Novi Sad J. Math.
Vol. 35, No. 1, 2005, 127-135

On Analytic Integrated Semigroups

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Abstract. The known definition of an analytic n-times integrated semigroup is reconsidered and one superfluous condition is removed. It is proved that every densely defined generator of an exponentially bounded, analytic n-times integrated semigroup of angle α with the appropriate growth rate at zero is also the generator of an analytic C_0-semigroup of the same angle.

AMS Mathematics Subject Classification (2000): 47D03, 47D99, 47D62.

Key words and phrases: analytic integrated semigroups, analytic C_0-semigroups, distribution semigroups, semigroups strongly continuous for $t > 0$.

1. Introduction

We analyze the definition of an analytic integrated semigroup given in [3]. With the notation explained in Sections 2 and 3, we will prove the following.

Theorem 1.1. *Let A be a densely defined operator and $\alpha \in (0, \frac{\pi}{2}]$. Then the following assertions are equivalent.*

(a) *A is the generator of an exponentially bounded, analytic n-times integrated semigroup $(S_n(t))_{t\geq 0}$ of angle α with $||S_n(z)|| = O(|z|^n),\ z \to 0,\ z \in \Sigma_\gamma$, for all $\gamma \in (0,\ \alpha)$ and some (all) $n \in \mathbb{N}$.*

(b) *A is the generator of an analytic C_0-semigroup of angle α.*

Theorem 1.2. *Let A be a closed linear operator and $0 < \alpha \leq \frac{\pi}{2},\ n \in \mathbb{N}$. Then the following assertions are equivalent.*

(a') *A is the generator of an exponentially bounded, analytic n-times integrated semigroup $(S_n(t))_{t\geq 0}$ of angle α.*

(b') *For all $\gamma \in (0,\ \alpha)$, there exist $C_\gamma > 0$ and $\omega_\gamma > 0$ so that $\omega_\gamma + \Sigma_{\frac{\pi}{2}+\gamma} \subset \rho(A)$ and that the following holds:*

$$||R(\lambda : A)|| \leq C_\gamma(1 + |\lambda|)^{n-1},\ \lambda \in \omega_\gamma + \Sigma_{\frac{\pi}{2}+\gamma},\ \text{and} \tag{1}$$

$$\lim_{\lambda\to+\infty} \frac{R(\lambda : A)x}{\lambda^{n-1}} = 0,\ x \in E. \tag{2}$$

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2. Preliminaries

We use the standard terminology: E denotes a non-trivial complex Banach space and $L(E)$ denotes the space of all bounded linear operators from E into E. For a linear operator A, its domain, range and null space are denoted by $D(A)$, $Im(A)$ and $Kern(A)$, respectively. We will always assume that A is a closed operator.

We refer to [1] and [2] for the material related to integrated semigroups.

Schwartz space of test functions on the real line $\mathbb{R}$ is denoted by $\mathcal{D} = C_0^\infty$. Its dual $\mathcal{D}'$ is equipped with the strong topology; we denote by $\mathcal{D}_0$ the subspace of $\mathcal{D}$ which consists of the elements supported by $[0, \infty)$. Further on, $\mathcal{D}'(L(E)) = L(\mathcal{D}, L(E))$ is the space of continuous linear functions from $\mathcal{D}$ into $L(E)$, equipped with the topology of uniform convergence on bounded subsets of $\mathcal{D}$; $\mathcal{D}'_0(L(E))$ is the subspace of $\mathcal{D}'(L(E))$ containing the elements supported by $[0, \infty)$. A distribution $\delta_t \in \mathcal{D}'$, $t \in \mathbb{R}$, is defined by $\langle \delta_t,\ \varphi\rangle := \varphi(t),\ \varphi \in \mathcal{D}$.

