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DIFFERENT STRUCTURES IN $Osc^k M$ AND ITS SUBSPACES

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Abstract. The theory of $Osc^k M$ was introduced by R. Miron and Gh. Atanasiu in [3], [4]. R. Miron in [5], [6] gave the comprehensive theory of higher order geometry and its application. In [1] the subspaces of Miron's $Osc^k M$ was introduced and in [2] special adapted basis was constructed. Using the above results we examine different structures in the subspaces of $Osc^k M$.[2]

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1. Special adapted basis in $T(Osc^k M)$ and $T^*(Osc^k M)$

Here $Osc^k M$ will be defined as a C^∞ manifold in which the transformations of the form (1.1) are allowed. It is formed as a tangent space of higher order of the base manifold M.

Let $E = Osc^k M$ be a $(k+1)n$-dimensional C^∞ manifold. In some local chart (U, φ) some point $u \in E$ has the coordinates

$$(x^a, y^{1a}, y^{2a}, \ldots, y^{ka}) = (y^{0a}, y^{1a}, y^{2a}, \ldots, y^{ka}) = (y^{Aa}),$$

where $x^a = y^{0a}$ and

$$a, b, c, d, e, \ldots = 1, 2, \ldots, n, \quad A, B, C, D, \ldots = 0, 1, 2, \ldots, k.$$

The following abbreviations:

$$\partial_{Aa} = \frac{\partial}{\partial y^{Aa}}, \quad A = 1, 2, \ldots, k, \quad \partial_a = \partial_{0a} = \frac{\partial}{\partial x^a} = \frac{\partial}{\partial y^{0a}}$$

will be used.

If in some other chart (U', φ') the point $u \in E$ has the coordinates $(x^{a'}, y^{1a'}, y^{2a'}, \ldots, y^{ka'})$, then in $U \cap U'$ the allowable coordinate transformations

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are given by:

$$
\begin{aligned}
(1)\qquad x^{a'} &= x^{a'}(x^1, x^2, \dots, x^n),\\
y^{1a'} &= (\partial_a x^{a'}) y^{1a} = (\partial_{0a} y^{0a'}) y^{1a},\\
y^{2a'} &= (\partial_{0a} y^{1a'}) y^{1a} + (\partial_{1a} y^{1a'}) y^{2a}, \dots,\\
y^{ka'} &= (\partial_{0a} y^{(k-1)a}) y^{1a} + (\partial_{1a} y^{(k-1)a}) y^{2a} + \dots + (\partial_{(k-1)a} y^{(k-1)a}) y^{ka}.
\end{aligned}
$$

The natural basis $\bar{B}$ of $T(E)$ is

$$\bar{B} = \{\partial_{0a}, \partial_{1a}, \dots, \partial_{ka}\}. \tag{2}$$

The natural basis $\bar{B}^*$ of $T^*(E)$ is

$$\bar{B}^* = \{dy^{0a}, dy^{1a}, \dots, dy^{ka}\}. \tag{3}$$

The special adapted basis

$$B^* = \{\delta y^{0a}, \delta y^{1a}, \delta y^{2a}, \dots, \delta y^{ka}\} \tag{4}$$

of $T^*(E)$ is given by [2]

$$
\begin{aligned}
(5)\quad \delta y^{0a} &= dx^a = dy^{0a}\\
\delta y^{1a} &= \binom{1}{1} dy^{1a} + \binom{1}{0} M^{1a}_{0b} dy^{0b},\\
\delta y^{2a} &= \binom{2}{2} dy^{2a} + \binom{2}{1} M^{1a}_{0b} dy^{1b} + \binom{2}{0} M^{2a}_{0b} dy^{0b},\\
\delta y^{3a} &= \binom{3}{3} dy^{3a} + \binom{3}{2} M^{1a}_{0b} dy^{2b} + \binom{3}{1} M^{2a}_{0b} dy^{1b} + \binom{3}{0} M^{3a}_{0b} dy^{0b},\\
\delta y^{4a} &= \binom{4}{4} dy^{4a} + \binom{4}{3} M^{1a}_{0b} dy^{3b}\\
&+ \binom{4}{2} M^{2a}_{0b} dy^{2b} + \binom{4}{1} M^{3a}_{0b} dy^{1b} + \binom{4}{0} M^{4a}_{0b} dy^{0b}, \dots,\\
\delta y^{ka} &= \binom{k}{k} dy^{ka} + \binom{k}{k-1} M^{1a}_{0b} dy^{(k-1)b} +\\
&+ \binom{k}{k-2} M^{(k-2)a}_{0b} dy^{(k-2)b} + \dots + \binom{k}{0} M^{ka}_{0b} dy^{0b}.
\end{aligned}
$$

