Novi Sad J. Math.
Vol. 35, No. 1, 2005, 187-195

TOPOLOGIES GENERATED BY CLOSED INTERVALS

Miloš S. Kurilić[1] and Aleksandar Pavlović[1]

Abstract. If $\langle L, < \rangle$ is a dense linear ordering without end points and A and B disjoint dense subsets of L, then the topology $\mathcal{O}_{AB}$ on the set L generated by closed intervals $[a, b]$, where $a \in A$ and $b \in B$, is finer than the standard topology, $\mathcal{O}_<$, generated by all open intervals and $\langle L, \mathcal{O}_{AB} \rangle$ is a GO-space. The basic properties of the topology $\mathcal{O}_{AB}$ (separation axioms, cardinal functions, metrizability) are investigated and compared with the corresponding results concerning the standard topology.

AMS Mathematics Subject Classification (2000): 54F05, 03E35, 03E75, 54A10, 54A25, 54D15

Key words and phrases: Linear orderings, topological spaces, closed intervals, GO-spaces, separation axioms, cardinal functions

1. Introduction

If $\langle L, < \rangle$ is a linear order there are several ways to define a topology on the set L using the ordering $<$. Firstly, *the standard topology*, $\mathcal{O}_<$, is generated by the family of all open intervals. Then the space $\langle L, \mathcal{O}_< \rangle$ is called a *linearly ordered topological space* (LOTS). Secondly, following the idea of Sorgenfrey (see [11]), we can observe the topology generated by the family of half-open intervals, i.e. the sets of the form $[x, y)$, where $x, y \in L$. The third way is to generate a topology by closed intervals $[a, b]$, where $a \in A$ and $b \in B$ and where A and B are some subsets of L. Some examples of such a construction are "the two arrows space" of Alexandroff and Uryson ([1], see [3]) and some subspaces of the spaces constructed by Todorčević in [13].

Throughout the paper $\langle L, < \rangle$ will be a dense linear order without end points and A and B disjoint dense subsets of L. Under these assumptions the family

$$\mathcal{B}_{AB} = \{[a, b] : a \in A \wedge b \in B \wedge a < b\}$$

is a base for a topology on the set L, say $\mathcal{O}_{AB}$. (The condition $A \cap B = \emptyset$ ensures the space $\langle L, \mathcal{O}_{AB} \rangle$ has no isolated points and the density of the sets A and B provides the space is Hausdorff.) The aim of the paper is to investigate topological properties of spaces of the form $\langle L, \mathcal{O}_{AB} \rangle$.

[1]Department of Mathematics and Informatics, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia and Montenegro, email: milos@im.ns.ac.yu, apavlovic@im.ns.ac.yu

2. Suborderability and separation axioms

It is easy to prove that, in the spaces of the form $\langle L, \mathcal{O}_{AB}\rangle$, all open intervals are open sets and that the intervals $[a,b]$, where $a \in A$ and $b \in B$, are clopen. So, we have

Fact 1. *Each space of the form $\langle L, \mathcal{O}_{AB}\rangle$ is zero-dimensional, non-compact and the topology $\mathcal{O}_{AB}$ is finer than the standard topology, $\mathcal{O}_<$, on L.*

Consequently, these spaces are $\mathrm{T}_{3\frac{1}{2}}$. But in Theorem 2 we will show that they have much stronger separation properties.

We remind the reader that a topological space $\langle X, \mathcal{O}\rangle$ is called a *suborderable space* if it can be topologically embedded in some LOTS and that a Hausdorff space $\langle X, \mathcal{O}\rangle$ is called *a generalized orderable* (briefly GO-)*space* if there exists a linear order $<$ on X such that $\mathcal{O}_< \subset \mathcal{O}$ and if at each point a neighborhood base consists of intervals. In [2] Čech showed that GO-spaces are the same as suborderable spaces. Now we have

Theorem 1. *Each space of the form $\langle L, \mathcal{O}_{AB}\rangle$ is a GO-space.*

Proof. Let $\tilde{A} = A \times \{1\}$, $\tilde{B} = B \times \{-1\}$, $K = \tilde{A} \cup (L \times \{0\}) \cup \tilde{B}$ and

$$\tilde{L} = \tilde{A} \cup \Big((L \setminus (A \cup B)) \times \{0\}\Big) \cup \tilde{B} = K \setminus \Big((A \cup B) \times \{0\}\Big).$$

