Novi Sad J. Math.
Vol. 35, No. 1, 2005, 123-125

METRIC LOCALLY CONSTANT FUNCTION ON SOME SUBSET OF ULTRAMETRIC SPACE

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Abstract. We prove some sufficient condition for the mapping $T : X \to X$, (X, d) being an ultrametric space, such that there exists a ball $B \subseteq X$, $T : B \to B$, with property that function $f(x) = d(x, Tx)$, $x \in B$, is metric locally constant.

AMS Mathematics Subject Classification (2000): 11S99.

Key words and phrases: ultrametric space, metric locally constant function, contraction.

1. Introduction

Let (X, d) be a metric space. If the metric d satisfies strong triangle inequality: for all $x, y, z \in X$

$$d(x, y) \leq max\{d(x, z), d(z, y)\}$$

it is called **ultrametric** on X [4].

The pair (X, d) is now an **ultrametric space.**

Remark. Let $X \neq \emptyset$, metric d defined on X by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y, \end{cases}$$

so called discrete metric is ultrametric. This example looks to be trivial but it is very important in theory of ultrametric spaces since topological space is ultrametrizable if and only if it is homeomorphic to a subspace of countable product of discrete spaces.

For some other examples see [4].

Let d be an ultrametric on X. The closed ball on $a \in X$ with a radius $\varepsilon > 0$ is the set

$$B(a; \varepsilon) = \{x \in X : d(x, a) \leq \varepsilon\}.$$

It is easy to verify the next properties.

Proposition. *Let d be an ultrametric on X.*

1) *If $a, b \in X$, $\varepsilon > 0$, and $b \in B(a; \varepsilon)$, than $B(a; \varepsilon) = B(b; \varepsilon)$;*
2) *If $a, b \in X$, $0 < \delta \leq \varepsilon$, then either $B(a; \varepsilon) \cap B(b; \delta) = \emptyset$ or $B(b; \delta) \subseteq B(a; \varepsilon)$. Hence, if a ball $B(a; \varepsilon)$ contains a ball $B(b; \delta)$, then either the balls are the same or $\delta < \varepsilon$; (This is not true for every metric space!)*
3) *Every ball is clopen (closed and open) in the topology defined by d. (Every ultrametrizable topology is zero–dimensional).*

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We denote by F_X the set of maps $f : X \to [0, +\infty)$.

Definition. *A function $f \in F_X$ is said to be* **metric locally constant** *(m.l.c. for short) provided that for any $x \in X$ and any y in the open ball $B(x; f(x))$ one has $f(x) = f(y)$.*

Remark. The notion of metric locally constant function has been introduced in [1] in order to study certain groups of isometries on a given ultrametric space. In particular, various Galois groups over local fields can be described in this way. For more information about properties of m.l.c. function see [2].

For instance, it is easy to see that for any ultrametric space X and any isometry $h : X \to X$, that the function $f : X \to [0, +\infty)$ given by $f(x) = d(x, h(x))$, $x \in X$, is m.l.c. function [2]. But this assertion is not true for contractive mappings.

As we know, the mapping $T : X \to X$ is contraction (1–Lipschitzian) provided that for any $x, y \in X$

$$d(T(x), T(y)) \leq d(x, y).$$

2. Result

Theorem. *Let (X, d) be a spherically complete ultrametric space and $T : X \to X$ contractive mapping. Then there exists subset $B \subseteq X$ such that $T : B \to B$ and that the function $f(x) = d(x, Tx)$, $x \in B$, is m.l.c..*

Proof. Let $B_a = B(a; d(a, Ta))$ denote the closed ball centered at a, with the radius $d(a, Ta)$ and let $\mathcal{A}$ be the collection of these spheres for any $a \in X$.

The relation

$$B_a \leq B_b \quad \text{iff} \quad B_b \subseteq B_a$$

is a partial order on $\mathcal{A}$.

Let $\mathcal{A}_1$ be a totally ordered subfamily of $\mathcal{A}$. Since (X, d) is spherically complete,

$$\bigcap_{B_a \in \mathcal{A}_1} B_a = B \neq \emptyset.$$

Let $b \in B$ and $B_a \in \mathcal{A}_1$.

Obviously $b \in B_a$ so $d(b, a) \leq d(a, Ta)$.

For any $x \in B_b$

$$d(x, a) \leq max\{d(x, b), d(b, a)\} \leq d(a, Ta)$$

and

$$d(x, b) \leq d(b, Tb) \leq max\{d(b, a), d(a, Ta), d(Ta, Tb)\} = d(a, Ta).$$

So $B_b \subseteq B_a$ for every $B_a \in \mathcal{A}_1$ and B_b is an upper bound in $\mathcal{A}$ for the family $\mathcal{A}_1$. By Zorn's lemma there is a maximal element in $\mathcal{A}_1$, say B_z.

For any $b \in B_z$

$$\begin{aligned} d(b, Tb) &\leq max\{d(b, z), d(z, Tz), d(Tz, Tb)\} \\ &\leq max\{d(b, z), d(z, Tz), d(z, b)\} = d(z, Tz), \end{aligned}$$

$B_b \cap B_z$ is nonempty (contains b) so by Proposition 1.(2)

$$B_b \subseteq B_z.$$

Since $Tb \in B_b$ we just prove that $T : B_z \to B_z$.

For $z = Tz \quad f(x) = 0$ so theorem is proved.

For $z \neq Tz$ we are going to prove that $f(b) = f(z)$ for every $b \in B_z$.

We know that $d(b, Tb) \leq d(z, Tz)$ for any $b \in B_z$. Let us suppose that for some $b \in B_z$

$$d(b, Tb) < d(z, Tz).$$

As

$$d(b, z) \leq d(z, Tz) \quad \text{and}$$

$$\begin{aligned} d(z, Tz) &\leq max\{d(z, b), d(b, Tz)\} \\ &\leq max\{d(z, b), d(b, Tb), d(Tb, Tz)\} \\ &\leq max\{d(z, b), d(b, Tb), d(b, z)\} \\ &= max\{d(z, b), d(b, Tb)\} \\ &= d(z, b) \end{aligned}$$

we obtain that $d(z, Tz) = d(b, z)$.

But

$$d(b, z) = d(z, Tz) > d(b, Tb)$$

implies that $z \in B_z$ but $z \notin B_b$ and hence

$$B_b \subsetneqq B_z$$

which contradicts the maximality of B_z.

Thus we proved that f is m.l.c. on $B = B_z$.

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Received by the editors February 2, 2005