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SOME OPEN PROBLEMS CONCERNING INDEPENDENCE NOTIONS[1]

(Proposed by Kazimierz Głazek, University of Zielona Góra, Poland)

Let $(A;\mathcal{C})$ be a closure space ($\mathcal{C} : \in^{\mathcal{A}} \to \in^{\mathcal{A}}$ is a closure operator in the sense of E.H. Moore, that is, $\mathcal{C}$ is extensive, monotonic and idempotent). Let $X \subseteq A$.

Consider conditions:

(C_1) $(\forall\ a \in X)\,[\mathcal{C}\,(\{\dashv\}) \cap \mathcal{C}\,(\mathcal{X} \setminus \{\dashv\}) = \mathcal{C}\,(\phi)]$;

(C_2) $(\forall\ Y, Z \subseteq X)\,[Y \cap Z = \phi \Rightarrow \mathcal{C}\,(\mathcal{Y}) \cap \mathcal{C}\,(\mathcal{Z}) = \mathcal{C}\,(\phi)]$;

(C_3) $(\forall\ Y, Z \subseteq X)\,[\mathcal{C}\,(\mathcal{Y} \cap \mathcal{Z}) = \mathcal{C}\,(\mathcal{Y}) \cap \mathcal{C}\,(\mathcal{Z})]$.

If X fulfils condition (C_i) and $\mathcal{C}\,(\phi) \cap \mathcal{X} = \phi$, then X is said to be C_i-**independent**. The C_1-independence is also called the "direct independence".

1) Investigate the notions of C_i-independence (i =1,2 and 3).

2) Characterize families of all C_i-independent subsets or all C_i-bases in closure spaces with closure operators of finite character, that is, with the following property:

(FC) $a \in \mathcal{C}\,(\mathcal{X})\,,\, X \subseteq A \Rightarrow a \in \mathcal{C}\,(\mathcal{Y})$ for some finite subset Y of X.

In particular, investigate these families for general algebras $\mathfrak{A} = (A;\mathbb{F})$ (or classes of algebras) if $\mathcal{C}\,(\mathcal{X}) = \langle \mathcal{X} \rangle_{\mathfrak{A}}$ - the subalgebra of $\mathfrak{A}$ generated by $X \subseteq A$.

3) For which general algebras $\mathfrak{A} = (A;\mathbb{F})$ (or classes of algebras) the C_3-independence has the following property:

$(JIS)_3$ if I and J are C_3-independent, then $I \cup J$ is also C_3-independent (for all $I, J \subseteq A$), where $\mathcal{C}\,(\mathcal{X}) = \langle \mathcal{X} \rangle_{\mathfrak{A}}$ for all $X \subseteq A$?

Characterize in these cases the families of all C_3-independent subsets of A.

4) Assume that a closure space $(A;\mathcal{C})$ is defined for all algebras $\mathfrak{A} = (A;\mathbb{F})$ as generating of subalgebras. The algebra $\mathfrak{A}$ (or the closure space $(A;\mathcal{C})$) is said to have $(JIS)_i$-property for C_i-independence, i =1 or 2, if for arbitrary C_i-independent sets I and J, $I \cup J$ is also C_i-independent whenever

$$\mathcal{C}\,(\mathcal{I}) \cap \mathcal{C}\,(\mathcal{J}) = \mathcal{C}\,(\mathcal{I} \cap \mathcal{J})\,.$$

For which algebras these properties (for $i = 1, 2$) hold?

5) Assume that a closure space $(A;\mathcal{C})$ has $(JIS)_i$-property (for C_i-independence), $i = 1, 2$ or 3. Characterize the families of allC_i-independent subsets and all C_i-bases. Compare the resuilts with $C-$independence in Matriod Theory.

[1]Some of the problems posed in the text were presented at the NSAC'03 problem session.

Consider the notion of **independence with respect to a family Q of mappings** defined on subsets of the carrier A of some algebra $\mathfrak{A} = (A; \mathbb{F})$ (Q-independence, for short).

6) For which families Q the following property $(JIS)_Q$ of Q-independence holds?

$(JIS)_Q$ for arbitrary Q-independent sets I and J $(I, J \subseteq A)$, the set $I \cup J$ is also Q-independent, whenever

$$\mathcal{C}(\mathcal{I}) \cap \mathcal{C}(\mathcal{J}) = \mathcal{C}(\mathcal{I} \cap \mathcal{J})?$$

7) For which algebras this property $(IJS)_Q$ holds for a well-defined family Q?

8) For algebras with $(JIS)_Q$ property for Q-independent sets, characterize the families $Ind(\mathfrak{A}; Q)$ of all Q-independent sets in the algebra $\mathfrak{A}$.

9) Investigate algebras $\mathfrak{A}$ with $(JIS)_M$ property for M-independent sets (that is, independent sets in the sense of Marczewski).

10) Give a common generalization of the

α) Equicardinality Theorem of bases in matroid theory,
β) Tarski Interpolation Theorem for closure systems of rank n,
γ) general algebraic Marczewski Theorem about arithmetical progressions of cardinalities of M-bases.

