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ON PROBABILISTIC 2-NORMED SPACES

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Abstract. In [16] K. Menger proposed the probabilistic concept of distance by replacing the number $d(p,q)$, as the distance between points p,q, by a distribution function $F_{p,q}$. This idea led to development of probabilistic analysis [3], [11] [18]. In this paper, generalized probabilistic 2-normed spaces are studied and topological properties of these spaces are given. As an example, a space of random variables is considered, connections with the generalized deterministic 2-normed spaces studied in [14] being also given.

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1. Introduction

The theory of probabilistic normed spaces was initiated and developed in [19],[15],[17],[12].This theory is important as a generalization of deterministic results of linear normed spaces and also in the study of random operator equations. For more results of probabilistic functional analysis we refer to [1],[3],[11],[18].

The linear 2-normed spaces were first introduced in [7], since these were studied in many papers, we mention [4],[5],[13].

In this paper we start from the results obtained [14] and we introduce a generalization of probabilistic 2-normed spaces studied in a previous paper [9]. Topological properties of these spaces and their connections with deterministic 2-normed spaces are considered. Examples of probabilistic 2-normed spaces are also given.

Let $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $I = [0,1]$ the closed unit interval. A mapping $F : \mathbb{R} \to I$ is called a distribution function if it is non decreasing, left-continuous with $\inf F = 0$ and $\sup F = 1$.

D_+ denotes the set of all distribution functions for that $F(0) = 0$. Let F, G be in D_+, then we write $F \leq G$ if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. If $a \in \mathbb{R}_+$, then H_a will be an element of D_+, defined by $H_a(t) = 0$ if $t \leq a$ and $H_a(t) = 1$ if $t > a$. It is obvious that $H_0 \geq F$ for all $F \in D_+$. The set D_+ will be endowed with the natural topology defined by the modified Lévy metric d_L [18].

A 2-normed space is a pair $(L, ||\cdot,\cdot||)$ ([7]), where L is a linear space of a dimension greater than one and $||\cdot,\cdot||$ is a real valued mapping on $L \times L$ such that the following conditions be satisfied:

(1) $||x, y|| = 0$ if and only if x and y are linearly dependent,

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(2) $||x,y|| = ||y,x||$, for all $x, y \in L$,
(3) $||\alpha \cdot x, y|| = |\alpha| ||x,y||$, whenever $x, y \in L$ and $\alpha \in \mathbb{R}$,
(4) $||x+y,z|| \leq ||x,z|| + ||y,z||$, for all $x, y, z \in L$.

A t-norm T is a two-place function $T : I \times I \to I$ which is associative, commutative, non decreasing in each place and such that $T(a,1) = a$, for all $a \in [0,1]$. A triangle function τ is a binary operation on D_+ which is commutative, associative and for which H_0 is the identity, that is, $\tau(F, H_{0)}) = F$, for every $F \in D_+$. The terminology and notations are standard as in [3],[18].

Definition 1. Let L be a linear space of a dimension greater than one, τ a triangle function , and let $\mathcal{F}$ be a mapping from $L \times L$ into D_+. If the following conditions are satisfied :

(5) $F_{x,y} = H_0$ if x and y are linearly dependent,
(6) $F_{x,y} \neq H_0$ if x and y are linearly independent,
(7) $F_{x,y} = F_{y,x}$, for every x, y in L,
(8) $F_{\alpha x,y}(t) = F_{x,y}(\frac{t}{|\alpha|})$, for every $t > 0, \alpha \neq 0$ and $x, y \in L$,
(9) $F_{x+y,z} \geq \tau(F_{xz}, F_{yz})$, whenever $x, y, z \in L$,

then $\mathcal{F}$ is called a probabilistic 2-norm on L and $(L, \mathcal{F}, \tau)$ is called a probabilistic 2-normed space ([9]). If (5)-(9) are satisfied and the probabilistic triangle inequality (13) is formulated under a t-norm T :

(9′) $F_{x+y,z}(t_1 + t_2) \geq T(F_{xz}(t_1), F_{yz}(t_2))$, for all $x, y, z, \in L, t_1, t_2 \in \mathbb{R}_+$ then $(L, \mathcal{F}, T)$ is called a random 2-normed space.

Remark 1. It is easy to check that every 2-normed space $(L, ||\cdot,\cdot||)$ can be made a random 2-normed space, in a natural way, by setting $F_{x,y}(t) = H_0(t - ||x,y||)$, for every $x, y \in L$, $t \in \mathbb{R}_+$ and $T = Min$.