Let $\prec$ be the lexicographic order on the set K. One can easily verify that $\tilde{A}$ and $\tilde{B}$ are dense subsets of the set $\tilde{L}$, and therefore $\tilde{L}$ is a dense linear ordering. Also, $\tilde{L}$ has no end points. Let $\mathcal{O}_{\tilde{A}\tilde{B}}$ be the topology on the set $\tilde{L}$ generated by the base $\mathcal{B}_{\tilde{A}\tilde{B}}$ consisting of all sets of the form

$$[\langle a,1\rangle, \langle b,-1\rangle]_{\tilde{L}} = \{x \in \tilde{L} : \langle a,1\rangle \preceq x \preceq \langle b,-1\rangle\},$$

where $a \in A$, $b \in B$ and $a < b$.

It can be easily verified that the mapping $\pi_1 : \langle \tilde{L}, \mathcal{O}_{\tilde{A}\tilde{B}}\rangle \to \langle L, \mathcal{O}_{AB}\rangle$ given by $\pi_1(\langle x, i\rangle) = x$ is a homeomorphism.

Let $\mathcal{O}_\prec$ be the standard topology on K (generated by the ordering $\prec$), and $(\mathcal{O}_\prec)_{\tilde{L}}$ the corresponding induced topology on $\tilde{L}$. Let us show that

$$\mathcal{O}_{\tilde{A}\tilde{B}} = (\mathcal{O}_\prec)_{\tilde{L}}.$$

($\subset$) For $a \in A$, $b \in B$ where $a < b$ it is clear that

$$[\langle a,1\rangle, \langle b,-1\rangle]_{\tilde{L}} = (\langle a,0\rangle, \langle b,0\rangle)_K \cap \tilde{L} \in (\mathcal{O}_\prec)_{\tilde{L}},$$

and therefore $\mathcal{B}_{\tilde{A}\tilde{B}} \subset (\mathcal{O}_\prec)_{\tilde{L}}$, which completes the proof of the first inclusion.

($\supset$) Let $\mathcal{P}_\prec$ be the subbase of the topology $\mathcal{O}_\prec$ consisting of all sets of the form $(x, \rightarrow)_K$ and $(\leftarrow, x)_K$, where $x \in K$. It if sufficient to show that

$$(\mathcal{P}_\prec)_{\tilde{L}} = \{P \cap \tilde{L} : P \in \mathcal{P}_\prec\} \subset \mathcal{O}_{\tilde{A}\tilde{B}}.$$

If $x \in \tilde{L}$, then $(x, \to)_K \cap \tilde{L} = (x, \to)_{\tilde{L}} \in \mathcal{O}_{\tilde{A}\tilde{B}}$ and $(\leftarrow, x)_K \cap \tilde{L} = (\leftarrow, x)_{\tilde{L}} \in \mathcal{O}_{\tilde{A}\tilde{B}}$. Otherwise, $P \cap \tilde{L} \in \mathcal{O}_{\tilde{A}\tilde{B}}$ since

$$\Big(\langle a,0\rangle, \to\Big)_K \cap \tilde{L} = \Big[\langle a,1\rangle, \to\Big)_{\tilde{L}}, \quad \Big(\leftarrow, \langle a,0\rangle\Big)_K \cap \tilde{L} = \Big(\leftarrow, \langle a,1\rangle\Big)_{\tilde{L}},$$

$$\Big(\langle b,0\rangle, \to\Big)_K \cap \tilde{L} = \Big(\langle b,-1\rangle, \to\Big)_{\tilde{L}}, \quad \Big(\leftarrow, \langle b,0\rangle\Big)_K \cap \tilde{L} = \Big(\leftarrow, \langle b,-1\rangle, \Big]_{\tilde{L}},$$

which completes the proof of the second inclusion. □

We note that LOTS are collectionwise normal (see [8] or [12]), hereditarily normal and that some of them are not perfectly normal (T_6) spaces (see [3] 3.12.3.d). Since, by [7], GO-spaces are collectionwise normal and hereditarily normal, by Theorem 1 we have

Theorem 2. *Each space of the form $\langle L, \mathcal{O}_{AB}\rangle$ is collectionwise normal and hereditarily normal.*

Generally, GO-spaces need not to be perfectly normal and the next example shows the same for the spaces of the form $\langle L, \mathcal{O}_{AB}\rangle$.

