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ERROR ESTIMATION OF PSEUDOSPECTRAL METHOD FOR SOLVING THE BAROTROPIC VORTICITY EQUATION

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Abstract. This paper is concerned with pseudospectral method using Legendre polynomials. The pseudospectral approximation for the barotropic vorticity equation is considered. The stability and rate of convergence are obtained. As a consequence, it is shown that this method achieves spectral accuracy if the solution to the barotropic vorticity equation is smooth.

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1. Introduction

The barotropic vorticity equation model is very important in the research of meteorological science and applied mathematics. Many scientists pay attention to the research of numerical methods of this equation. The early works were mainly concerned with finite difference methods [2, 4, 8, 9, 13].

The spectral method has a convergence rate of "infinite" order, i.e., the error decays faster than algebraically when the solution is infinitely smooth. This method has become one of the most powerful tools for the numerical solution of nonlinear partial differential equations arising in the fluid dynamics [1, 5, 10]. Many authors provide various spectral schemes and analyze the errors. Usually, only nonlinear problems in Cartesian coordinates are considered. But in meteorological science and some other fields [7, 14], one also has to deal with the problem defined on the spherical surface.

Since spectral methods usually entail too much computation work, sometimes it is even impossible to implement them strictly [3, 6]. The pseudospectral methods are preferred over spectral methods in practice, due to their numerical efficiency. However, there is a price to pay in the form of aliasing error which cause poor accuracy and instability in resolution demanding problems. Therefore, various techniques have already been developed for de-aliasing [11, 12].

In this paper, by taking the Barotropic Vorticity equation as an example, the pseudospectral method for solving partial differential equations on spherical surface is discussed. An interpolation procedure, which is different from that in the ordinary sense, is proposed. Based on such an interpolation, a pseudospectral scheme is constructed. Its generalized stability and convergence are analyzed rigorously.

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2. The Pseudospectral Scheme

Let S be the unit spherical surface

$$S = \left\{(\lambda, \theta) : 0 \leq \lambda < 2\pi, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right\},$$

where λ and θ are the longitude and latitude coordinate on the spherical surface, respectively. Let $\xi(\lambda,\ \theta, t)$, $\psi(\lambda, \theta, t)$, and $\Omega > 0$ be the vorticity, the stream function and angular velocity of the earth respectively. The gradient, the jacobian operator and the laplacian are as follows:

$$\nabla u = \left(\frac{1}{cos\theta}\frac{\partial u}{\partial \lambda}, \frac{\partial u}{\partial \theta}\right), \qquad J(u, v) = \frac{1}{cos\theta}\left(\frac{\partial u}{\partial \lambda}\frac{\partial v}{\partial \theta} - \frac{\partial u}{\partial \theta}\frac{\partial v}{\partial \lambda}\right),$$
$$\triangle u = \frac{1}{cos\theta}\frac{\partial}{\partial \theta}\left(cos\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{cos^2\theta}\frac{\partial^2 u}{\partial \lambda^2}.$$

The barotropic vorticity equation on S is as follows:

$$(1) \quad \begin{cases} \frac{\partial \xi}{\partial t}(t) + J(\xi(t), \psi(t)) - 2\Omega\frac{\partial}{\partial \lambda}\psi(t) = 0, & (\lambda, \theta) \in S, \quad t \in (0, T], \\ -\triangle\psi(t) = \xi(t), & (\lambda, \theta) \in S, \quad t \in (0, T], \\ \xi(\lambda, \theta, 0) = \xi_0(\lambda, \theta), & (\lambda, \theta) \in S, \end{cases}$$

where the initial value $\xi_0(\lambda, \theta)$ is given. For fixed ψ, we require

$$(2) \qquad \mu(\psi(t)) = \int\int_S \psi(\lambda, \theta, t) dS \equiv 0.$$

Now, we are going to construct the pseudospectral scheme for (1). First we introduce some approximation subspaces and define an interpolation operator. For a non-negative integer $n \geq 0$, denoted by $L_n(x)$, the nth degree Legendre polynomial is defined on $[-1, 1]$. Recall the orthogonality relation

$$(L_i(x), L_j(x)) = \frac{2}{2i+1}\delta_{ij}, \qquad \forall i, j \geq 0,$$

where $(f, g) = \int_{-1}^{1} f(x)g(x)dx$. Also recall that $L'_n(x)$ satisfies the recurrence relation

$$L'_n(x) = \sum_{k=0}^{n-1}(2k+1)L_k(x).$$

For an integer $m(|m| \leq n)$, the associated Legendre polynomials are defined as

$$L_{m,n}(x) = \sqrt{\frac{(2n+1)(n-m)!}{2(n-m)!}}(1-x^2)^{m/2}\frac{d^m}{dx^m}L_n(x), \qquad 0 \leq m \leq n,$$
$$L_{m,n}(x) = L_{-m,n}(x), \qquad -n \leq m \leq 0.$$