$\{\delta y^{0a}, \delta y^{1a}, \dots, \delta y^{ka}\}$ form the adapted basis B^* of $T^*(E)$. In the special adapted bases B^* (1.5) and B (1.7) the J structure has a simpler form.

Theorem 1.1. *The necessary and sufficient conditions that δy^{Aa} are transformed as d-tensor field, i.e.*

$$\delta y^{Aa'} = \frac{\partial x^{a'}}{\partial x^a} \delta y^{Aa}, \quad A = 0, 1, \dots, k$$

are the following equations

$$(6)\quad \begin{aligned}
M^{1a}_{0b}\partial_{0a}y^{0b'} &= \binom{1}{0}M^{1b'}_{0c'}\partial_{0b}y^{0c'} + \partial_{0b}y^{1b'},\\
M^{2a}_{0b}\partial_{0a}y^{0b'} &= \binom{2}{0}M^{2b'}_{0c'}\partial_{0b}y^{0c'} + \binom{2}{1}M^{1b'}_{0c'}\partial_{0b}y^{1c'} + \binom{2}{2}\partial_{0b}y^{2b'},\\
M^{3a}_{0b}\partial_{0a}y^{0b'} &= \binom{3}{0}M^{3b'}_{0c'}\partial_{0b}y^{0c'} + \binom{3}{1}M^{2b'}_{0c'}\partial_{0b}y^{1c'} +\\
&+ \binom{3}{2}M^{1b'}_{0c'}\partial_{0b}y^{2c'} + \binom{3}{3}\partial_{0b}y^{3b'}, \dots,\\
M^{ka}_{0b}\partial_{0a}y^{0b'} &= \binom{k}{0}M^{kb'}_{0c'}\partial_{0b}y^{0c'} + \binom{k}{1}M^{(k-1)b'}_{0c'}\partial_{0b}y^{1c'}\\
&+ \binom{k}{2}M^{(k-2)b'}_{0c'}\partial_{0b}y^{2c'} + \dots +\\
&+ \binom{k}{k-1}M^{1b'}_{0c'}\partial_{0b}y^{(k-1)c'} + \binom{k}{k}\partial_{0b}y^{kb'}.
\end{aligned}$$

The special adapted basis $B = \{\delta_{0a}, \delta_{1a}, \dots, \delta_{ka}\}$ of $T(E)$ is given by [2]

$$(7)\quad \begin{aligned}
\delta_{0a} &= \binom{0}{0}\partial_{0a} - \binom{1}{0}N^{1b}_{0a}\partial_{1b} - \binom{2}{0}N^{2b}_{0a}\partial_{2b} -\\
&- \binom{3}{0}N^{3b}_{0a}\partial_{3b} - \dots - \binom{k}{0}N^{kb}_{0a}\partial_{kb},
\end{aligned}$$

$$(8)\quad \begin{aligned}
\delta_{1a} &= \binom{1}{1}\partial_{1a} - \binom{2}{1}N^{1b}_{0a}\partial_{2b} -\\
&- \binom{3}{1}N^{2b}_{0a}\partial_{3b} - \dots - \binom{k}{1}N^{(k-1)b}_{0a}\partial_{kb}, \dots,
\end{aligned}$$