Following [8] (cf. also [11] and [6]), a distribution semigroup G is an element $G \in \mathcal{D}'_0(L(E))$ with the properties

$$G(\varphi *_0 \psi) = G(\varphi)G(\psi),\ \varphi,\ \psi \in \mathcal{D},\ \text{and } \mathcal{N}(G) := \bigcap_{\varphi \in \mathcal{D}_0} KernG(\varphi) = \{0\},$$

where $*_0$ is the convolution $f *_0 g(t) := \int\limits_0^t f(t-s)g(s)ds,\ t \in \mathbb{R}$. We denote such a semigroup as (DSG). The infinitesimal generator of G is defined by $A = \{(x, y) \in E^2 : G(-\varphi')x = G(\varphi)y,\ \varphi \in \mathcal{D}_0\}$; it is a closed linear operator in E satisfying $G(\varphi)A \subset AG(\varphi)$, $\varphi \in \mathcal{D}$ and $\{G(\varphi)x : x \in E,\ \varphi \in \mathcal{D}\} \subset D(A)$.

Following [2, Definition 3.2.5], a strongly continuous operator family $(T(t))_{t>0}$ is called a semigroup if the following conditions are satisfied:

(i) $T(t+s) = T(t)T(s),\ t,\ s > 0$,

(ii) $T(t)x = 0$, for all $t > 0$, implies $x = 0$,

(iii) $\sup_{t \in (0,1]} ||T(t)|| < \infty$.

Obviously, if $(T(t))_{t>0}$ is a semigroup, then there exist constants $M > 0$ and $\omega > 0$ such that $||T(t)|| \le Me^{\omega t},\ t > 0$. Define $S_1(t) := \int\limits_0^t T(s)ds,\ t \ge 0$. Then $(S_1(t))_{t\ge 0}$ is an exponentially bounded once integrated semigroup. A closed linear operator A is said to be the generator of $(T(t))_{t>0}$ if and only if A is the generator of $(S_1(t))_{t\ge 0}$, or equivalently, if there exists $\omega_1 \ge \omega$ such that

$$R(\lambda : A) = \lambda \int\limits_0^\infty e^{-\lambda t} S_1(t)dt,\ \lambda > \omega_1.$$

3. Analytic integrated semigroups

We refer to [2] for the material closely related to analytic semigroups. Especially, we need [2, Corollary 3.9.9]: *Let $\gamma \in (0, \frac{\pi}{2})$. Suppose that $e^{\pm i\gamma}A$ generate C_0-semigroups. Then A generates an analytic C_0-semigroup of angle γ.*

Let $\alpha \in (0, \pi]$. We will use the notation $\Sigma_\alpha := \{z \in \mathbb{C} : z \neq 0,\ |\arg z| < \alpha\}$.

Let $K \in L^1_{loc}([0,\infty))$ be an exponentially bounded function. Analytic convoluted K-semigroups as well as their relations with convoluted cosine functions, ultradistribution and hyperfunction sines are investigated in [7]. Here is the definition with $K(t) = \frac{t^{n-1}}{(n-1)!}$, $n \in \mathbb{N}$; actually, we reformulate [4, Definition 21.2] (cf. also [3]) given by deLaubenfels so that one of his conditions is neglected.

Definition 3.1. Let $0 < \alpha \leq \frac{\pi}{2}$, $n \in \mathbb{N}$, and let $(S_n(t))_{t\geq 0}$ be an exponentially bounded n-times integrated semigroup. Then $(S_n(t))_{t\geq 0}$ is an analytic n-times integrated semigroup of angle α, if there exists an analytic function $\mathbf{S}_n : \Sigma_\alpha \to L(E)$ which satisfies:

(i) $\mathbf{S}_n(t) = S_n(t),\ t > 0$, and

(ii) $\lim_{z\to 0,\ z\in\Sigma_\gamma} \mathbf{S}_n(z)x = 0$, for all $\gamma \in (0,\ \alpha)$ and $x \in E$.