Then the spherical harmonic functions $Y_{m,n}(\lambda, \theta)$ are

$$(3) \qquad Y_{m,n}(\lambda, \theta) = \frac{1}{\sqrt{2\pi}}e^{im\lambda}L_{m,n}(sin\theta), \qquad n \geq |m|.$$

Let N be any positive integer. We define the trail function space for pseudospectral approximation as

$$(4) \qquad \widetilde{V}_N = \left\{ v : v = \sum_{n=0}^{N} \sum_{|m| \leq n} \widehat{v}_{m,n} Y_{m,n}(\lambda, \theta) \right\}.$$

Furthermore, define V_N as the real subspace of $\widetilde{V}_N$, and V_N^0 as the subspace of V_N, whose average on the spherical surface vanishes.

Next, we consider the interpolation from $C(S)$ onto V_N, where $C(S)$ is the space containing the set of all continuous functions on S . Let $x_j (0 \leq j \leq N)$ be the $N+1$ roots of the Legendre polynomial $L_{N+1}(x)$. Clearly, $x_j \in [-1, 1]$. Let

$$w_j = \frac{1}{(1 - x_j^2)[L'_{N+1}(x_j)]^2}, \qquad 0 \leq j \leq N,$$

which are the $N+1$ weights in the Legendre-Gauss quadrature formula associated with the $N+1$ roots. Define S_N as a set of grid points on S,

$$(5) \qquad S_N = \left\{ (\lambda_l, \theta_j) : \lambda_l = \frac{2l\pi}{2N+1}, 0 \leq l \leq 2N; \theta_j = sin^{-1} x_j, 0 \leq j \leq N \right\}.$$

Then we define the interpolation operator I_N, from $C(S)$ onto V_N, as follows

$$(6) \qquad I_N v = \sum_{n=0}^{N} \sum_{|m| \leq n} v_{m,n} Y_{m,n}(\lambda, \theta),$$

where

$$v_{m,n} = \frac{2\pi}{2N+1} \sum_{l=0}^{2N} \sum_{j=0}^{N} w_j v(\lambda_l, \theta_j) Y_{m,n}^*(\lambda_l, \theta_j).$$

Moreover, we introduce the discrete inner product $(\cdot, \cdot)_N$ as

$$(7) \qquad (u, v)_N = \frac{2\pi}{2N+1} \sum_{l=0}^{2N} \sum_{j=0}^{N} w_j u(\lambda_l, \theta_j) v^*(\lambda_l, \theta_j), \qquad \forall u, v \in C(S).$$

It is obvious that $v_{m,n} = (v, Y_{m,n})_N$. The degree of freedom of S_N is $(2N+1)(N+1)$, while the dimension of V_N is $\sum_{n=0}^{N}(2n+1) = (N+1)^2$, thus generally $I_N v(\lambda_l, \theta_j) \neq v(\lambda_l, \theta_j)$, Consequently I_N is not the interpolation operator in the ordinary sense. It is actually the projection from $C(S)$ onto V_N corresponding to the discrete inner product $(\cdot, \cdot)_N$.

Finally, we consider the finite difference discretization in the temporal direction. Let τ be the step size in time t, and

$$R_\tau = \left\{ t = k\tau : 1 \leq k \leq \left[\frac{T}{\tau} \right] \right\}.$$

Define

$$u_{\hat{t}}(\lambda,\theta,t)=\frac{1}{2\tau}[u(\lambda,\theta,t+\tau)-u(\lambda,\theta,t-\tau)],$$
$$\widehat{u}(\lambda,\theta,t)=\frac{1}{2}[u(\lambda,\theta,t+\tau)+u(\lambda,\theta,t-\tau)].$$

The fully discrete Legendre pseudospectral scheme for solving (1)-(2) is to find $\eta(t)\in V_N$, $\phi(t)\in V_N^0$, approximating to $\xi(t)$ and $\psi(t)$ respectively.

$$(8)\qquad \begin{cases} \eta_{\hat{t}}(t)+I_N J(\eta(t),\phi(t))-2\Omega\frac{\partial}{\partial\lambda}\phi(t)=0, & \text{on } S\times R_\tau,\\ -\triangle\phi(t)=\eta(t), & \text{on } S\times R_\tau,\\ \mu(\phi(t))=0, & \text{on } S,\\ \eta(0)=I_N\xi_0, \quad \eta(\tau)=I_N\left[\xi_0+\tau\frac{\partial}{\partial t}\xi(0)\right] & \text{on } S. \end{cases}$$