Example 1. A space $\langle L, \mathcal{O}_{AB}\rangle$ which is not perfectly normal. Let $K = [0,1]^2 \setminus \{\langle 0,0\rangle, \langle 1,1\rangle\}$ and let $<$ be the lexicographic order on K. Then, clearly, $\langle K, <\rangle$ is a dense linear order without end points. Let us divide the set of rational numbers from the interval (0,1) into two disjoint sets Q_1 and Q_2 dense in $(0,1)$. Clearly, the sets $A = [0,1] \times Q_1$ and $B = [0,1] \times Q_2$ are disjoint order-dense subsets of K and we will prove that the space $\langle K, \mathcal{O}_{AB}\rangle$ is not perfectly normal.

It can be easily verified that, for $0 < x < 1$, neighborhood bases at points $\langle x,0\rangle$ and $\langle x,1\rangle$ are $\mathcal{B}(\langle x,0\rangle) = \{(\langle y,0\rangle, \langle x,q_2\rangle] : 0 < y < x \wedge q_2 \in Q_2\}$ and $\mathcal{B}(\langle x,1\rangle) = \{[\langle x,q_1\rangle, \langle y,1\rangle) : q_1 \in Q_1 \wedge 1 > y > x\}$ respectively. The set $[0,1] \times (0,1)$ is open. Let us suppose that it can be represented as a countable union of closed sets. Then some of them, say F, intersects $\mathfrak{c}$ many sets of shape $\{x\} \times [0,1]$.

Let $F_1 = \pi_1[F]$. Clearly $|F_1| = \mathfrak{c}$ and $F_1 \subset [0,1]$. So, regarding the standard topology on $[0,1]$, there exists an accumulation point $x \in (0,1)$ of the set F_1 and also a sequence $\langle x_n : n \in \omega\rangle$ of the elements of the set F_1 which converges to the point x. Without loss of generality we can assume that $\langle x_n : n \in \omega\rangle$ is an increasing, or a decreasing sequence.

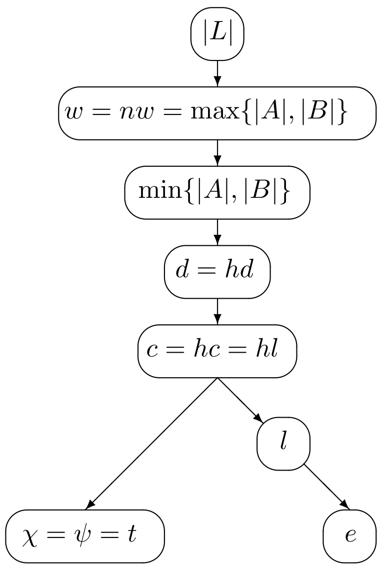
If $\langle x_n : n \in \omega\rangle$ is an increasing sequence then we will show that the set F intersects arbitrary basic neighborhood $U = (\langle y,0\rangle, \langle x,q_2\rangle]$ of the point $\langle x,0\rangle$. Since $y < x$ there exists x_n such that $y < x_n < x$. Also, there exists a point $z \in (\{x_n\} \times (0,1)) \cap F$, and clearly $z \in U$. Since F is a closed set, it contains all of its accumulation points, so $\langle x,0\rangle \in F$, which contradicts the fact that $F \subset [0,1] \times (0,1)$.

Analogously, if $\langle x_n : n \in \omega\rangle$ is a decreasing sequence, it can be proved that $\langle x,1\rangle \in F$, a contradiction again.

3. Cardinal functions

The basic facts concerning cardinal functions can be found in [3] or [4]. If L is a LOTS, then $|L| \geq w = nw \geq d = hd \geq c = hc = hl \geq \chi = \psi = t$ and $c \geq l \geq e$ (see [3] p. 222), and, in addition, for dense LOTS we have $nw = d$. In this section we firstly investigate these cardinal invariants of the topology $\mathcal{O}_{AB}$ and compare them with the corresponding invariants of the standard topology. After that we give some notes on metrizability.