$(S_n(t))_{t\geq 0}$ is an exponentially bounded, analytic n-times integrated semigroup of angle α, if for all $\gamma \in (0,\ \alpha)$ there exist constants $M_\gamma > 0$ and $\omega_\gamma > 0$ such that $||\mathbf{S}_n(z)|| \leq M_\gamma e^{\omega_\gamma Re z},\ z \in \Sigma_\gamma$.

It is clear that an analytic n-times integrated semigroup $(S_n(t))_{t\geq 0}$ of angle α is exponentially bounded if and only if for all $\gamma \in (0,\ \alpha)$, there exist appropriate constants $M_\gamma > 0$ and $\omega_\gamma > 0$ so that $||\mathbf{S}_n(z)|| \leq M_\gamma e^{\omega_\gamma |z|},\ z \in \Sigma_\gamma$.

From now on, we shall denote $\mathbf{S}_n$ by S_n since it will not cause any confusion. Put $T(z) := \frac{d^n}{dz^n} S_n(z),\ z \in \Sigma_\alpha$.

The definition of an analytic n-times integrated semigroup $(S_n(t))_{t\geq 0}$ of angle α in [4] contains, in addition, the next condition: $T(z_1+z_2) = T(z_1)T(z_2),\ z_1,\ z_2 \in \Sigma_\alpha$. We prove in the following proposition that this condition automatically holds.

Proposition 3.2. *Let $(S_n(t))_{t\geq 0}$ be an analytic n-times integrated semigroup of angle α. Then $(T(z))_{z\in\Sigma_\alpha}$ is an analytic operator family with:*

1. $T(z_1 + z_2) = T(z_1)T(z_2),\ z_1,\ z_2 \in \Sigma_\alpha$,
2. $T(t)x = 0$, for all $t > 0$, *implies* $x = 0$, *and*
3. $\lim_{z\to 0,\ z\in\Sigma_\gamma} T(z)x = x,\ x \in D(A^n)$.

Proof. Let A be the generator of $(S_n(t))_{t\geq 0}$ and let $t_1 > 0$ and $t_2 > 0$ be fixed. Put

$$G(\varphi)x = (-1)^n \int_0^\infty \varphi^{(n)}(t) S_n(t)x dt,\ \varphi \in \mathcal{D},\ x \in E.$$

Then G is a (DSG) generated by A (cf. [6]). Integration by parts gives that for every $\varphi \in \mathcal{D}$ with $\operatorname{supp}\varphi \subset (a,\infty)$, $0 < a < \min(t_1,t_2)$, one has

$$G(\varphi)x = \int\limits_0^\infty \varphi(t)T(t)x dt,\ x \in E.$$

Put $\varphi_k(t) = \varphi(k(t-t_1)),\ \psi_k(t) = \varphi(k(t-t_2)),\ t \in \mathbb{R},\ k \in \mathbb{N}$, where φ is an arbitrary nonnegative test function satisfying $\int\limits_{\mathbb{R}} \varphi(t)dt = 1$ and $\operatorname{supp}\varphi \subset [0,1]$. Then (φ_k) and (ψ_k) are sequences in $\{\varphi \in \mathcal{D} : \operatorname{supp}\varphi \subset (a,\infty)\}$ which converge in distributional sense to δ_{t_1} and δ_{t_2}, respectively. Consequently, $(\varphi_k * \delta_{t_2})$ is a sequence in $\{\varphi \in \mathcal{D} : \operatorname{supp}\varphi \subset (a,\infty)\}$ which converges in distributional sense to $\delta_{t_1+t_2}$, and

$$\begin{aligned} T(t_1)T(t_2)x &= \lim_{k\to\infty} G(\varphi_k)T(t_2)x = \lim_{k\to\infty} G(\varphi_k)\lim_{j\to\infty} G(\psi_j)x \\ &= \lim_{k\to\infty}\lim_{j\to\infty} G(\varphi_k)G(\psi_j)x = \lim_{k\to\infty}\lim_{j\to\infty} G(\varphi_k *_0 \psi_j)x \\ &= \lim_{k\to\infty} G(\varphi_k * \delta_{t_2})x = T(t_1+t_2)x,\ x \in E. \end{aligned}$$