Proposition 1. If T is a left continuous t-norm and τ_T is the triangle function defined by $\tau_T(F,G)(t) = \sup_{t_1+t_2<t} T(F(t_1), G(t_2)), \quad t > 0$, then $(L, \mathcal{F}, \tau_T)$ is a probabilistic 2-normed space iff $(L, \mathcal{F}, T)$ is a random 2-normed space.

Theorem 1. Let $(L, \mathcal{F}, T)$ be a random 2-normed space under a continuous t-norm T such that $T \geq T_m$, where $T_m = \max\{Sum - l, 0\}$, then
$(L, \mathcal{F}, T)$ becomes a Hausdorff linear topological space with a fundamental system of neighborhoods of the null vector θ given by
$\mathcal{V}_\Theta = \{V(t,A) : t > 0, A \in \mathcal{A}, \}$, where $V(t,A) = \{y \in L : F_{ya}(t) > 1-t, a \in A\}$, and $\mathcal{A}$ is the finite subset family of L.

2. Probabilistic 2-normed spaces

Now, we define a probabilistic 2-norm on a pair of different linear spaces. The results obtained are an extension of those from [9] and a probabilistic generalization of those from [14].

Definition 2. Let L, M be two real linear spaces of dimension greater than one, and let $\mathcal{F}$ be a function defined on the Cartesian product $L \times M$ into D_+ satisfying the following properties:

(10) $F_{\alpha x,y}(t) = F_{x,\alpha y}(t) = F_{x,y}(\frac{t}{|\alpha|})$, for every $t > 0, \alpha \in \mathbb{R} - \{0\}$ and $(x, y) \in L \times M$

(11) $F_{x+y,z} \geq \tau(F_{x,z}, F_{y,z})$, for every $x, y \in L$ and $z \in M$.
(12) $F_{x,y+z} \geq \tau(F_{x,y}, F_{x,z})$, for every $x \in L$ and $y, z \in M$.

The function $\mathcal{F}$ is called a generalized probabilistic 2-norm on $L \times M$ and the triple $(L \times M, \mathcal{F}, \tau)$ is called a generalized probabilistic 2-normed space (briefly GP-2-N space).

The triangle inequalities $(11), (12)$ can be formulated using a t-norm T.

(13) $F_{x+y,z}(t_1 + t_2) \geq T(F_{x,z}(t_1), F_{y,z}(t_2))$, for every $t_1, t_2 \in \mathbb{R}_+, x \in L$ and $y, z \in M$;
(14) $F_{x,y+z}(t_1 + t_2) \geq T(F_{x,y}(t_1), F_{x,z}(t_2))$, for every $t_1, t_2 \in \mathbb{R}_+, x, y \in L$ and $z \in M$.

If $(14), (17)$ and (18) are satisfied then the triple $(L \times M, \mathcal{F}, T)$ is called a generalized random 2-normed spaces (briefly GR-2-N space).

Proposition 2. If $(L \times M, \mathcal{F}, T)$ is GR-2-N space then the probabilistic 2-norm $\mathcal{F}$ has the following properties :

(15) $F_{x,\theta}(t) = H_0(t)$ for all $t > 0$ and $x \in L$, where θ is the null vector in M;
(16) $F_{\theta,y}(t) = H_0$, for all $t \in \mathbb{R}_+$ and $y \in M$, where θ is the null vector in L.

Proof. Indeed, $F_{x,\theta}(t) = F_{x,\alpha\theta}(t) = F_{x,\theta}(\frac{t}{|\alpha|})$, for all $\alpha \in \mathbb{R} - \{0\}$. Then

$$F_{x,\theta}(t) = \lim_{\alpha \to 0} F_{x,\theta}(\frac{t}{|\alpha|}) = F_{x,\theta}(\infty) = H_0(t)$$

□

The GR-2-norm $\mathcal{F}$ induces a topology on each linear spaces L and M, hence we can define the product topology on $L \times M$.