Theorem 3. *The relations between the basic cardinal functions of the spaces of the form $\langle L, \mathcal{O}_{AB} \rangle$ are described by the following diagram.*



Proof. It is known (see [4] or [3] p. 225) that the following inequalities hold in each topological space: $nw \leq w$, $c \leq d \leq hd$, $c \leq hc \leq hl$, $e \leq l \leq hl$, $\psi \leq \chi$ and $t \leq \chi$. Clearly, in spaces $\langle L, \mathcal{O}_{AB} \rangle$ we have $d \leq \min\{|A|, |B|\} \leq \max\{|A|, |B|\} \leq |L|$.

$w = \max\{|A|, |B|\}$. Obviously, $w \leq |\mathcal{B}_{AB}| = \max\{|A|, |B|\}$. Let us suppose that $w = \lambda < \max\{|A|, |B|\} = |A| = \kappa$. Then, by [3], Theorem 1.1.15, there exists a base $\mathcal{B}' \subset \mathcal{B}_{AB}$ such that $|\mathcal{B}'| = \lambda$. Let $\mathcal{B}' = \{[a_i, b_i] : i \in \lambda\}$, let $a' \in A \setminus \{a_i : i \in \lambda\}$ and let b' be an arbitrary element of B greater than a'. The set $[a', b']$ is open, but it can not be represented as the union of some subfamily of $\mathcal{B}'$. A contradiction. If $\max\{|A|, |B|\} = |B|$, the proof is similar.

$nw \geq w$. Suppose $|A| \leq |B|$ and let $\mathcal{N}$ be a net in the space $\langle L, \mathcal{O}_{AB} \rangle$. If $b \in B$, then there is $a \in A$ such that $a < b$ and, clearly, $[a, b]$ is a neighborhood of b. Since $\mathcal{N}$ is a net we can choose $N_b \in \mathcal{N}$ such that $b \in N_b \subset [a, b]$. If $b, b' \in B$ and $b < b'$, then $b' \in N_{b'} \setminus N_b$ so $|\mathcal{N}| \geq |B| = \max\{|A|, |B|\} = w$. If $|A| > |B|$ the proof is similar.

The equality $d = hd$ for GO-spaces has been proved by Skula in [10]. Lutzer in [7] has obtained that $c = hl$ in GO spaces. The proof of $\chi \leq c$ and $\psi = \chi = t$ for GO-spaces can be found in [9] or [5]. □

In the following examples we show that, in ZFC, all the inequalities from the diagram, except $c < d$, can be strict.

Example 2. $w < |L|$. If we divide the set of rational numbers into two disjoint dense subsets Q_1 and Q_2, then $w(\langle \mathbb{R}, \mathcal{O}_{Q_1 Q_2} \rangle) = \aleph_0 < \mathfrak{c} = |\mathbb{R}|$.

Example 3. $d < \min\{|A|, |B|\}$. If we divide the set of irrational numbers into two disjoint dense subsets A and B of cardinality $\mathfrak{c}$, then $d(\langle \mathbb{R}, \mathcal{O}_{AB} \rangle) = |\mathbb{Q}| = \aleph_0 < \mathfrak{c} = |A| = |B|$.

Example 4. $l < c$. Let us consider the space K defined in Example 1. Since $\{(\langle x, 0\rangle, \langle x, 1\rangle) : x \in [0,1]\}$ is a $\mathfrak{c}$-sized cellular family, we have $c(K) = \mathfrak{c}$.

