Novi Sad J. Math.
Vol. 34, No. 2, 2004, 167-174
Proc. Novi Sad Algebraic Conf. 2003
(eds. I. Dolinka, A. Tepavčević)


# A NOTE ON SEMILATTICE DECOMPOSITIONS OF COMPLETELY $\pi$-REGULAR SEMIGROUPS

**Žarko Popović[1], Stojan Bogdanović[1], Miroslav Ćirić[2]**

**Abstract.** We study completely $\pi$-regular semigroups admitting a decomposition into a semilattice of $\sigma_n$-simple semigroups, and describe them in terms of properties of their idempotents. In the general case, semigroups admitting a decomposition into a semilattice of $\sigma_n$-simple semigroups were characterized by M. Ćirić and S. Bogdanović in [3] (see Theorem 1 below), in terms of paths of length $n$ in the graph corresponding to the relation $\longrightarrow$, and in terms of principal filters and $n$-radicals. Here we prove that in the completely $\pi$-regular case, it suffices to consider only those paths of length n starting and/or ending with and idempotent, as well as principal filters and n-radicals generated by idempotents.

*AMS Mathematics Subject Classification (2000):* 20M10

*Key words and phrases:* semilattice decompositions, completely $\pi$-regular semigroups, $\sigma_n$-simple semigroups

## 1. Introduction and preliminaries

The relation $\longrightarrow$, introduced by M. S. Putcha in [6] and T. Tamura in [10], plays a crucial role in semilattice decompositions of semigroups. In the mentioned papers it was used in study of semigroups decomposable into a semilattice of Archimedean semigroups, whereas T. Tamura in [11] showed how the least semilattice congruence on a semigroup can be constructed starting from $\longrightarrow$. He proved that the transitive closure of $\longrightarrow$ is a quasi-order whose symmetric opening (that is, its natural equivalence) is equal to the least semilattice congruence on a considered semigroup. Furthermore, M. S. Putcha in [7], proved that the action of the transitive closure and the symmetric opening operators in the Tamura's procedure can be permuted.

The hardest step in Tamura's procedure is the application of transitive closure operator to the relation $\longrightarrow$. As known, the transitive closure of a relation one obtains using an iteration procedure. In the general case, the number of

[1]Faculty of Economics, University of Niš, Trg Kralja Aleksandra 11, P. O. Box 121, 18000 Niš, Serbia, e-mail: `zpopovic@orion.eknfak.ni.ac.yu, sbogdan@pmf.ni.ac.yu`

[2]Department of Mathematics and Informatics, Faculty of Sciences and Mathematics, University of Niš, Višegradska 33, P. O. Box 224, 18000 Niš, Serbia and Montenegro, e-mail: `ciricm@bankerinter.net`

iterations applied may be infinite, and a natural problem that arises here is: Under what conditions on a semigroup $S$, the least semilattice congruence on $S$ can be obtained applying only a finite number of iterations to $\longrightarrow$? This problem has been considered first by M. S. Putcha in [7], and later by M. Ćirić and S. Bogdanović in [3], who gave structural characterizations (cited here as Theorem 1) of all semigroups in which this number can be bounded by a given natural number $n$, the so-called semilattices of $\sigma_n$-simple semigroups. These semigroups have been also studied by S. Bogdanović, M. Ćirić and Ž. Popović in [2].

On the other hand, semilattice decompositions are especially interesting when they are considered for some particular types of semigroups. For example, many authors studied semilattice decompositions of completely $\pi$-regular semigroups. Another characterization of the least semilattice congruence on such semigroups was given by M. S. Putcha in [5], and by L. N. Shevrin [8, 9] and M. L. Veronesi [12], and in a series of papers by S. Bogdanović and M. Ćirić (see the survey paper [1]), decompositions of completely $\pi$-regular semigroups into a semilattice of Archimedean semigroups (that is, $\sigma_1$-simple semigroups) were studied and described in terms of idempotents, regular elements and other special elements.

In this paper we study completely $\pi$-regular semigroups admitting a decomposition into a semilattice of $\sigma_n$-simple semigroups and describe them in terms of properties of their idempotents. In the general case, semigroups admitting a decomposition into a semilattice of $\sigma_n$-simple semigroups were characterized by M. Ćirić and S. Bogdanović in [3] (see Theorem 1 below), in terms of paths of length $n$ in the graph corresponding to the relation $\longrightarrow$, and in terms of principal filters and $n$-radicals. Here we prove that in the completely $\pi$-regular case, it suffices to consider only those paths of length n starting and/or ending with and idempotent, as well as principal filters and n-radicals generated by idempotents.