Remark 2.1. The Fourier coefficients of unknown functions $\eta(t+2\tau)$ and $\phi(t+2\tau)$ can actually be evaluated explicitly. Besides, in calculating the nonlinear terms, Fast Fourier Transform technique can also be used in the λ direction. The scheme (8) is of second order accuracy in the temporal direction, except for the calculation for an extra initial value $\eta(\tau)$, its total computational work remain the same as two-level scheme

$$\frac{1}{\tau}[\eta(t+\tau)-\eta(\tau)]+I_N J(\eta(t),\phi(t))-2\Omega\frac{\partial}{\partial\lambda}\phi(t)=0.$$

3. Lemmas

Let D(S) be the set of all infinitely differentiable functions defined on S, which are regular at two poles $\theta=\pm\frac{\pi}{2}$. $D'(S)$ is the dual space of $D(S)$. For any $u\in D'(S)$, we can defined the generalized derivatives, generalized gradient and Laplacian operators as in [15]. The inner product and L^2-norm are defined as follows:

$$(u,v)=\int\int_S uvds=\int_0^{2\pi}\int_{-\pi/2}^{\pi/2}u(\lambda,\theta)v(\lambda,\theta)cos\theta d\theta d\lambda, \qquad \|u\|=(u,u)^{\frac{1}{2}}.$$

The symbol L^2 denotes the set of all functions for which this integral is finite:

$$L^2(S)=\{u\in D'(S):\|u\|<\infty\}.$$

Furthermore, define

$$H^1(S)=\left\{u:u,\frac{1}{cos\theta}\frac{\partial u}{\partial\lambda},\frac{\partial u}{\partial\theta}\in L^2(S)\right\}$$

be the space equipped with the following semi-norm and norm as

$$(9)\qquad |u|_1=\left(\left\|\frac{1}{cos\theta}\frac{\partial u}{\partial\lambda}\right\|^2+\left\|\frac{\partial u}{\partial\theta}\right\|^2\right)^{\frac{1}{2}}, \qquad \|u\|_1=(\|u\|^2+|u|_1^2)^{\frac{1}{2}}.$$

For the integer $r \geq 0$, $H^r(S)$ can be defined similarly. In particular, the norm of $H^2(S)$ is equivalent to $(\|u\|^2 + \|\triangle u\|^2)^{1/2}$.
For any real number $r \geq 0$, we define $H^r(S)$ as the complex interpolation between the two spaces $H^{[r]}(S)$ and $H^{[r+1]}(S)$, where $[r]$ is the integral part of r. The $Y_{m,n}$ are the eigenfunctions of the spherical Laplace operator $\triangle$, corresponding to the eigenvalues $n(n+1)$. Thus in the space norm $H^r(S)$, the norm $\|u\|_r^2$ is equivalent to [6]

$$\sum_{n=0}^{\infty} \sum_{|m| \leq n} n^r (n+1)^r |\widehat{u}_{m,n}|^2, \tag{10}$$

where $\widehat{u}_{m,n}$ are the Fourier coefficients related to the spherical harmonic functions $Y_{m,n}$. Particularly, $H^0(S) = L^2(S)$ and $\|u\|_0 = \|u\|$. It can be verified for any $u, v \in H^2(S)$

$$-(\triangle u, v) = (\nabla u, \nabla v). \tag{11}$$

The $L^2(S)$ orthogonal projection $P_N : L^2(S) \longrightarrow V_N$ is such mapping that for any $u \in L^2(S)$

$$(u - P_N u, v) = 0, \quad \forall v \in V_N,$$

or equivalently

$$P_N u(\lambda, \theta) = \left(\sum_{l=-N}^{N} \sum_{|m| \leq N} \widehat{u}_{m,n} \right) Y_{l,m}(\lambda, \theta).$$

Throughout this paper we shall use c to denote a general positive constant independent of τ and N. It can be of different values in different cases.

Lemma 1. *If $0 \leq \beta \leq r$, then for all $u \in H^r(S)$,*

$$\|u - P_N u\|_\beta \leq c N^{\beta - r} \|u\|_r$$

Proof. By (10)

$$\begin{aligned}
\|u - P_N u\|_\beta^2 &\leq c \sum_{m=-N}^{N} \sum_{n=N+1}^{\infty} n^\beta (n+1)^\beta |\widehat{u}_{m,n}|^2 + \\
&+ c \sum_{|m|>N} \sum_{n=|m|}^{\infty} n^\beta (n+1)^\beta |\widehat{u}_{m,n}|^2 \\
&\leq c \sum_{m=-N}^{N} \sum_{n=N+1}^{\infty} n^\beta (n+1)^\beta |\widehat{u}_{m,n}|^2 + \\
&+ c \sum_{|m|>N} \sum_{n=N+1}^{\infty} n^\beta (n+1)^\beta |\widehat{u}_{m,n}|^2 \\
&\leq c N^{2\beta - 2r} \sum_{m=-\infty}^{\infty} \sum_{n=N+1}^{\infty} n^r (n+1)^r |\widehat{u}_{m,n}|^2
\end{aligned}$$

$$\leq \quad cN^{2\beta-2r}\|u\|_r^2$$

□

Lemma 2. *(Inverse Inequality) If $0 \leq r \leq \beta$, then $\forall u \in V_N$,*

$$\|u\|_\beta \leq cN^{\beta-r}\|u\|_r$$

Proof. Let

$$u = \sum_{m=-N}^{N} \sum_{n=|m|}^{N} \widehat{u}_{m,n} Y_{m,n}(\lambda, \theta).$$