It extends to all z_1 and z_2 belonging Σ_α by the uniqueness theorem for analytic functions. It proves 1. The assumption $T(t)x = 0$, for all $t > 0$, implies that for every $\varphi \in \mathcal{D}$ with $\operatorname{supp}\varphi \subset (0,\infty)$, the following holds $G(\varphi)x = 0$. Since the translation is a continuous linear mapping from $\mathcal{D}$ into $\mathcal{D}$, the previous remains true for all $\varphi \in \mathcal{D}_0$. It implies $x \in \mathcal{N}(G)$ and $x = 0$. Thus 2. is proved.
Let us prove 3. Let $x \in D(A^n)$ be fixed. Then it is well known that

$$\frac{d^n}{dt^n} S_n(t)x = S_n(t)A^n x + \frac{t^{n-1}}{(n-1)!}A^{n-1}x + \ldots + tAx + x,\ t \geq 0.$$

By the uniqueness theorem for analytic functions, we have

$$\frac{d^n}{dz^n} S_n(z)x = S_n(z)A^n x + \frac{z^{n-1}}{(n-1)!}A^{n-1}x + \ldots + zAx + x,\ z \in \Sigma_\gamma,$$

and 3. follows from Definition 3.1. □

The proof of Proposition 3.2 implies the next statement.

Corollary 3.3. *Let $(S_n(t))_{t\geq 0}$ be an n-times integrated semigroup generated by A. If $S_n(\cdot) \in C^n((0,\infty) : L(E))$, then $(T(t))_{t>0}$ satisfies* 2. *Furthermore,* 1. *holds for all positive real numbers t_1 and t_2.*

The following lemma will be used in the proof of Proposition 3.5 given below.

Lemma 3.4. *Let $n \in \mathbb{N}$. Suppose that A generates an n-times integrated semigroup $(S_n(t))_{t\geq 0}$ and $a_i \in \mathbb{C}$, $i = 0,1,\ldots,n-1$. Let $V_n(t) := S_n(t) + \sum_{i=0}^{n-1} a_i t^i I,\ t \geq 0$. If $(V_n(t))_{t\geq 0}$ is an n-times integrated semigroup generated by a closed linear operator B, then $A = B$ and $a_i = 0,\ i = 0,1,\ldots,n-1$.*

Proof. Let $x \in E,\ \varphi \in \mathcal{D}$. Define

$$G(\varphi)x = (-1)^n \int_0^\infty \varphi^{(n)}(t) S_n(t)x dt \text{ and } H(\varphi)x = (-1)^n \int_0^\infty \varphi^{(n)}(t) V_n(t)x dt.$$

Then G and H are (DSG) generated by A and B, respectively. Moreover, it follows that $G(\varphi) = H(\varphi),\ \varphi \in \mathcal{D}_0$, and it implies

$$A = \{(x,y) \in E^2 : G(-\varphi')x = G(\varphi)y,\ \varphi \in \mathcal{D}_0\}$$

$$= \{(x,y) \in E^2 : H(-\varphi')x = H(\varphi)y,\ \varphi \in \mathcal{D}_0\} = B.$$

Since every (local) n-times integrated semigroup is uniquely determined by its generator, one obtains $S_n(t) = V_n(t),\ t \geq 0$. This ends the proof. □

Proposition 3.5. *Assume that A is a densely defined operator and that A is the generator of an exponentially bounded, analytic n-times integrated semigroup $(S_n(t))_{t\geq 0}$ of angle α, $0 < \alpha \leq \frac{\pi}{2}$, which satisfies $||S_n(z)|| = O(|z|^n),\ z \to 0,\ z \in \Sigma_\gamma$, for all $\gamma \in (0,\ \alpha)$. Then A generates a C_0-semigroup.*