Now we give precise definitions of the notions used above and the ones that will be used in the further text.

$\mathbb{N}$ will be used in the sequel to denote the set of all positive integers. Let $S$ be a semigroup. For a subset $A$ of $S$, we define $\sqrt{A} = \{x \in S \,|\, (\exists n \in \mathbb{N})\, x^n \in A\}$. A subset $A$ of $S$ is *completely semiprime* if for any $x \in S$, $x^2 \in A$ implies $x \in A$. If $A$ is an ideal of $S$, then it is completely semiprime if and only if $\sqrt{A} \subseteq A$. A subsemigroup $T$ of $S$ is a *filter* of $S$ if for all $x, y \in S$, $xy \in T$ implies $x, y \in T$. The least filter of $S$ containing an element $a$ (the intersection of all filters of $S$ containing $a$) is denoted by $N(a)$ and called the *principal filter* of $S$ generated by $a$.

The *division relation* $\mid$ and the relation $\longrightarrow$ on $S$ are defined by

$$a \mid b \;\Leftrightarrow\; (\exists x, y \in S^1)\; b = xay, \qquad a \longrightarrow b \;\Leftrightarrow\; (\exists k \in \mathbb{N})\; a \mid b^k.$$

For $n \in \mathbb{N}$, $n \geq 2$, the relation $\longrightarrow^n$ on $S$ is defined by

$$a \longrightarrow^n b \;\Leftrightarrow\; (\exists x \in S)\; a \longrightarrow^{n-1} x \longrightarrow b,$$

and for $n = 1$, $\longrightarrow^n = \longrightarrow$. In other words, $\longrightarrow^n$ is the $n$-th power of $\longrightarrow$ in the semigroup of binary relations on $S$. The transitive closure of $\longrightarrow$ is denoted by $\longrightarrow^\infty$. For $n \in \mathbb{N}$ and $a \in S$, the sets $\Sigma_n(a)$ and $\Sigma(a)$ are defined by

$$\Sigma_n(a) = \{x \in S \mid a \longrightarrow^n x\}, \qquad \Sigma(a) = \{x \in S \mid a \longrightarrow^\infty x\},$$

and the equivalence relations $\sigma_n$ and $\sigma$ on $S$ are defined by

$$(a, b) \in \sigma_n \;\Leftrightarrow\; \Sigma_n(a) = \Sigma_n(b), \qquad (a, b) \in \sigma \;\Leftrightarrow\; \Sigma(a) = \Sigma(b).$$

In other words,

$$\Sigma_1(a) = \sqrt{SaS}, \quad \Sigma_{n+1}(a) = \sqrt{S\Sigma_n(a)S} \supseteq \Sigma_n(a), \quad \text{and} \quad \Sigma(a) = \bigcup_{n \in \mathbb{N}} \Sigma_n(a).$$

As it was proved by M. Ćirić and S. Bogdanović in [8], $\sigma$ is the least semilattice congruence on $S$, $N(a) = \{x \in S \mid x \longrightarrow^\infty a\}$ and $\Sigma(a)$ is the least completely semiprime ideal of $S$ containing $a$, called the *principal radical* of $S$ generated by $a$. The set $\Sigma_n(a)$ is called the *$n$-radical* generated by $a$. A semigroup $S$ is *$\sigma_n$-simple* if $\sigma_n$ coincides with the universal relation on $S$, and $\sigma_1$-simple semigroups are also called *Archimedean semigroups*.

The set of all idempotents of a semigroup $S$ is denoted by $E(S)$. If $e \in E(S)$, then $G_e = \{x \in S \mid x \in eS \cap Se,\ e \in xS \cap Sx\}$ is the largest subgroup of $S$ having $e$ as its identity, called the *maximal subgroup* of $S$ determined by $e$, and the set $T_e$ is defined by $T_e = \sqrt{G_e}$. An element $a$ of $S$ is *completely $\pi$-regular* if at least one of its powers lies in some subgroup of $S$. There is exactly one such subgroup, and its identity is denoted by $a^0$.

For undefined notions and notations we refer to the book [4].

## 2. The main results

We start this section by recalling a theorem from the paper [3] by M. Ćirić and S. Bogdanović, which characterizes semilattices of $\sigma_n$-simple semigroups.