Then by (10)

$$\begin{aligned}
\|u\|_\beta^2 &\leq c \sum_{m=-N}^{N} \sum_{n=|m|}^{N} n^\beta (n+1)^\beta |\widehat{u}_{m,n}|^2 \\
&\leq cN^{2\beta-2r} \sum_{m=-N}^{N} \sum_{n=|m|}^{N} n^r (n+1)^r |\widehat{u}_{m,n}|^2 \\
&\leq cN^{2\beta-2r}\|u\|_r^2
\end{aligned}$$

□

Lemma 3. *[11] If $u \in C(S)$ and $v \in V_N$, then:*

$$(i)\ I_N v = v, \qquad (ii)\ (I_N u, v) = (I_N u, v)_N = (u, v)_N.$$

Lemma 4. *[5] For all $u \in C(S)$ we have*

$$\begin{aligned}
&\|I_N u\| = \|I_N u\|_N \leq \|u\|_N, \\
&\|u - I_N u\|_N = \inf_{\forall v \in V_N} \|u - v\|_N.
\end{aligned}$$

Lemma 5. *[5] For all $u \in V_N$*

$$(i) \left\| \frac{1}{cos\theta} \frac{\partial u}{\partial \lambda} \right\|_N = \left\| \frac{1}{cos\theta} \frac{\partial u}{\partial \lambda} \right\|, \qquad (ii) \left\| \frac{\partial u}{\partial \theta} \right\|_N = \left\| \frac{\partial u}{\partial \theta} \right\|,$$

Lemma 6. *Assume $0 \leq r \leq \beta$ and $\beta > 1$. Then for all $u \in H^\beta(S)$,*

$$\|u - I_N u\|_r \leq cN^{1+r+\varepsilon-\beta}\|u\|_\beta$$

where ε is an arbitrary small number.

Proof. First we consider the case $r = 0$. By the embedding theorem on spherical surface, we have

$$\|u\|_\infty \le c\|u\|_{1+\varepsilon}, \qquad \forall \varepsilon > 0 \tag{12}$$

Thus $H^\beta(S) \subset C(S)$ and $I_N u$ is well defined. It follows from Lemma 4 that

$$\|u - I_N u\| \le \|u - P_N u\|_N \le c\|u - P_N u\|_\infty.$$

Because $I_N u \in V_N$, $P_N u \in V_N$, we have form Lemma 3 that

$$\begin{aligned}\|u - I_N u\| &\le \|u - P_N u\| + \|I_N u - P_N u\| = \|u - P_N u\| + \|I_N u - P_N u\|_N \\ &\le \|u - P_N u\| + \|u - I_N u\|_N + \|u - P_N u\|_N \\ &\le 3\|u - P_N u\|_\infty.\end{aligned}$$

The combination of (12) and Lemma 2 leads to

$$\|u - P_N u\|_\infty \le c\|u - P_N u\|_{1+\varepsilon} \le cN^{1+\varepsilon-\beta}\|u\|_\beta,$$

where ε is an arbitrary small number. If $r > 0$, then we have from Lemma 2 that

$$\begin{aligned}\|I_N u - P_N u\|_r &\le cN^r\|I_N u - P_N u\| \\ &\le cN^r(\|u - P_N u\| + \|u - I_N u\|) \\ &\le cN^{1+r+\varepsilon-\beta}\|u\|_\beta,\end{aligned}$$

which, together with Lemma 1, implies the conclusion of this lemma. □

Lemma 7. *[14] There exists a positive constant c such that $\|u\|^2 \le c|u|_1^2$, $\forall u \in H^1(S)$ with $\mu(u) = 0$*

Lemma 8. *[3] Suppose that $E(t)$ and $\rho(t)$ are two non-negative, non-decreasing functions defined on R_τ, b_1, b_2 and c are non-negative constants, $t_1 \in R_\tau$, and the following conditions are fulfilled:*

(i) $E(t) \le \rho(t) + c\sum_{t'=\tau}^{t-\tau}(E(t') + N^{b_1}E^{b_2+1}(t'))$, $\quad \forall t \in R_\tau$