Proof. Let us fix $\gamma \in (0,\ \alpha)$ and $t \in (0, \frac{1}{1+\sin\gamma})$. Assume that $||S_n(z)|| \leq M|z|^n,\ |z| \leq 1,\ z \in \overline{\Sigma_\gamma}$, for some $M > 0$. The Cauchy integral formula implies

$$T(t) = \frac{n!}{2\pi i} \oint_{|z-t| = t\sin\gamma} \frac{S_n(z)}{(z-t)^{n+1}} dz.$$

Since $t + t\sin\gamma e^{i\theta} \in \overline{\Sigma_\gamma} \cap \{z \in \mathbb{C} : 0 < |z| \leq 1\}$, $\theta \in [0, 2\pi]$, we obtain

$$||T(t)|| \leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{||S_n(t + t\sin\gamma e^{i\theta})||}{t^{n+1}\sin^{n+1}\gamma} t\sin\gamma d\theta \leq \frac{n!}{2\pi t^n \sin^n\gamma} 2\pi M (t + t\sin\gamma)^n;$$

hence $\sup_{t\in(0,1]} ||T(t)|| < \infty$. Corollary 3.3 implies that $(T(t))_{t>0}$ is a semigroup. Let $S_1(t) = \int_0^t T(s)ds,\ t \geq 0$. Then $(S_1(t))_{t\geq 0}$ is an exponentially bounded once integrated semigroup. Define $V_n(t) := \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} T(s)x ds,\ t \geq 0$. Then $(V_n(t))_{t\geq 0}$ is an exponentially bounded n-times integrated semigroup. Let $x \in E$ be fixed. Then $V_n(\cdot)x \in C^{n-1}([0,\infty) : E) \cap C^n((0,\infty) : E)$ and $\frac{d^n}{dt^n} V_n(t)x = T(t)x,\ t > 0$. Moreover, $\frac{d^n}{dt^n} S_n(t)x = T(t)x,\ t > 0$. Lemma 3.4 implies $S_n(t) = V_n(t),\ t \geq 0$. Then we obtain

$$\lambda \int_0^\infty e^{-\lambda t} S_1(t)x dt = \lambda^n \int_0^\infty e^{-\lambda t} \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} T(s)x ds dt$$

$$= \lambda^n \int_0^\infty e^{-\lambda t} S_n(t)x dt = R(\lambda : A)x,\ \lambda \text{ sufficiently large},$$

and it implies that A is the generator of $(T(t))_{t>0}$. Define $T(0) := I$. By virtue of [2, Corollary 3.3.11], $(T(t))_{t\geq 0}$ is a C_0-semigroup generated by A. $\square$

The next proposition is a slight improvement of [3, Proposition 3.7(a)].

Proposition 3.6. *Let A be the generator of an exponentially bounded, analytic n-times integrated semigroup $(S_n(t))_{t\geq 0}$ of angle α, $0 < \alpha \leq \frac{\pi}{2}$, $n \in \mathbb{N}$. Let $\theta \in (-\alpha,\ \alpha)$. Then $e^{i\theta}A$ generates an exponentially bounded, analytic n-times integrated semigroup $(e^{-in\theta}S(te^{i\theta}))_{t\geq 0}$ of angle $\alpha - |\theta|$.*

Proof. It is proved in [3, Proposition 3.7(a)] that $(e^{-in\theta}S(te^{i\theta}))_{t\geq 0}$ is an exponentially bounded $n-$times integrated semigroup generated by $e^{i\theta}A$. Put $S_\theta(z) := e^{-in\theta}S(ze^{i\theta}),\ z \in \Sigma_{\alpha-|\theta|}$. By Definition 3.1, $(S_\theta(t))_{t\geq 0}$ is an exponentially bounded, analytic n-times integrated semigroup of angle $\alpha - |\theta|$. $\square$