**Theorem 1.** *Let $n \in \mathbb{N}$. Then the following conditions on a semigroup $S$ are equivalent:*

(i) *$S$ is a semilattice of $\sigma_n$-simple semigroups;*

(ii) $(\forall a \in S)\; a\,\sigma_n\, a^2$*;*

(iii) $(\forall a, b \in S)\; a \longrightarrow^n b \;\Rightarrow\; a^2 \longrightarrow^n b$*;*

(iv) $(\forall a, b, c \in S)\; a \longrightarrow^n c \;\&\; b \longrightarrow^n c \;\Rightarrow\; ab \longrightarrow^n c$*;*

(v) *for every* $a \in S$, $\Sigma_n(a)$ *is an ideal od* $S$*;*

(vi) $(\forall a, b \in S)\ \Sigma_n(ef) = \Sigma_n(e) \cap \Sigma_n(f)$*;*

(vii) *for every* $a \in S$, $N(a) = \{x \in S \mid x \longrightarrow^n a\}$*;*

(viii) $(\forall a, b, c \in S)\ a \longrightarrow^n b \ \&\ b \longrightarrow^n c \ \Rightarrow\ a \longrightarrow^n c$*;*

(ix) $\sigma_n = \ \longrightarrow^n \cap\ (\longrightarrow^n)^{-1}$ *on* $S$.

Note that (iii) is known as the *power property* for $\longrightarrow^n$, (iv) is the *common multiple property* for $\longrightarrow^n$, briefly the *cm-property*, and (viii) is the transitivity of $\longrightarrow^n$.

Next we prove two auxiliary lemmas.

**Lemma 1.** *Let* $a$ *be a completely* $\pi$*-regular element of a semigroup* $S$*. Then for every* $b \in S$ *and every* $n \in \mathbb{N}$*,*

$$a^0 \longrightarrow^n b \quad \textit{implies} \quad a \longrightarrow^n b.$$

*In other words, for every* $n \in \mathbb{N}$*,*

$$\Sigma_n(a^0) \subseteq \Sigma_n(a).$$

*Proof.* Let $m \in \mathbb{N}$ such that $a^m \in G_{a_0}$, and let $(a^m)^{-1}$ be the inverse of $a^m$ in the group $G_{a_0}$. Then $a^0 = (a^m(a^m)^{-1})^2 \in SaS$, which yields $Sa^0S \subseteq SaS$, and hence

$$\Sigma_1(a^0) = \sqrt{Sa^0S} \subseteq \sqrt{SaS} = \Sigma_1(a).$$

Now, by induction we easily verify that $\Sigma_n(a^0) \subseteq \Sigma_n(a)$, for every $n \in \mathbb{N}$. □

**Lemma 2.** *Let* $b$ *be a completely* $\pi$*-regular element of a semigroup* $S$*. Then for every* $a \in S$ *and every* $n \in \mathbb{N}$*,*

$$a \longrightarrow^n b \quad \textit{if and only if} \quad a \longrightarrow^n b^0.$$

*Proof.* Let $m \in \mathbb{N}$ such that $b^m \in G_{b_0}$. Consider an arbitrary $a \in S$.

Suppose that $a \longrightarrow b$. Then $b^k \in SaS$, for some $k \in \mathbb{N}$, and hence $b^{mk} \in G_{b_0} \cap SaS$. Let $(b^{mk})^{-1}$ be the inverse of $b^{mk}$ in the group $G_{b_0}$. Now $b^0 = (b^{mk}(b^{mk})^{-1})^2 \in SaS$ so we obtain that $a \mid b^0$, which is equivalent to $a \longrightarrow b^0$, because $b^0$ is an idempotent. Conversely, let $a \longrightarrow b^0$, i.e. $a \mid b^0$. Then $b^m = b^0 b^m \in SaSb^m \subseteq SaS$, and hence $a \longrightarrow b$.

Therefore, we have proved that our assertion holds for $n = 1$. By induction we easily verify that this assertion holds for every $n \in \mathbb{N}$. □

Note that if $b$ is completely $\pi$-regular we have $a \longrightarrow b^0$ if and only if $a \mid b^0$. Therefore, in such a case we obtain

$$a \longrightarrow b \quad \text{if and only if} \quad a \mid b^0.$$

Now we are prepared for the main result of the paper.