(ii) $\rho(t_1)e^{2ct} \le N^{-b_1/b_2}$

Then, for all $t \in R_\tau$, $t \le t_1$, we have

$$E(t) \le \rho(t)e^{2ct}.$$

4. The Stability

We now analyze the generalized stability of scheme (8). Suppose that $\eta(0)$ has the error $\widetilde{\eta}_0$, while the right-hand side of first and second equation of

(8) have errors $\widetilde{f}_1$ and $\widetilde{f}_2$ respectively. They induce the error of $\eta(t)$ and $\phi(t)$ denoted by $\widetilde{\eta}(t)$ and $\widetilde{\phi}(t)$. Then

$$(13)\qquad \begin{cases} \widetilde{\eta}_{\widehat{t}}(t) + I_N J(\eta(t),\widetilde{\phi}(t)) + I_N J(\widetilde{\eta}(t),\phi(t)) + I_N J(\widetilde{\eta}(t),\widetilde{\phi}(t)) \\ \qquad -2\Omega\frac{\partial}{\partial\lambda}(\widetilde{\phi}(t)) = \widetilde{f}_1(t), \\ -\triangle\widetilde{\phi}(t) = \widetilde{\eta}(t) + \widetilde{f}_2(t), \\ \mu(\widetilde{\phi}(t)) = 0, \\ \widetilde{\eta}(0) = \widetilde{\eta}_0, \quad \widetilde{\eta}(\tau) = \widetilde{\eta}_1. \end{cases}$$

By taking the L^2 inner product with $2\widehat{\widetilde{\eta}}(t)$ on both sides of the first equation of (13), we obtain

$$(14)\qquad (\|\widetilde{\eta}(t)\|^2)_{\widehat{t}} + \sum_{i=1}^{3} F_i(t) - 4\Omega\left(\frac{\partial}{\partial\lambda}\widetilde{\phi}(t),\widehat{\widetilde{\eta}}(t)\right) = 2\left(\widetilde{f}_1(t),\widehat{\widetilde{\eta}}(t)\right),$$

where

$$\begin{aligned} F_1(t) &= \left(I_N J(\eta(t),\widetilde{\phi}(t)), 2\widehat{\widetilde{\eta}}(t)\right), \\ F_2(t) &= \left(I_N J(\widetilde{\eta}(t),\phi(t)), 2\widehat{\widetilde{\eta}}(t)\right), \\ F_3(t) &= \left(I_N J(\widetilde{\eta}(t),\widetilde{\phi}(t)), 2\widehat{\widetilde{\eta}}(t)\right), \\ \left(\frac{\partial}{\partial\lambda}\widetilde{\phi}(t),\widehat{\widetilde{\eta}}(t)\right) &= \frac{1}{2}\left(\frac{\partial}{\partial\lambda}\widetilde{\phi}(t),\widetilde{\eta}(t+\tau)\right) + \frac{1}{2}\left(\frac{\partial}{\partial\lambda}\widetilde{\phi}(t),\widetilde{\eta}(t-\tau)\right). \end{aligned}$$

By taking the L^2 inner product with $\widetilde{\phi}(t)$ on both sides of the second equation of (13), and obtain

$$|\widetilde{\phi}(t)|_1^2 \le \frac{1}{2c}\|\widetilde{\phi}(t)\|^2 + c(\|\widetilde{\eta}(t)\|^2 + \|\widetilde{f}_2(t)\|^2).$$

Thus Lemma 7 leads to

$$(15)\qquad |\widetilde{\phi}(t)|_1^2 \le c(\|\widetilde{\eta}(t)\|^2 + \|\widetilde{f}_2(t)\|^2).$$

Moreover, by Lemma 7 and (15)

$$(16)\qquad \|\widetilde{\phi}(t)\|_2^2 \le c(\|\widetilde{\phi}(t)\|^2 + \|\triangle\widetilde{\phi}(t)\|^2) \le c(\|\widetilde{\eta}(t)\|^2 + \|\widetilde{f}_2(t)\|^2).$$

Now we are going to estimate $|F_j(t)|, j = 1,2,3$. Assume that ε is suitably small number, and $\|u\|_{1,\infty} = \|u\|_{C^1(S)} = \sup_{x\in S}|u(x)|_1$, $\|u\|_\infty = \|u\|_{C(S)}$, we have from (16) that

$$\begin{aligned} |F_1(t)| &\le 2\left|\left(J(\eta(t),\widetilde{\phi}(t)),\widehat{\widetilde{\eta}}(t)\right)\right| \le 2\sqrt{2}\|\eta(t)\|_{1,\infty}\|\widehat{\widetilde{\eta}}(t)\||\widetilde{\phi}(t)|_1 \\ &\le c\|\eta(t)\|_{1,\infty}\|\widehat{\widetilde{\eta}}(t)\|\left(\|\widetilde{\eta}(t)\|^2 + \|\widetilde{f}_2(t)\|^2\right)^{1/2} \end{aligned}$$

$$(17)\qquad |F_1(t)| \le \|\widehat{\widetilde{\eta}}(t)\|^2 + c\|\eta(t)\|_{1,\infty}\left(\|\widetilde{\eta}(t)\|^2 + \|\widetilde{f}_2(t)\|^2\right).$$