$$(9)\quad \delta_{ka} = \binom{k}{k}\partial_{ka}.$$

Theorem 1.2. *The elements of the natural basis of $T(E) : \{\partial_{0a}, \partial_{1a}, \dots, \partial_{ka}\}$ and special adapted basis $B : \{\delta_{0a}, \delta_{1a}, \dots, \delta_{ka}\}$ of $T(E)$ and the coefficients M^{Bb}_{0a} of B^* are connected by*

$$(10)\quad \begin{array}{rclllll}
\partial_{0a} &=& \delta_{0a} + & M^{1b}_{0a}\delta_{1b} + & M^{2b}_{0a}\delta_{2b} & + \dots + & M^{kb}_{0a}\delta_{kb},\\
\partial_{1a} &=& & \delta_{1a} + & \binom{2}{1}M^{1b}_{0a}\delta_{2b} & + \dots + & \binom{k}{1}M^{(k-1)b}_{0a}\delta_{kb},\\
\partial_{2a} &=& & & \delta_{2a} & + \dots + & \binom{k}{2}M^{(k-2)b}_{0a}\delta_{kb},\\
\vdots & & & & & & \\
\partial_{ka} &=& & & & & \delta_{ka}.
\end{array}$$

There are also the conditions when the special adapted bases B and B^* are dual to each other, further when the elements of B and B^* are transforming as tensors.

The proof of Theorems 1.1 and 1.2 can be found in [2].

2. The J structure

Definition 2.1. *The k-tangent structure J is an $\mathcal{F}(E)$-linear mapping*

$$J : \chi(E) \to \chi(E)$$

defined by

(1) $J\partial_{0i} = \partial_{1i},\ J\partial_{1i} = 2\partial_{2i}, \ldots, J\partial_{\alpha i} = (\alpha+1)\partial_{(\alpha+1)i}, \cdots,\ J\partial_{(k-1)i} = k\partial_{ki},$

$$J\partial_{ki} = 0.$$

Its representation in the basis $\bar{B} = \{\partial_{0i}, \partial_{1i}, \ldots, \partial_{ki}\}$ is

$$J = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 2 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & k & 0 \end{bmatrix} \tag{2}$$

The k-structure J determined by Definition 2.1 is the same as J used in [5], [6], but there it is represented in different basis of the tangent space.

For the k-tangent structure J the relation

$$J^{k+1} = 0 \tag{3}$$

is valid. In the natural bases $\bar{B}$ and $\bar{B}^*$ of $T(E)$ and $T^*(E)$ it can be written in the form

$$J = [\partial_{0a}\partial_{1a}\ldots\partial_{ka}] \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & & 0 & 0 \\ 0 & 2 & 0 & & 0 & 0 \\ 0 & 0 & 3 & & 0 & 0 \\ \vdots & & \vdots & & k & 0 \end{bmatrix} \otimes \begin{bmatrix} dy^{0a} \\ dy^{1a} \\ dy^{2a} \\ \vdots \\ dy^{ka} \end{bmatrix} = \tag{4}$$

$$\partial_{1a} \otimes dy^{0a} + 2\partial_{2a} \otimes dy^{1a} + 3\partial_{3a} \otimes dy^{2a} + \cdots + k\partial_{ka} \otimes dy^{(k-1)a}$$

Theorem 2.1. *The k-tangent structure J defined by Definition (2.1) the elements of the basis $B = \{\delta_{0a}, \delta_{1a}, \ldots, \delta_{ka}\}$ determined by (1.7) transform in the following way*

$$J\delta_{0a} = \delta_{1a},\ J\delta_{1a} = 2\delta_{2a},\ J\delta_{Aa} = (A+1)\delta_{(A+1)a}, \ldots \tag{5}$$

$$J\delta_{(k-1)a} = k\partial_{ka}, J\delta_{ka} = 0.$$

Theorem 2.2. *The k-tangent structure J given by (2.1) satisfies the relations*

(6) $$dy^{0b}J = 0, dy^{1b}J = dy^{0b}, dy^{2b}J = 2dy^{1b}, \dots, dy^{kb}J = kdy^{(k-1)b}.$$

Theorem 2.3. *For the k-tangent structure J given by (2.1) we have*

(7) $$\delta y^{0b}J = 0, \delta y^{1b}J = \delta y^{0b}, \delta y^{2b}J = 2\delta y^{1b}, \dots, \delta y^{kb}J = k\delta y^{(k-1)b}$$

where $\{\delta y^{0b}, \delta y^{1b}, \dots, \delta y^{kb}\}$ is the special adapted basis B^* of $T(E)$ determined by (1.5).