Let $\mathcal{A}$ be the family of all finite and non-empty subsets of the linear space M, $A \in \mathcal{A}$, $\varepsilon > 0$ and $\lambda \in (0, 1)$. By a neighborhood of zero in the linear space L we mean a subset of L defined by

$$V(\varepsilon, \lambda, A) = \{x \in L : F_{x,a}(\varepsilon) > 1 - \lambda, a \in A\}$$

If $\mathcal{B}$ is the family of all finite and non-empty subsets of the linear space L and $B \in \mathcal{B}$, then by a neighborhood of zero in the linear space M we mean a subset of M defined by

$$W(\varepsilon, \lambda, A) = \{x \in M : F_{b,x}(\varepsilon) > 1 - \lambda, b \in \mathcal{B}\}$$

Theorem 2. Let $(L \times M, \mathcal{F}, T)$ be a GR-2-N space under a continuous t-norm T, $T \geqslant T_m$, where $T_m = Max(Sum - 1, 0)$. Then:

a) The family

$$\mathcal{V}_M = \{V(\varepsilon, \lambda, A) : \varepsilon > 0, \ \lambda \in (0, 1), A \in \mathcal{A}\}$$

is a base system of neighborhoods of zero in the linear space L.

b) The family

$$\mathcal{W}_L = \{W(\varepsilon, \lambda, A) : \varepsilon > 0,\ \lambda \in (0,1), B \in \mathcal{B}\}$$

is a base for a system of neighborhoods of zero in the linear space M.

Proof. First, we will prove the statement (a). Let $V(\varepsilon_k, \lambda_k, A_k), k = 1, 2$ be in $\mathcal{V}_M$. We consider $A = A_1 \cup A_2$, $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, $\lambda = \min\{\lambda_1, \lambda_2\}$, then $V(\varepsilon, \lambda, A) \subset V(\varepsilon_1, \lambda_1, A_1) \cap V(\varepsilon_2, \lambda_2, A_2)$.

Let $\alpha \in \mathbb{R}$ such that $0 \leq |\alpha| \leq 1$ and $x \in \alpha V(\varepsilon, \lambda, A)$, then $x = \alpha y$, where $y \in V(\varepsilon, \lambda, A)$. For every $a \in A$ we have

$$F_{x,a}(\varepsilon) = F_{\alpha y,a}(\varepsilon) = F_{y,a}(\frac{\varepsilon}{|\alpha|}) \geq F_{y,a}(\varepsilon) > 1 - \lambda.$$

This shows us that $x \in V(\varepsilon, \lambda, A)$, hence $\alpha V(\varepsilon, \lambda, A) \subset V(\varepsilon, \lambda, A)$.

Now, let us show that, for every $V \subset \mathcal{V}_M$ and $x \in L$ there exists $\beta \in \mathbb{R},\ \beta \neq 0$ such that $\beta x \in V$. If $V \in \mathcal{V}_M$ then there exists $\varepsilon > 0$, $\lambda \in (0,1)$ and $A \in \mathcal{A}$ such that $V = V(\varepsilon, \lambda, A)$. Let x be arbitrarily fixed in L and $\alpha \in \mathbb{R}$, $\alpha \neq 0$, then $F_{\alpha x,a}(\varepsilon) = F_{x,a}(\frac{\varepsilon}{|\alpha|})$. Since $\lim_{|\alpha| \to 0} F_{x,a}(\frac{\varepsilon}{|\alpha|}) = 1$ it follows that, for every $a \in A$ there exists $\alpha(a) \in \mathbb{R}$ such that $F_{x,a}(\frac{\varepsilon}{|\alpha(a)|}) > 1 - \lambda$. If we choose $\beta = \min\{|\alpha(a)| : a \in A\}$, then we have

$$F_{\beta x,a}(\varepsilon) = F_{x,a}(\frac{\varepsilon}{\beta}) \geq F_{x,a}(\frac{\varepsilon}{|\alpha(a)|}) > 1 - \lambda,$$

for all $a \in A$, hence $\beta x \in V$.

Let us prove that, for any $V \in \mathcal{V}_M$, there exists $V_0 \in \mathcal{V}_M$ such that $V_0 + V_0 \subset V$.

If $V = V(\varepsilon, \lambda, A)$ and $x \in V(\varepsilon, \lambda, A)$, then there exists $\eta > 0$ such that $F_{x,a}(\varepsilon) > 1 - \eta > 1 - \lambda$, for every $a \in A$. If $V_0 = V(\frac{\varepsilon}{2}, \frac{\eta}{2}, A)$ and $x, y \in V_0$, $a \in A$ by triangle inequality we have

$$F_{x+y,a}(\varepsilon) \geq T(F_{x,a}(\frac{\varepsilon}{2}), F_{y,a}(\frac{\varepsilon}{2})) \geq T(1 - \frac{\eta}{2}, 1 - \frac{\eta}{2}) \geq T_m(1 - \frac{\eta}{2}, 1 - \frac{\eta}{2}) > 1 - \eta > 1 - \lambda.$$

The above inequalities show us that $V_0 + V_0 \subset V$.