Furthermore by (12) and Lemma 5, we get

$$
\begin{aligned}
|F_2(t)| &\le 2\left|\left(J(\widetilde{\eta}(t),\phi(t)),\widehat{\widetilde{\eta}}(t)\right)\right| \\
&\le c\|\phi(t)\|_{1,\infty}\|\widehat{\widetilde{\eta}}(t)\|\left(\left\|\frac{1}{cos\theta}\frac{\partial\widetilde{\eta}(t)}{\partial\lambda}\right\|_N+\left\|\frac{\partial\widetilde{\eta}(t)}{\partial\theta}\right\|_N\right) \\
&\le 2\sqrt{2}\|\phi(t)\|_{1,\infty}\|\widehat{\widetilde{\eta}}(t)\|\,|\widetilde{\eta}(t)|_1
\end{aligned}
$$

$$
|F_2(t)| \le c\|\widehat{\widetilde{\eta}}(t)\|^2+\|\phi(t)\|^2_{1,\infty}\|\widetilde{\eta}(t)\|^2. \tag{18}
$$

On the other hand, it follows from (12) and Lemma 2 that

$$
\|\widetilde{\phi}(t)\|_{1,\infty}\le c\|\widetilde{\phi}(t)\|_{2+\varepsilon}\le cN^{\varepsilon}\|\widetilde{\phi}(t)\|_2.
$$

Thus from the above estimate and (16) we get

$$
\begin{aligned}
|F_3(t)| &\le 2\sqrt{2}\|\widetilde{\phi}(t)\|_{1,\infty}\|\widehat{\widetilde{\eta}}(t)\|\,|\widetilde{\eta}(t)|_1 \\
&\le cN^{1+\varepsilon}\|\widehat{\widetilde{\eta}}(t)\|\|\widetilde{\eta}(t)\|\|\widetilde{\phi}(t)\|_2
\end{aligned}
$$

$$
|F_3(t)| \le \|\widehat{\widetilde{\eta}}(t)\|^2+cN^{2(1+\varepsilon)}\left(\|\widetilde{\eta}(t)\|^4+\|\widetilde{f}_2(t)\|^4\right). \tag{19}
$$

Finally, we have

$$
\begin{aligned}
\left|4\Omega\left(\frac{\partial\widetilde{\phi}}{\partial\lambda}(t),\widehat{\widetilde{\eta}}(t)\right)\right| &\le 4\Omega\|\widehat{\widetilde{\eta}}(t)\|\left|\widetilde{\phi}(t)\right|_1\le c\|\widehat{\widetilde{\eta}}(t)\|^2+c|\widetilde{\phi}(t)|_1^2 \\
&\le c\|\widehat{\widetilde{\eta}}(t)\|^2+c\left(\|\widetilde{\eta}(t)\|^2+\|\widetilde{f}_2(t)\|^2\right).
\end{aligned} \tag{20}
$$

$$
2\left|\left(\widetilde{f}_1(t),\widehat{\widetilde{\eta}}(t)\right)\right|\le 2\|\widetilde{f}_1(t)\|\|\widehat{\widetilde{\eta}}(t)\|\le\|\widehat{\widetilde{\eta}}(t)\|^2+\|\widetilde{f}_1(t)\|^2 \tag{21}
$$

By substituting the above estimates (17)−(21) into (14), we get

$$
\left(\|\widetilde{\eta}(t)\|^2\right)_{\hat{t}}\le c\left\|\widehat{\widetilde{\eta}}(t)\right\|^2+M\|\widetilde{\eta}(t)\|^2+cN^{2(1+\varepsilon)}\|\widetilde{\eta}(t)\|^4+G(t), \tag{22}
$$

where

$$
M=c\|\eta\|_{1,\infty}+c\|\phi\|^2_{1,\infty}+c,
$$

$$
G(t)=c(1+\|\eta\|_{1,\infty})\left\|\widetilde{f}_2(t)\right\|^2+cN^{2(1+\varepsilon)}\left\|\widetilde{f}_2(t)\right\|^4+c\left\|\widetilde{f}_1(t)\right\|^2.
$$

In fact

$$
\left\|\widehat{\widetilde{\eta}}(t)\right\|^2\le\frac{1}{2}\left(\|\widetilde{\eta}(t+\tau)\|^2+\|\widetilde{\eta}(t-\tau)\|^2\right).
$$