Theorem 2.4. *The structure J in the adapted basis $B = \{\delta_{0a}, \delta_{1a}, \dots, \delta_{ka}\}$ and*
$B^* = \{\delta y^{0a}, \delta y^{1a}, \dots, \delta y^{ka}\}$ is given by

(8) $$J = \delta_{1a} \otimes \delta y^{0a} + 2\delta_{2a} \otimes \delta y^{1a} + 3\delta_{3a} \otimes \delta y^{2a} + \dots k\delta_{ka} \otimes \delta y^{(k-1)a}.$$

The proof of Theorems 2.1-2.4 can be found in [2].

3. $f(2t+1,-1)$-structure in Osc^kM and the structure on the hypersurface

In the special adapted basis $B = \{\delta_{0a}, \delta_{1a}, \dots, \delta_{ka}\}$ of $T(E)$, the vectors $\{\delta_{0a}\}$ span the n-dimensional space $T_H(E)$, and the vectors $\{\delta_{1a}, \delta_{2a}, \dots, \delta_{ka}\}$ the $k \cdot n$-dimensional $T_V(E)$ and,

$$T(E) = T_H(E) + T_V(E).$$

On the space $T(E) \otimes T(E)$, a metric tensor G is defined such that $T(E)$ can be decomposed into two orthogonal parts $T_H(E)$ and $T_V(E)$, where

$$G = g_{0a\ 0b}\delta y^{0a} \otimes \delta y^{0b} + g_{Aa\ Bb}\delta y^{Aa} \otimes \delta y^{Bb},\ A = 1, 2, \dots, k.$$

Definition 3.1. *Let $E = Osc^kM$ be an $m = (k+1)n$-dimensional differentiable manifold of class C^∞, and let there be given a tensor field $f \neq 0$ of the type (1,1) and of class C^∞ such that*

(1) $$f^{2t+1} - f = 0,\ \ f^{2i+1} - f \neq 0 \ \textit{for}\ 1 \leq i < t,$$

where t is a fixed integer greater than 1. Let rank $f = r$ be constant. We call such a structure an $f(2t+1,-1)$-structure or an f-structure of the rank r and of degree $2t+1$.

Theorem 3.1. *For a tensor field $f, f \neq 0$ satisfying (2.1), the operators*

(2) $$\mathbf{m} = I - f^{2t},\ \ \mathbf{l} = f^{2t}$$

are the complementary projection operators where I denotes the identity operator applied to the tangent space at a point of the manifold.

Proof. We have

$$\mathbf{l}+\mathbf{m}=I,\ \mathbf{l}^2=\mathbf{l},\ \mathbf{m}^2=\mathbf{m},\ \mathbf{ml}=\mathbf{lm}=0$$

by virtue of (3.1), which proves the theorem. □

Let L and M be the complementary distributions corresponding to the operators $\mathbf{l}$ and $\mathbf{m}$, respectively. If rank $f=r$ is constant and dim $L=r$, then dim $M=m-r$.