Proof of Theorem 1.1. Let us prove (b) → (a). Fix $n \in \mathbb{N}$; then it is clear that A generates an exponentially bounded, analytic n-times integrated semigroup $(S_n(t))_{t\geq 0}$ of angle α. Let $\gamma \in (0,\ \alpha)$. Then there exists a constant $M \geq 1$ such that $||T(z)|| \leq M,\ z \in \overline{\Sigma_\gamma} \ \cap\ \{z \in \mathbb{C} : |z| \leq 1\}$. Moreover, for all $\beta \in (-\gamma,\ \gamma)$, $e^{i\beta}A$ is the generator of a C_0-semigroup $(T_\beta(t))_{t\geq 0}$ given by $T_\beta(t) = T(e^{i\beta}t),\ t \geq 0$. By Proposition 3.6,

$$S_n(z)x = e^{in\arg z}\int\limits_0^{|z|}\frac{(|z|-s)^{n-1}}{(n-1)!}T(se^{i\arg z})x\,ds,\ x \in E,\ z \in \Sigma_\gamma.$$

Hence, $||S_n(z)|| = O(|z|^n),\ z \to 0,\ z \in \Sigma_\gamma$. In order to prove (a) → (b), let us suppose that there exists $n \in \mathbb{N}$ such that A generates an exponentially bounded, analytic n-times integrated semigroup $(S_n(t))_{t\geq 0}$ of angle α satisfying $||S_n(z)|| = O(|z|^n),\ z \to 0,\ z \in \Sigma_\gamma$, for all $\gamma \in (0,\ \alpha)$. By the uniqueness of analytic continuation, it is sufficient to prove that A generates an analytic C_0-semigroup of angle γ, for all $\gamma \in (0,\ \alpha)$. By Proposition 3.6, for all $\gamma \in (0,\ \alpha)$, $e^{\pm i\gamma}A$ generate exponentially bounded, analytic n-times integrated semigroups of angle $\alpha - |\theta|$ which satisfies the assumptions of Proposition 3.5. Thus, $e^{\pm i\gamma}A$ generate (differentiable) C_0-semigroups $(T(te^{\pm i\gamma}))_{t\geq 0}$ and A generates an analytic C_0-semigroup of angle γ. The proof is completed. $\square$

Next we prove Theorem 1.2; it is a generalization of [4, Theorem 21.13(a)] and [3, Lemma 5.5(a)].

Proof of Theorem 1.2. In order to see that (b') is a consequence of (a'), let us fix $\gamma \in (0,\ \alpha)$ and choose $\gamma_1 \in (\gamma,\ \alpha)$ after that. Suppose that $||S_n(z)|| \leq Me^{\omega|z|},\ z \in \Sigma_{\gamma_1}$, for some $\omega > 0$. Clearly, $\frac{R(\lambda:A)}{\lambda^n} = \int\limits_0^\infty e^{-\lambda t}S_n(t)dt,\ \lambda > \omega$. Then [2, Theorem 2.6.1] implies that there exists an analytic function

$R : \omega + \Sigma_{\gamma_1 + \frac{\pi}{2}} \to L(E)$ which satisfies $R(\lambda) = R(\lambda : A),\ Re\lambda > \omega$ and

$$\sup_{\lambda \in \omega + \Sigma_{\gamma + \frac{\pi}{2}}} ||(\lambda - \omega) \frac{R(\lambda)}{\lambda^n}|| < \infty.$$

Proposition B.5 in [2] implies $\omega + \Sigma_{\gamma_1 + \frac{\pi}{2}} \subset \rho(A)$ and $R(\lambda : A) = R(\lambda),\ \lambda \in \omega + \Sigma_{\gamma_1 + \frac{\pi}{2}}$. Hence, (1) holds for any $\omega_\gamma > \omega$ and appropriate $C_\gamma > 0$. Since $\lim_{t \to 0+} S_n(t)x = 0,\ x \in E$, (2) holds by [2, Theorem 2.6.4(a)]. Assume conversely that (b') holds. Fix $\gamma \in (0,\ \alpha)$ and choose $\omega > \omega_\gamma$ after that. Then the function $q : (\omega,\ \infty) \to \mathbb{C},\ q(\lambda) = \frac{R(\lambda:A)}{\lambda^n},\ \lambda > \omega$, can be analytically extended to a function $\tilde{q} : \omega + \Sigma_{\gamma + \frac{\pi}{2}} \to \mathbb{C}$. Furthermore,