Summing up (22) for all $t'\in R_\tau$, $\tau\le t'\le t-\tau$, we have

$$
\begin{aligned}
\|\widetilde{\eta}(t)\|^2 \le \|\widetilde{\eta}(0)\|^2+\|\widetilde{\eta}(\tau)\|^2+c\tau\sum_{t'=\tau}^{t-\tau}\|\widehat{\widetilde{\eta}}(t')\|^2+M\tau\sum_{t'=\tau}^{t-\tau}\|\widetilde{\eta}(t')\|^2 \\
+cN^{2(1+\varepsilon)}\tau\sum_{t'=\tau}^{t-\tau}\|\widetilde{\eta}(t')\|^2++\tau\sum_{t'=\tau}^{t-\tau}\|G(t')\|^2.
\end{aligned} \tag{23}
$$

Let

$$E(t)=\|\widetilde{\eta}(t)\|^2\,,$$
$$\rho(t)=\|\widetilde{\eta}(0)\|^2+\|\widetilde{\eta}(t)\|^2+2\sum_{t'=\tau}^{t-\tau}G(t'),$$

then (23) implies

$$E(t)\le\rho(t)+c\tau\sum_{t'=\tau}^{t-\tau}\left[E(t')+cN^{2(1+\varepsilon)}E^2(t')\right].$$

Finally, the application of Lemma 8 (with $b_1=2(1+\varepsilon), b_2=1$) leads to the following theorem

Theorem 1. *Suppose that in scheme (8) the mesh size τ and N^{-1} are sufficiently small and there exist $t_1\in R_\tau$ such that*

$$\rho(t_1)e^{2ct}\le N^{-2(1+\varepsilon)},$$

ε being a suitably small number. Then we have for all $t\in R_\tau$, $t\le t_1$, we have

$$E(t)\le\rho(t)e^{2ct}.$$

5. Convergence

In this section we discuss the convergence of the scheme (8). Let $\xi^N(t)=P_N\xi(t),\quad \psi^N(t)=P_N\psi(t)$. It follows from (1) that

$$\text{(24)}\qquad\begin{cases}\xi^N_{\hat{t}}(t)+I_NJ(\xi^N(t),\psi^N(t))-2\Omega\frac{\partial}{\partial\lambda}\psi^N(t)=\sum_{j=1}^2\widetilde{g}_j(t),\\ -\triangle\psi^N(t)=\xi^N(t),\\ \mu(\psi^N(t))=0,\\ \xi^N(0)=P_N\xi_0,\quad \xi^N(\tau)=I_N\left[\xi_0+\tau\frac{\partial}{\partial t}\xi(0)\right],\end{cases}$$

where

$$\widetilde{g}_1(t)=\frac{\partial\xi^N(t)}{\partial t}-\xi^N_{\hat{t}}(t),$$
$$\widetilde{g}_2(t)=I_NJ\left(\xi^N(t),\psi^N(t)\right)-P_NJ(\xi(t),\psi(t)).$$

Furthermore, let $\widetilde{\xi}(t)=\eta(t)-\xi^N(t)$ and $\widetilde{\psi}(t)=\phi(t)-\psi^N(t)$, then by (1) and (8), we get

$$\text{(25)}\qquad\begin{cases}\widetilde{\xi}_{\hat{t}}(t)+I_NJ(\xi^N(t),\widetilde{\psi}(t))+I_NJ(\widetilde{\xi}(t),\psi^N(t))+I_NJ(\widetilde{\xi}(t),\widetilde{\psi}(t))\\ \qquad\qquad-2\Omega\frac{\partial}{\partial\lambda}\widetilde{\psi}(t)=-\sum_{j=1}^2\widetilde{g}_j(t),\\ -\triangle\widetilde{\psi}(t)=\widetilde{\xi}(t),\\ \mu(\widetilde{\psi}(t))=0,\\ \widetilde{\xi}(0)=(I_N-P_N)\xi_0,\quad \widetilde{\xi}(\tau)=(I_N-P_N)\left[\xi_0+\tau\frac{\partial}{\partial t}\xi(0)\right].\end{cases}$$

Take the inner product with $2\widehat{\widetilde{\xi}}(t)$ and $\widehat{\widetilde{\psi}}(t)$ in the first and second equation of (25) respectively. Then, by a procedure similar to the proof of Theorem 1 we get an estimate for the error $\widetilde{\xi}(t)$. Hence in order to get the convergence rate, we need only to estimate $\|\widetilde{g}_j(t)\|^2$. Let

$$\rho_1(t) = \left\|\widetilde{\xi}(0)\right\|^2 + \left\|\widetilde{\xi}(\tau)\right\|^2 + \tau d \sum_{t'=\tau}^{t-\tau} \left[\|\widetilde{g}_1(t')\|^2 + \|\widetilde{g}_2(t')\|^2\right].$$