11) Formulate (in the "language" of the Q-independence) and prove the general theorem, which contains as special cases the femous representation theorems for so-called v**-algebras**, v***-algebras**, **separable variables algebras**, and **abelian algebras** (in the sense used by R. McKenzie and his co-workers).

12) Let $\mathfrak{A}$ be a linear space over an arbitrary field. Consider special kinds of the Q-independence for

A) $Q = M$ (the Marczewski independence),
B) $Q = S$ (the local independece or independence-in-itself in the sense of J. Schmidt),
C) $Q = S_0$ (the weak independence in the sense of S. Swierczkowski),
D) $Q = I$ (the independence with respect the family of injective mappings),
E) $Q = G$ (the Grätzer independence),
and
F) the independence with respect to the closure operator defined by generating of subalgebras (i.e. sbspaces in the considered case).

It is well-known (see K.Głazek, Dissert. Math. **81 (**1971)) that the following relations hold:

$Ind(\mathfrak{A}, M) \cup \{\{0\}\} = Ind(\mathfrak{A}, S) = Ind(\mathfrak{A}, S_0)$,
$Ind(\mathfrak{A}, M) = <>_{\mathfrak{A}}\text{-}ind = Ind(\mathfrak{A}, I)$,
$X \in Ind(\mathfrak{A}, G) \Longleftrightarrow (X \setminus \{0\}) \in Ind(\mathfrak{A}, M)$.

Does the above relations characterize linear spaces? (It is the problem posed by K.Głazek and F. Pastijn at the algebraic seminar in Gent University, Belgium, in 1980).

13) For which general algebras $\mathfrak{A}$, $Ind(\mathfrak{A}, I) = Ind(\mathfrak{A}, M)$?

14) For which algebras is the S-independence equivalent to the M-independence for subsets consisting of at least two elements?

This holds for linear spaces, affine spaces (and, more generally,for v^{**}-algebras), torsion-free groups and regular reducts of Boolean algebras.

15) For which algebras is the S_0-independence equivalent to the S-independence for subsets consisting of at least two elements?

16) Characterize the sets of cardinal numbers of Q-bases of an algebra for special cases of Q-independence, e.g., for $Q = S_0$, S, G, and I.

17) Describe algebras $\mathfrak{A} = (A; \mathbb{F})$, in which the whole set A is Q-independent, for special families of, e.g., $Q = S$, S_0, G, I, etc.

18) Work out the notion of independence with respect of the closure operator defined by generating of subalgebras for special semirings with idempotent addition, e.g., for dioïds (in the sence of J. Kuntzmann)

a) $(R \cup \{-\infty\}; \max, +)$, the schedule algebra or the exotic semiring,
b) $(R \cup \{-\infty\}; \min, +)$,
c) $([0, 1]; \max, \cdot)$,
d) $(\mathbb{N} \cup \{-\infty\}; \max, +)$, the polar semiring,
e) $(\mathbb{N} \cup \{+\infty\}; \min, +)$, the tropical semiring,
f) $(\mathbb{Z} \cup \{+\infty\}; \min, +)$, the equatorial semiring.

19) Work out the notion of Marczewski independence for semirings described in a)-f) above.

20) Work out the notion of Q-independence (for special families Q) for the above defined semirings.

21) Work out generalized matrices (in the sense defined in K. Głazek, Colloq. Math. **52** (1979),127-189) over algebras with the abelian property (in the sense of A.G.Kurosh; the commutativity property for operations in another terminology). Compare with G. Ricci results.

22) Characterize families of Q-independent subsets of n-ary groups (abelian or commutative) for different families Q.

These problems was formulated by me during series of my seminars at COMSATS I.I.T. in Islamabad (February-June 2003).

Some research problems on similar topics, which were published by K. Głazek in the following articles:

[1] K. Głazek, *Independence with respect to family of mappings in abstract algebras*, Dissert. Math., vol. 81 (1971), PWN (Inst. of Math. of Polish Acad. of Sci.), Warsaw 1971, 55 pages (see pages: 16, 27, 32, 43, and 45).

[2] K. Głazek, Q-independence and various notions of independence in regular reducts of Boolean algebras, Acta Fac. Rerum Nat. Univ. Comenianae - Mathematica, 1971, a special no., p. 25-37 (Problems 1 - 4, see pages: 29, 30, 34, and 36).

[3] K. Głazek and A. Iwanik, Quasi-constants in general algebras, Colloq. Math. **29** (1974), 45-50 (Problems 1-3, see pages: 47 and 50).

[4] K. Głazek, Quasi-constants in universal algebras and independent subalgebras, Acta Fac. Rerum Nat. Univ. Comenianae - Mathematica, 1975, a special no., p. 9-16 (Problems 1- 9, see pages: 11, 12, and 15).

[5] K. Głazek, *Some old and new problems in the independence theory*, Colloq. Math. 42 (1979), 127-189 [there are original author's problems P-1086-1132; and 35 open problems posed earlier by him or other authors (quoted from the literature or seminar talks); some problems from the paper are solved or partially solved: problem 4.1 is solved by A. Kisielewicz, there are several papers by G. Ricci concerning problem 4.6, problem 5.9 is solved by E. Graczyńska and F. Pastijn, problems 7.7, 7.8 and 7.10 were answered negatively by S. Niwczyk].