**Theorem 2.** *Let $S$ be a completely $\pi$-regular semigroup and $n \in \mathbb{N}$. Then the following conditions are equivalent:*

(i) *$S$ is a semilattice of $\sigma_n$-simple semigroups;*

(ii) $(\forall a \in S)\ a\,\sigma_n\,a^0$;

(iii-a) $(\forall a, b \in S)\ a \longrightarrow^n b \ \Rightarrow\ a^0 \longrightarrow^n b$;

(iii-b) $(\forall a \in S)(\forall f \in E(S))\ a \longrightarrow^n f \ \Rightarrow\ a^2 \longrightarrow^n f$;

(iv-a) $(\forall a, b \in S)(\forall g \in E(S))\ a \longrightarrow^n g\ \&\ b \longrightarrow^n g \ \Rightarrow\ ab \longrightarrow^n g$;

(iv-b) $(\forall e, f \in E(S))(\forall c \in S)\ e \longrightarrow^n c\ \&\ f \longrightarrow^n c \ \Rightarrow\ ef \longrightarrow^n c$;

(iv-c) $(\forall e, f, g \in E(S))\ e \longrightarrow^n g\ \&\ f \longrightarrow^n g \ \Rightarrow\ ef \longrightarrow^n g$.

(v) *for every $e \in E(S)$, $\Sigma_n(e)$ is an ideal od $S$;*

(vi) $(\forall e, f \in E(S))\ \Sigma_n(ef) = \Sigma_n(e) \cap \Sigma_n(f)$;

(vii) *for every $e \in E(S)$, $N(e) = \{x \in S \mid x \longrightarrow^n e\}$.*

*If $n \geq 2$, then any of the above conditions is equivalent to*

(viii) $(\forall e, f, g \in E(S))\ e \longrightarrow^n f\ \&\ f \longrightarrow^n g \ \Rightarrow\ e \longrightarrow^n g$.

*Proof.* (i)⇒(ii). For an arbitrary $a \in S$, $a^0 \longrightarrow a$ and $a \mid a^0$, which implies $a \longrightarrow a^0$, and if (i) holds, then by (ix) of Theorem 1 it follows $a\,\sigma_n\,a^0$.

(ii)⇒(iii-a). The condition (ii) is equivalent to $\Sigma_n(a) = \Sigma_n(a^0)$, whereas (iii-a) is equivalent to $\Sigma_n(a) \subseteq \Sigma_n(a^0)$, so it is evident that (ii) implies (iii-a).

(iii-a)⇒(i). Let $a, b \in S$ such that $a \longrightarrow^n b$. By the assumption (iii-a), $a^0 \longrightarrow^n b$, and since $(a^2)^0 = a^0$, we have that $(a^2)^0 \longrightarrow^n b$, so by Lemma 1, $a^2 \longrightarrow^n b$. Hence, by Theorem 1, $S$ is a semilattice of $\sigma_n$-simple semigroups.

(i)⇒(iii-b). This is an immediate consequence of Theorem 1.

(iii-b)⇒(i). Consider $a, b \in S$ such that $a \longrightarrow^n b$. By Lemma 2, $a \longrightarrow^n b$ implies $a \longrightarrow^n b^0$, and by (iii-b), $a \longrightarrow^n b^0$ implies $a^2 \longrightarrow^n b^0$, so again by Lemma 2, $a^2 \longrightarrow^n b$. By this and by Theorem 1 it follows that (i) holds.

(i)⇒(iv-c). This is an immediate consequence of Theorem 1.

(iv-c)⇒(iv-a). Let $a, b \in S$ and $g \in E(S)$ such that $a \longrightarrow^n g$ and $b \longrightarrow^n g$. This means that $a \longrightarrow x \longrightarrow^{n-1} g$ and $b \longrightarrow y \longrightarrow^{n-1} g$, for some $x, y \in S$. By the hypothesis, $S$ is a completely $\pi$-regular semigroup, so $x \in T_{e_0}$ and $y \in T_{f_0}$, for some $e_0, f_0 \in E(S)$, and by Lemma 2, we have that $a \longrightarrow x$ is equivalent to $a \mid e_0$ and $b \longrightarrow y$ is equivalent to $b \mid f_0$. But, $a \mid e_0$ and $b \mid f_0$ yield $e_0 = uav$ and $f_0 = pbq$, for some $u, v, p, q \in S$. Set $e = (vua)^2$ and $f = (bqp)^2$. Then $e, f \in E(S)$ and

$$e_0 = e_0^3 = ua(vua)^2 v = uaev,$$

so we have that $e \mid e_0$, and similarly, $f \mid f_0$. Again by Lemma 2, $e \mid e_0$ is equivalent to $e \longrightarrow x$ and $f \mid f_0$ is equivalent to $f \longrightarrow y$, which yields