$$\sup_{\lambda \in \omega + \Sigma_{\gamma + \frac{\pi}{2}}} ||(\lambda - \omega)\tilde{q}(\lambda)|| \leq \sup_{\lambda \in \omega + \Sigma_{\gamma + \frac{\pi}{2}}} C_\gamma (|\lambda| + \omega)(1 + |\lambda|)^{n-1} / |\lambda|^n < \infty.$$

An application of [2, Theorem 2.6.1] gives that there exists an analytic function $S_n : \Sigma_\gamma \to L(E)$ such that $\frac{R(\lambda:A)}{\lambda^n} = \int\limits_0^\infty e^{-\lambda t} S_n(t)dt,\ \lambda > \omega$, and that $||S_n(z)|| \leq M_\beta e^{wRez},\ z \in \Sigma_\beta$, for all $\beta \in (0,\ \gamma)$. Define $S_n(0) := 0$. Assumption (2) and [2, Theorem 2.6.4] imply that $\lim_{t \to 0+} S_n(t)x = 0$. Thus, $(S_n(t))_{t \geq 0}$ is a strongly continuous operator family, and consequently, $(S_n(t))_{t \geq 0}$ is an exponentially bounded n-times integrated semigroup generated by A. Moreover, for every fixed $\beta \in (0,\ \gamma)$ and $x \in E$, $\sup_{z \in \Sigma_\beta} ||e^{-\omega z} S_n(z)x|| < \infty$ and [2, Proposition 2.6.3(b)] implies $\lim_{z \to 0,\ z \in \Sigma_\beta} S_n(z)x = 0$. Accordingly, A is the generator of an exponentially bounded, analytic n-times integrated semigroup $(S_n(t))_{t \geq 0}$ of angle γ. This completes the proof of the theorem. □

Theorem 1.2 and [4, Theorem 21.17] immediately imply:

Corollary 3.7. *Let A be a densely defined operator. Suppose that there exist $\alpha \in (0, \frac{\pi}{2}]$ and $n \in \mathbb{N}$ such that for all $\gamma \in (0,\ \alpha)$ there exist $\omega_\gamma > 0$ and $C_\gamma > 0$ so that (1) holds. Then (2) is automatically fulfilled.*

The Hille-Yosida type estimates for generators of analytic convoluted semigroups have been recently proved in [7]. Note that there exist an exponentially bounded, continuous function K with $0 \in \operatorname{supp} K$ and a closed linear operator A in $L^2[0, \pi]$ such that the resolvent set of A does not contain any half-line (ω, ∞), $\omega \in \mathbb{R}$ and that A generates an analytic convoluted K-semigroup of angle $\frac{\pi}{2}$ (cf. [7]).

Proposition 3.8. *If A generates an exponentially bounded, analytic n-times integrated semigroup $(S_n(t))_{t \geq 0}$ of angle α, then for all $\gamma \in (0,\ \alpha)$, the following holds: (we continue the numbering)*

4. $S_n(z)A \subset AS_n(z),\ z \in \Sigma_\alpha$,
5. $T(z)A \subset AT(z),\ z \in \Sigma_\alpha$; *if $x \in D(A)$, then* $\frac{d}{dz}T(z)x = T(z)Ax,\ z \in \Sigma_\alpha$,