According to Taylor's formula

$$\left\|\frac{\partial \xi^N(t)}{\partial t} - \xi^N_{\hat{t}}(t)\right\| \le \left\|\frac{\partial \xi(t)}{\partial t} - \xi_{\hat{t}}(t)\right\| \le \tau^{3/2}\left(\int_t^{t+\tau}\left\|\frac{\partial^3\xi}{\partial t^3}(t')\right\|^2 dt'\right)^{1/2}$$

we get

$$\tau \sum_{t'=\tau}^{t-\tau} \|g_1(t')\|^2 \le c\tau^4 \|\xi(t)\|^2_{H^3(0,T;L^2(S))}.$$

We separate $\widetilde{g}_2(t)$ as $\widetilde{g}_2(t) = A_1(t) + A_2(t) + A_3(t)$,

$$\begin{aligned} A_1(t) &= (I_N - P_N)J(\xi^N(t), \psi^N(t)), \qquad P_N J(\xi^N(t) - \xi(t), \psi^N(t)),\\ A_2(t) &= P_N J(\xi(t), \psi^N(t) - \psi(t)). \end{aligned}$$

By Lemma 6, Lemma 1 and embedding theorem on spherical surface, it is not difficult to see that

$$\begin{aligned} |A_1(t)| &\le cN^{1+\varepsilon-r}\left\|J(\xi^N(t), \psi^N(t))\right\|_r \le cN^{1+\varepsilon-r}\left\|\xi^N(t)\right\|_{1+r}\left\|\psi^N(t))\right\|_{1+r}\\ &\le cN^{1+\varepsilon-r}\left\|\xi(t)\right\|_{1+r}\left\|\psi(t))\right\|_{1+r}. \end{aligned}$$

Also, it follows from Lemma 1 and inverse inequality that

$$\begin{aligned} |A_2(t)| &\le \left\|J(\xi^N(t) - \xi^N(t), \psi^N(t))\right\| \le \left\|\xi^N(t) - \xi(t)\right\|_1 \left\|\psi^N(t))\right\|_{1,\infty}\\ &\le cN^{1-r}\left\|\xi(t)\right\|_r \left\|\psi(t))\right\|_r,\\ |A_3(t)| &\le cN^{1-r}\left\|\xi(t)\right\|_r \left\|\psi(t)\right\|_r. \end{aligned}$$

By substituting the above estimate into the expression $\rho_1(t)$ we get

$$\begin{aligned} \|\widetilde{g}_2(t)\|^2 &\le \|A_1(t)\|^2 + \|A_2(t)\|^2 + \|A_3(t)\|^2\\ &\le cN^{2(1+\varepsilon-r)}\left\|\xi(t)\right\|^2_{1+r}\left\|\psi(t))\right\|^2_{1+r} + cN^{2(1-r)}\left\|\xi(t)\right\|^2_r\left\|\psi(t)\right\|^2_r, \end{aligned}$$

$$\begin{aligned} \tau\sum_{t'=\tau}^{t-\tau}\|\widetilde{g}_2(t')\|^2 &\le cN^{2(1+\varepsilon-r)}\left(\|\xi(t)\|^4_{C(0,T;H^{r+1}(S))} + \|\psi(t))\|^4_{C(0,T;H^{r+1}(S))}\right)\\ &\quad + cN^{2(1-r)}\left(\|\xi(t)\|^4_{C(0,T;H^{r+1}(S))} + \|\psi(t)\|^4_{C(0,T;H^{r+1}(S))}\right), \end{aligned}$$

$$\left\|\widetilde{\xi}(0)\right\|^2 \leq cN^{2(\varepsilon-r)}\left\|\xi_0\right\|_{r+1}^2, \qquad \left\|\widetilde{\xi}(\tau)\right\|^2 \leq c\tau^4\left\|\xi(t)\right\|_{H^2(0,T;H^1(S))}^2.$$

Consequently

$$\rho_1(t) \leq d\left(\tau^4 + N^{2(1+\varepsilon-r)}\right),$$

where d is a positive constant depending only on the norm of ξ and ψ in the spaces mentioned above. Finally, we reach the following theorem for convergence rate

Theorem 2. *Assume that the exact solution of (1) satisfies the following smoothness,* $\xi \in H^3(0,T;L^2(S)) \bigcap C(0,T;H^{r+1}(S)) \bigcap C(0,T;H^r(S)) \bigcap H^2(0,T;H^1(S))$, $\psi \in C(0,T;H^{r+1}(S)) \bigcap C(0,T;H^{r+1}(S))$ *with* $r > 2$. *Then there exist a positive constant d, such that for all* $t \in R_\tau$

$$\|\xi(t) - \eta(t)\|^2 \leq d(\tau^4 + N^{2(1+\varepsilon-r)})$$

where ε is suitably small.

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