Theorem 3.2. *For f satisfying (3.1) and $\mathbf{l},\mathbf{m}$, defined by (3.2), we have*

$$\begin{aligned}&(a)\ \mathbf{l}f=f\mathbf{l}=f,\\&(b)\ \mathbf{m}f=f\mathbf{m}=0,\\&(c)\ f^{2t}\mathbf{m}=0,\\&(d)\ (\mathbf{m}+f^t)^2=I.\end{aligned}\tag{3}$$

Theorem 3.3. *Suppose that there is a projection operator $\mathbf{m}$ on E and that there exists a tensor field f such that (3.3b) and (3.3d) are satisfied, then f satisfies (3.1).*

Proposition 3.1. *Let an f-structure of the rank r and degree $2t+1$ be given on E, then $f^{2t}\mathbf{l}=\mathbf{l}$ and $f^{2t}\mathbf{m}=0$, i.e. f^t acts on L as an almost product structure operator and on M as a null operator.*

We shall assume that E is a Osc^kM space of dimension $m=(k+1)n$, and that rank $f=r=k\cdot n$. Then $\dim L=k\cdot n, \dim M=n$ and $M=T_H(E), L=T_V(E)$.

If we denote by h the projection morphism of $T(E)$ to $T_H(E)$, we can construct the mapping α which is defined in [10] by

$$\alpha(X,Y)=\frac{1}{2}[\overline{h}(\mathbf{l}X,\mathbf{l}Y)]+\overline{h}(\mathbf{m}X,\mathbf{m}Y)],\quad \forall X,Y\in T(E),$$

where $\overline{h}=Gh$, is a pseudo-Riemannian structure on $T(E)$, such that $\alpha(X,Y)=0, \forall X\in M$,
$Y\in L$.

If we put $g(X,Y)=\frac{1}{2t}[\alpha(X,Y)+\alpha(fX,fY)+\cdots+\alpha(f^{2t-1}X,f^{2t-1}Y)]$, it is easy to see that $g(X,Y)=0,\forall X\in M, Y\in L$.

Also, using (3.2) and Theorem 3.2 we get $g(fX,fY)=\frac{1}{2t}[\alpha(fX,fY)+\alpha(f^2X,f^2Y)+\cdots+\alpha(X,Y)]=g(X,Y)$. Thus f is an isometry with respect to g.

In [9] and [10] an adapted frame form $f(2t+1,-1)$-structure is chosen and matrices of tensors g_{ij} and f_i^j are given with respect to this adapted frame. According to the results in [9] and [10] we have for the $E=Osc^kM$ the following theorem

Theorem 3.4. *A necessary and sufficient condition for a space E of dimension $(k+1)n$ to admit a tensor field $f\neq 0$ of type (1,1) and of rank $k\cdot n$, such that $f^{2\cdot 2^k+1}-f=0$, is that* *i)* $r = k \cdot 2p = k \cdot n,$ *ii)* $2p = s \cdot 2^k = s \cdot t,\ s \in N,\ t = 2^k,$

iii) the group of the tangent bundle of the manifold be reduced to the group

$$\overline{S}_{(\frac{2p}{2^k})} \times \overline{S}_{(\frac{2p}{2^{k-1}})} \times \ldots \times \overline{S}_{(\frac{2p}{4})} \times U_p \times O_{2p} \times O_{m-r}.$$

Theorem 3.5. *Denote $-f^k$ by ϕ. The structure ϕ satisfies the condition $\phi^3 - \phi = 0$, i.e. ϕ is an $\phi(3,-1)$-structure.*

Theorem 3.6. *The structure ϕ is an almost paracontact Riemannian structure if rank $f = m - 1$.*

Proof. Let

$$\mathbf{m} = I - f^{2k} = \begin{bmatrix} 0 & 0 & & & \\ 0 & 0 & & & \\ & & \ddots & & \\ & & 0 & 0 & \\ & & 0 & 0 & \\ & & & & 1 \end{bmatrix}, \quad \xi = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \mu = (0, 0, \ldots, 0, 1)$$

M is an 1-dimensional distribution.