In what follows we show that $V \in \mathcal{V}_M$ and $\alpha \in \mathbb{R},\ \alpha \neq 0$ implies $\alpha V \in \mathcal{V}_M$.

Let us remark that $\alpha V = \alpha V(\varepsilon, \lambda, A) = \{\alpha x : F_{x,a}(\varepsilon) > 1 - \lambda, a \in A)$ and $F_{x,a}(\varepsilon) > 1 - \lambda \Leftrightarrow F_{x,a}(\frac{|\alpha|\varepsilon}{|\alpha|}) = F_{\alpha x,a}(|\alpha|\varepsilon) > 1 - \lambda$. This shows that $\alpha V = V(|\alpha|\varepsilon, \lambda, A)$, hence $\alpha V \in \mathcal{V}_M$.

The above statements show us that $\mathcal{V}_M$ is a base for a system neighborhoods of the origin. The topology generated by this system on the linear space L is named $\mathcal{F}_M$-topology on L.

The proof of the statement (b) is similar and we omitted it. □

We now consider the following example of GR-2-N space having as base spaces sets of random variables with values in a Banach algebra.

The study of Banach algebra-valued random variables is of great importance in the theory of random equations because many of the Banach spaces encountered are also algebras.

Let $(X, \|.\|)$ be a separable Banach space which is also an algebra. Let $(\Omega, \mathcal{K}, P)$ be a complete probability measure space and let $(X, \mathcal{B})$ be the measurable space, where $\mathcal{B}$ is the σ-algebra of Borel subsets of the separable Banach algebra $(X, ||.||)$. We denote by E the linear space of all random variables defined on $(\Omega, \mathcal{K}, P)$ with values in $(X, \mathcal{B})$.

Since, in a Banach algebra, the operation of multiplication is continuous, the product of two X-valued random variables $x(\omega)y(\omega)$ is a well-defined X-valued random variable.

For all $x, y \in E$, $t \in \mathbb{R}$, and $t > 0$ we define

$$(17) \quad \mathcal{F}_{x,y}(t) = F_{x,y}(t) = P(\{\omega \in \Omega : ||x(\omega)y(\omega)|| < t\})$$

Theorem 3. Let L, M be two linear subspaces of E. Then the triple $(L \times M, \mathcal{F}, T_m)$ is a generalized random 2-normed space.

Proof. We have to show that conditions of Definition 2 are satisfied. $F_{\alpha x,y}(t) = P(\{\omega \in \Omega : ||\alpha x(\omega)y(\omega)|| < t\}) = P(\{\omega \in \Omega : |\alpha|||x(\omega)y(\omega)|| < t\}) = P(\{\omega \in \Omega : ||x(\omega)y(\omega)|| < \frac{t}{|\alpha|}\}) = F_{x,y}(\frac{t}{|\alpha|})$. Similarly, one shows that $F_{x,\alpha y}(t) = F_{x,y}(\frac{t}{|\alpha|})$. So, the condition (10) is satisfied. □

For each $x, y \in L, z \in M$, and $t_1, t_2 \in \mathbb{R}_+ - \{0\}$ we define the sets:

$$A = \{\omega \in \Omega : ||x(\omega)z(\omega)|| < t_1\}, \quad B = \{\omega \in \Omega : ||y(\omega)z(\omega)|| < t_2\},$$

$$C = \{\omega \in \Omega : ||[x(\omega) + y(\omega)]z(\omega)|| < t_1 + t_2\}$$

From the triangle inequality of the norm $||.||$ it follows that $A \cup B \subset C$. By properties of the measure of probability P we have

$$P(C) \geq P(A \cap B) \geq P(A) + P(B) - P(A \cap B) \geq P(A) + P(B) - 1$$

Taking into account (17) $P(A) = F_{xz}(t_1)$ $P(B) = F_{y,z}(t_1)$ and $P(C) = F_{x+y,z}(t_1 + t_2)$, hence, the inequality (13) is satisfied. Similarly, one proves the inequality (14). □

Theorem 4. Let L, M be two linear spaces over the field $\mathbb{R}$ of real numbers, let $T = Min$ and let us consider the mappings :