6. *if* $x \in E$ *and* $z \in \Sigma_\alpha$, *then* $\int_0^z S_n(\lambda)x d\lambda \in D(A)$ *and*

$$A \int_0^z S_n(\lambda)x d\lambda = S_n(z)x - \frac{z^n}{n!}x.$$

Proof. Let $x \in D(A)$ and $z \in \Sigma_\alpha$ be fixed. Proposition 3.6 implies that $e^{i \arg z}A$ generates an exponentially bounded n-times integrated semigroup $(e^{-in \arg z}S_n(te^{i \arg z}))_{t \geq 0}$. Thus, 4. is true. Note that 5. is proved in [3] with the help of regularized semigroups; we give a quite different proof. Proposition 3.6 implies that $e^{i \arg z}A$ is the generator of a distribution semigroup G_z given by

$$G_z(\varphi)x = (-1)^n \int_0^\infty \varphi^{(n)}(t)e^{-in \arg z}S_n(te^{i \arg z})x dt, \; x \in E, \; \varphi \in \mathcal{D}.$$

Integration by parts and the analyticity of $S_n(\cdot)$ imply that for all $\varphi \in \mathcal{D}$ with $\operatorname{supp} \varphi \subset (\frac{|z|}{2}, \infty)$, the following holds

$$G_z(\varphi)x = \int_0^\infty \varphi(t)T(te^{i \arg z})x dt, \; x \in E.$$

Let (φ_n) be a sequence in $\{\varphi \in \mathcal{D} : \operatorname{supp} \varphi \subset (\frac{|z|}{2}, \infty)\}$ which converges in distributional sense to $\delta_{|z|}$. Thus, $T(z)x = \lim_{n\to\infty} G_z(\varphi_n)x, \; x \in E$. Let $x \in D(A)$. Since $G_z(\varphi_n)x \in D(A), \; n \in \mathbb{N}$ and $\lim_{n\to\infty} AG_z(\varphi_n)x = \lim_{n\to\infty} G_z(\varphi_n)Ax = T(z)Ax$, the closedness of A implies $T(z)x \in D(A), \; z \in \Sigma_\alpha$ and $AT(z)x = T(z)Ax, \; z \in \Sigma_\alpha$. Since $x \in D(A)$, one also has $G_z(-\varphi')x = G_z(\varphi)e^{i \arg z}Ax, \; \varphi \in \mathcal{D}_0$ and it implies that for all $\varphi \in \mathcal{D}$ with $\operatorname{supp} \varphi \subset (\frac{|z|}{2}, \infty)$, the following holds

$$\int_0^\infty \varphi(t)\frac{d}{dt}T(te^{i \arg z})x dt = \int_0^\infty \varphi(t)T(te^{i \arg z})Ax dt.$$

Then the standard limit procedure implies:

$$\frac{d}{dz}T(z)x = T(z)Ax = \lim_{n\to\infty} G_z(\varphi_n)Ax.$$

Hence, 5. is proved. Let us prove 6. Let $x \in E$ and $z \in \Sigma_\alpha$. By Proposition 3.6, one obtains

$$\int_0^z S_n(\lambda)x d\lambda = e^{i \arg z} \int_0^{|z|} S_n(te^{i \arg z}) dt \in D(e^{i \arg z}A); \text{ therefore,}$$

$$
\begin{aligned}
A\int_0^z S_n(\lambda)x\,d\lambda &= e^{i\arg z}A\int_0^{|z|} S_n(te^{i\arg z})x\,dt \\
&= e^{in\arg z}(e^{-in\arg z}S_n(|z|e^{i\arg z})x - \frac{|z|^n}{n!}x) \\
&= S_n(z)x - e^{in\arg z}\frac{|z|^n}{n!}x = S_n(z)x - \frac{z^n}{n!}x.
\end{aligned}
$$

At the end, we note that several examples of non-densely defined generators of exponentially bounded, once integrated analytic semigroups can be derived through the analysis of the higher order elliptic differential operators in the spaces of Hölder continuous functions (cf. [9, Section 4]).

Acknowledgements

I express my sincere thanks to Professors Stevan Pilipović, Jizhou Zhang, Francisco Periago and Bernd Straub.

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Received by the editors February 5, 2005