$$e \longrightarrow x \longrightarrow^{n-1} g \ \text{ and } \ f \longrightarrow y \longrightarrow^{n-1} g,$$

i.e. $e \longrightarrow^n g$ and $f \longrightarrow^n g$. Now, by the assumption (iv), we obtain that $ef \longrightarrow^n g$, i.e. $ef \longrightarrow z \longrightarrow^{n-1} g$, for some $z \in S$, and hence

$$z^k \in SefS = S(vua)^2(bqp)^2S \subseteq SabS,$$

which means that $ab \longrightarrow z$. Therefore, $ab \longrightarrow z \longrightarrow^{n-1} g$, so $ab \longrightarrow^n g$. Hence, we have proved that (iv-a) holds.

(iv-a)⇒(iii-b). This implication is obvious.

(iv-b)⇒(iv-c). This implication is obvious.

(iv-c)⇒(iv-b). Let $e, f \in E(S)$ and $c \in S$ such that $e \longrightarrow^n c$ and $f \longrightarrow^n c$. By Lemma 2, $e \longrightarrow^n c^0$ and $f \longrightarrow^n c^0$, and (iv-c) yields $ef \longrightarrow^n c^0$, so again by Lemma 2 we obtain $ef \longrightarrow^n c$, which was to be proved.

(i)⇒(v). This follows immediately by Theorem 1.

(v)⇒(i). Consider an arbitrary $a \in S$. By the assumption (v), $\Sigma_n(a^0)$ is an ideal of $S$, and clearly, it is a completely semiprime ideal of $S$, so by Lemma 2 of [3], $\Sigma_n(a^0) = \Sigma(a^0)$. On the other hand, $a^0 \longrightarrow a$ implies $a \in \Sigma_1(a^0) \subseteq \Sigma(a^0)$, which, taken together with Lemma 2 of [3] and Lemma 1, gives

$$\Sigma(a) \subseteq \Sigma(a^0) = \Sigma_n(a^0) \subseteq \Sigma_n(a).$$

This yields $\Sigma_n(a) = \Sigma(a)$, which means that $\Sigma_n(a)$ is an ideal of $S$. Therefore, by (v) of Theorem 1 we conclude that (i) holds.

(i)⇒(vi). This is an immediate consequence of Theorem 1.

(vi)⇒(i). Let $e, f \in E(S)$ and $c \in S$ such that $e \longrightarrow^n c$ and $f \longrightarrow^n c$. By (vi), this implies $c \in \Sigma_n(e) \cap \Sigma_n(f) = \Sigma_n(ef)$, i.e. $ef \longrightarrow^n c$, which was to be proved.

(i)⇒(vii). This follows immediately by Theorem 1.

(vii)⇒(i). Let $a, b \in S$ and $g \in E(S)$ such that $a \longrightarrow^n g$ and $b \longrightarrow^n g$. Then $a, b \in N(g)$, and since $N(g)$ is a subsemigroup of $S$, then $ab \in N(g)$. However, by (vii), this means that $ab \longrightarrow^n g$, which was to be proved.

Further, let $n \geq 2$.

(i)⇒(viii). This is an immediate consequence of Theorem 1.

(viii)⇒(i). According to Theorem 1, in order to prove (i), it suffices to prove that $\longrightarrow^n$ is a transitive relation, and we will consider $a, b, c \in S$ such that $a \longrightarrow^n b$ and $b \longrightarrow^n c$.

First, by Lemma 2 we have that $a \longrightarrow^n b^0$ and $b \longrightarrow^n c^0$. Furthermore, $a \longrightarrow^n b^0$ yields $a \longrightarrow y \longrightarrow^{n-1} b^0$, for some $y \in S$, and since $y \in T_{e_0}$, for some $e_0 \in E(S)$, by Lemma 2 it follows that $a \longrightarrow y$ if and only if $a \mid e_0$, i.e. $e_0 = uav$, for some $u, v \in S$. If we set $e = (vua)^2$, then $e \in E(S)$ and $e_0 = uaev$ so $e \mid e_0$. But, by Lemma 2, $e \mid e_0$ is equivalent to $e \longrightarrow y$, so we have that $e \longrightarrow y \longrightarrow^{n-1} b^0$, i.e. $e \longrightarrow^n b^0$.