Multiplying the corresponding matrices, it is clear that $\mathbf{m} = \xi \otimes \mu$,

$$\phi^2 = I - \mathbf{m} = I - \xi \otimes \mu, \quad \phi\xi = 0, \quad \mu\phi = 0, \quad \mu(\xi) = 1, \quad \mu(X) = g(\xi, X)$$

and $g(\phi X, \phi Y) = g(X, Y) - \mu(X) \cdot \mu(Y)$, which prove the Theorem. □

Theorem 3.7. *Let E be a Osc^kM manifold with $\phi(3,-1)$-structure of rank $r = n \cdot k$ and let $\mathcal{N}^{m-1}$ be a hypersurface in E. If the dimension of $T(\mathcal{N}^{m-1})_u \cap f(T(\mathcal{N}^{m-1}))_u$ is constant, say s, for all $u \in \mathcal{N}^{m-1}$, then $\mathcal{N}^{m-1}$ possesses a natural $F(3,-1)$-structure of rank s.*

Proof. Let C be a transversal defined on $\mathcal{N}^{m-1}$, i.e. $C \in T(E)_u$ but $C \notin T(\mathcal{N}^{m-1})_u$ for all $u \in \mathcal{N}^{m-1}$. Let B be a differential of the imbedding of $\mathcal{N}^{m-1}$ in E. Then B is a map of $T(\mathcal{N}^{m-1})$ into $T_R(E)$, where $T_R(E)$ denotes the restriction of $T(E)$, the tangent bundle of E to $\mathcal{N}^{m-1}$. Then we can find a locally 1-form C^* defined on $\mathcal{N}^{m-1}$ such that:

$$B^{-1}B = I, \quad BB^{-1} = I - C^* \otimes C, \quad C^*B = B^{-1}C = 0, \quad C^*(C) = 1.$$

Let F be defined locally on $T(\mathcal{N}^{m-1})$ by $F = B^{-1}\phi B$. Then:

$$\begin{aligned} F^2X &= B^{-1}\phi BB^{-1}\phi BX = B^{-1}\phi(I - C^* \otimes C)\phi(BX) \\ &= B^{-1}\phi^2(BX) - C^*\phi(BX)B^{-1}\phi C. \end{aligned}$$

If C is in distribution E, then $\phi C = 0$; so we have that

$$\begin{aligned}(F^3 - F)X &= B^{-1}\phi B B^{-1}\phi^2 BX - B^{-1}\phi BX \\ &= B^{-1}\phi(I - C^* \otimes C)\phi^2 BX - B^{-1}\phi BX \\ &= B^{-1}((\phi^3 - \phi)BX) = 0\end{aligned}$$

for all X. On the other hand, suppose that C is in distribution L. Then:

$$\begin{aligned}(F^3 - F)X &= (B^{-1}\phi B)B^{-1}\phi^2(BX) \\ &- (B^{-1}\phi b)C^*(\phi BX)B^{-1}\phi C - B^{-1}\phi BX \\ &= B^{-1}(\phi^3 - \phi)BX - C^*(\phi^2 BX)B^{-1}\phi C \\ &- C^*(\phi BX)B^{-1}\phi^2 C + C^*(\phi BX)C^*(\phi C)B^{-1}\phi C = 0\end{aligned}$$

since $\phi^2 C = C$ on L and $C^* B = B^{-1}C = 0$, and since we can choose C^* so that $C^*(\phi C) = 0$. Also $C^*(\phi^2 BX) = C^*(BX + (\phi^2 - 1)BX) = 0$. □

Theorem 3.8. *If (ϕ, ξ, μ) is an almost paracontact structure on E, then $\mathcal{N}^{m-1}$ possesses a natural $F(3, -1)$-structure if ξ is tangent to $\mathcal{N}^{m-1}$. The hypersurface $\mathcal{N}^{m-1}$ possesses a natural almost product structure if ξ is not tangent to $\mathcal{N}^{m-1}$.*

Proof. When ξ is not tangent to $\mathcal{N}^{m-1}$, ξ can be chosen for a pseudonormal. Then we have from Theorem 3.7 that $T(\mathcal{N}^{m-1}) \cap f(T(\mathcal{N}^{m-1})) = T(\mathcal{N}^{m-1})$, and rank $F = \dim \mathcal{N}^{m-1} = m - 1$. The almost paracontact structure F has a maximal rank, i.e. F is an almost product structure. □

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