$f : L \times M \longrightarrow [0, \infty)$, $\quad \mathcal{F} : L \times M \longrightarrow D_+$ such that

$$\mathcal{F}_{x,y}(t) = F_{x,y}(t) = \begin{cases} H_0(t) & \text{if} \quad t \leq 0 \\ \min\{\frac{t}{f(x,y)}, 1\} & \text{if} \quad t > 0, \end{cases}$$

when we adopt the convention $\frac{x}{0} > 1$. Then :

a) $(L \times M, f)$ is a generalized 2-normed space if and only if $(L \times M, \mathcal{F}, T)$ is a generalized random 2-normed space.

b) Topologies generated by f and $\mathcal{F}$ on L and on M, respectively, are equivalent.

Proof. First, let suppose that f is a generalized 2-norm. If $(x, y) \in L \times M$, $t > 0$ and $\alpha \in \mathbb{R} - \{0\}$ then:

$$F_{\alpha x,y}(t) = \min\{\frac{t}{f(\alpha x, y)}, 1\} = \min\{\frac{t}{|\alpha| f(x, y)}, 1\} = \min\{\frac{\frac{t}{|\alpha|}}{f(x, y)}, 1\} =$$

$$= F_{xy}(\frac{t}{|\alpha|}).$$

Similarly, one proves that $F_{x,\alpha y}(t) = F_{xy}(\frac{t}{|\alpha|})$.

Let us prove the random triangle inequality (14). We suppose that there exists $t_1, t_2 > 0$, $x \in L$ and $y, z \in M$ such as

$$F_{x,y+z}(t_1 + t_2) < T(F_{x,y}(t_1), F_{x,z}(t_2)) = \min\left\{\frac{t_1}{f(x, y)}, \frac{t_2}{f(x, z)}, 1\right\},$$

then it follows

$$\frac{t_1 + t_2}{f(x, y + z)} < \frac{t_1}{f(x, y)}; \quad \frac{t_1 + t_2}{f(x, y + z)} < \frac{t_2}{f(x, z)}.$$

Hence $(t_1 + t_2)f(x, y) < t_1 f(x, y + z)$; $\quad (t_1 + t_2)f(x, z) < t_2 f(x, y + z)$. By addition it follows

$$(t_1 + t_2)(f(x, y) + f(x, z)) < (t_1 + t_2)f(x, y + z).$$

This implies that

$$f(x, y) + f(x, z) < f(x, y + z)$$

which is contrary to the fact that f is a generalized 2-norm. Consequently,

$$F_{x,y+z}(t_1 + t_2) \geq \min\{F_{x,y}(t_1), F_{x,z}(t_2)\} \quad (\forall)x \in L, y, z \in M, t_1, t_2 \in \mathbb{R}_+.$$

So, the triangle inequality (14) is verified. Similarly, one proves that the triangle inequality (13) is verified.

Conversely, let $\mathcal{F}$ be a random 2-norm defined on $L \times M$. Since $F_{\alpha x,y}(t) = F_{x,y}(\frac{t}{|\alpha|})$ then

$$\min\left\{\frac{t}{f(\alpha x, y)}, 1\right\} = \min\left\{\frac{\frac{t}{|\alpha|}}{f(x, y)}, 1\right\} = \min\left\{\frac{t}{|\alpha| f(x, y)}, 1\right\}$$

By the arbitrariness of t it follows that $f(\alpha x, y) = |\alpha| f(x, y)$. Let us suppose that there exist $x \in L, y, z \in M$ such that $f(x, y + z) > f(x, y) + f(x, z)$. If we take $t_1 > f(x, y)$, $\quad t_2 > f(x, z)$ such as $f(x, y + z) > t_1 + t_2 > f(x, y) + f(x, z)$, we have $F_{x,y}(t_1) = F_{x,z}(t_2) = 1$ and $F_{x,y+z}(t_1 + t_2) < 1$, which is a contradiction with the triangle inequality (14). Therefore

$$f(x, y + z) \leq f(x, y) + f(x, z)$$

for every $x \in L, y, z \in M$. So, f is a generalized 2-norm. This completes the proof of the point (a) .

The second statement (b) follows by taking into account that the following inequalities are equivalent :

$$F_{x,a}(\varepsilon) > 1 - \varepsilon \Leftrightarrow \min\left\{\frac{\varepsilon}{f(x,a)}, 1\right\} > 1 - \varepsilon \Leftrightarrow f(x,a) < \frac{\varepsilon}{1-\varepsilon}.$$

□

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