On the other hand, $b \longrightarrow^n c^0$ gives $b \longrightarrow z \longrightarrow^{n-1} c^0$, for some $z \in S$, and $z \in T_{h_0}$, for some $h_0 \in E(S)$. Now, by Lemma 2, $b \longrightarrow z$ if and only if $b \mid h_0$, i.e. $h_0 = pbq$, for some $p, q \in S$. Set $h = (bqp)^2$. Then $h \in E(S)$ and $h \mid h_0$, which is equivalent to $h \longrightarrow z$, again by Lemma 2. Thus, $h \longrightarrow z \longrightarrow^{n-1} c^0$, that means $h \longrightarrow^n c^0$.

Finally we have $b^0 \longrightarrow b$, and also $b \mid h$, so $b \longrightarrow h$. Hence, $b^0 \longrightarrow^2 h$, so $b^0 \longrightarrow^n h$, because $n \geq 2$. Therefore,

$$e \longrightarrow^n b^0, \quad b^0 \longrightarrow^n h \quad \text{and} \quad h \longrightarrow^n c^0,$$

so by the assumption (v) we conclude that $e \longrightarrow^n c^0$.

Now, in order to prove that $a \longrightarrow^n c$, we start with the relation $e \longrightarrow^n c^0$, and by Lemma 2 we obtain that $e \longrightarrow^n c$. But this means that $e \longrightarrow t \longrightarrow^{n-1} c$, for some $t \in S$. Further, $e \longrightarrow t$ implies

$$t^k \in SeS = S(vua)^2 S \subseteq SaS,$$

for some $k \in \mathbb{N}$, so $a \longrightarrow t$. Therefore, $a \longrightarrow t \longrightarrow^{n-1} c$, and we have that $a \longrightarrow^n c$, which was to be proved. □

*Remark 1.* The requirement $n \geq 2$ is crucial for the equivalence of (i) and (viii) in the previous theorem. Namely, every completely $\pi$-regular semigroup $S$ satisfies the condition

$$(\forall e, f, g \in E(S))\ e \longrightarrow f \ \&\ f \longrightarrow g \ \Rightarrow\ e \longrightarrow g,$$

because it is clearly equivalent to the condition

$$(\forall e, f, g \in E(S))\ e \mid f \ \&\ f \mid g \ \Rightarrow\ e \mid g,$$

and the division relation is transitive. But, $S$ is not necessarily a semilattice of $\sigma_1$-simple semigroups. For example, the five-element Brandt semigroup

$$\mathbb{B}_2 = \langle a, b \,|\, a^2 = b^2 = 0,\, aba = a,\, bab = b \rangle$$

is completely $\pi$-regular, and hence satisfies the above conditions. But $S$ is not a semilattice of $\sigma_1$-simple (Archimedean) semigroups.

## References


[1] Bogdanovic, S., Ćirić, M., Semilattices of Archimedean semigroups and (completely) $\pi$-regular semigroups, I ( A survey), Filomat (Niš) **7** (1993), 1-40.

[2] Bogdanović, S., Ćirić, M., Popović, Ž., Semilattice decompositions of semigroups revisited, Semigroup Forum **61** (2000), 263–276.

[3] Ćirić, M., Bogdanović, S., Semilattice decompositions of semigroups, Semigroup Forum **52** (1996), 119–132.

[4] Howie, J. M., Fundamentals of Semigroup Theory, London Math. Soc. Monographs, New Series, Clarendon Press, Oxford, 1995.

[5] Putcha, M. S., Semigroups in which power of each element lies in a subgroup, Semigroup Forum **5** (1973), 354–361.

[6] Putcha, M. S., Semilattice decompositions of semigroups, Semigroup Forum 6 (1973), 12–34.

[7] Putcha, M. S., Minimal sequences in semigroups, Trans. Amer. Math. Soc. 189 (1974), 93–106.

[8] Shevrin, L. N., Theory of epigroups I, Mat. Sbornik **185** (1994), no. 8, 129–160 (in Russian).

[9] Shevrin, L. N., Theory of epigroups II, Mat. Sbornik **185** (1994), no. 9, 153–176 (in Russian).

[10] Tamura, T., On Putcha's theorem concerning semilattice of Archimedean semigroups, Semigroup Forum **5** (1972), 83–86.

[11] Tamura, T., Note on the greatest semilattice decomposition of semigroups, Semigroup Forum **4** (1972), 255–261.

[12] Veronesi, M. L., Sui semigruppi quasi fortemente regolari, Riv. Mat. Univ. Parma **(4) 10** (1984), 319–329.