e \in IdV$ we can assume that $0 \leq a, b \leq p-1$. Because of $a+1 \equiv 1(p)$ and $b+1 \equiv 1(p)$ (since $M \subseteq H_p$) we get $a = b = 0$. Thus $\sigma(f) \approx yx$ if σ is y-odd and $\sigma(f) \approx xy$ otherwise. Clearly, $\sigma(g) \approx x^{rp-1}$ for some integer r. Using $x^p \approx e$ we get $\sigma(g) \approx x^{-1}$. Thus $\sigma \sim_V \sigma_d$ if σ is y-odd and $\sigma \sim_V \sigma_{id}$ otherwise. Consequently, any $\sigma \in M$ is V-equivalent to σ_{id} or σ_d.
If there is no y-odd hypersubsitution $\sigma \in M$ then all $\sigma \in M$ are V-equivalent to σ_{id} and V is M-solid (see [6]).
If there is some $\sigma \in M$ which is y-odd then V is self-dual and since each $\sigma \in M$ is V-equivalent to σ_{id} or σ_d, V is M-solid (see [6]).

Suppose that V is M-solid. Then $x^p \approx e \in IdV$ by Lemma 11. We have to consider the case that there is a y-odd hypersubstitution $\sigma \in M$. We have already shown that then $\sigma \sim_V \sigma_d$. Since V is M-solid, the application of σ to any identity in V gives again an identity in V. Thus the application of σ_d to any identity in V is also an identity in V (see [6]), i.e. V is self-dual. □

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