Let $K = \bigcup_{i \in I} [a_i, b_i]$, where $[a_i, b_i] \in \mathcal{B}_{AB}$, let $G = ((0,1] \times \{0\}) \cup ([0,1] \times \{1\})$ and $J = \{i \in I : [a_i, b_i] \cap G \neq \emptyset\}$. Then for $i \in J$ we have $\pi_1[[a_i, b_i] \cap G] = [x_i, y_i]_{[0,1]}$, where $x_i = \pi_1(a_i)$ and $y_i = \pi_1(b_i)$ and $0 \leq x_i < y_i \leq 1$. Clearly $[0,1] = \bigcup_{i \in J} [x_i, y_i]$ and there exists a countable set $J' \subset J$ such that $\bigcup_{i \in J} (x_i, y_i) = \bigcup_{i \in J'} (x_i, y_i)$. For the set $P = [0,1] \setminus \bigcup_{i \in J} (x_i, y_i)$ we have $P \subset \{x_i : i \in J\} \cup \{y_i : i \in J\}$. If $x_i, x_j \in P \cap \{x_i : i \in J\}$ and $x_i < x_j$, then $y_i \leq x_j$ (since otherwise $x_j \notin P$) so $(x_i, y_i) \cap (x_j, y_j) = \emptyset$. Consequently $P \cap \{x_i : i \in J\}$ (and similarly $P \cap \{y_i : i \in J\}$) is a countable set, so $|P| \leq \aleph_0$. Since $\bigcup_{i \in J'} (x_i, y_i) \times [0,1] \subset \bigcup_{i \in J'} [a_i, b_i]$, for a point $\langle x, y\rangle \in K \setminus \bigcup_{i \in J'} [a_i, b_i]$ we have $x \in P$. The set $P \times \mathbb{Q}$ is countable, so, it remains to cover such points $\langle x, y\rangle$, where $y \notin \mathbb{Q}$. For such $\langle x, y\rangle$ there are $q_1 \in Q_1$ and $q_2 \in Q_2$ such that $\langle x, y\rangle \in (\langle x, q_1\rangle, \langle x, q_2\rangle) \subset [a_i, b_i]$ for some $i \in I$. Since there are countably many open sets of the form $(\langle x, q_1\rangle, \langle x, q_2\rangle)$, where $x \in P$, $q_1 \in Q_1$ and $q_2 \in Q_2$, all the points $\langle x, y\rangle \in K \setminus \bigcup_{i \in J'} [a_i, b_i]$ can be covered by countably many intervals $[a_i, b_i]$. Thus $l(K, \mathcal{O}_{AB}) = \aleph_0$.

Example 5. $\chi < c$ and $\chi < l$. Let κ be an infinite cardinal. The set $Q(\kappa) = \kappa \times \mathbb{Q}$ with the lexicographic order $<$ is a dense linearly ordered set without end points. Let us divide the set of rational numbers into two disjoint dense subsets Q_1 and Q_2. The sets $A = \kappa \times Q_1$ and $B = \kappa \times Q_2$ are dense in $Q(\kappa)$. Clearly, the sets $Q_\alpha = \{\alpha\} \times \mathbb{Q}$, $\alpha \in \kappa$, are open in the space $\langle Q(\kappa), \mathcal{O}_{AB}\rangle$ and $Q_\alpha \cap Q_\beta = \emptyset$ for $\alpha \neq \beta$, so $Q(\kappa) = \bigcup_{\alpha < \kappa} Q_\alpha$. This implies $l(Q(\kappa)) = c(Q(\kappa)) = \kappa$.

Let $x = \langle \alpha, q\rangle \in Q(\kappa)$. Clearly, the family $\mathcal{B}(x) = \{[\langle \alpha, a\rangle, \langle \alpha, b\rangle] : a \leq q \leq b \wedge a \in Q_1 \wedge b \in Q_2\}$ is a countable neighborhood base at the point x, so, we have $\chi(Q(\kappa)) = \aleph_0$.

Now, for $\kappa > \aleph_0$ we have $\chi(Q(\kappa)) < c(Q(\kappa))$ and $\chi(Q(\kappa)) < l(Q(\kappa))$.

Example 6. $l < \chi$. Let the set $L = ((\omega_1 + 1) \times [0,1)_{\mathbb{Q}}) \setminus \{\langle 0, 0\rangle\}$ be ordered lexicographically and let $Q_1, Q_2 \subset (0,1)_{\mathbb{Q}}$ be disjoint sets dense in $(0,1)_{\mathbb{Q}}$. Then, clearly, the sets $A = (\omega_1 + 1) \times Q_1$ and $B = (\omega_1 + 1) \times Q_2$ are dense subsets of L.

Firstly, $\chi(\langle \omega_1, 0 \rangle) > \aleph_0$, since if the point $\langle \omega_1, 0 \rangle$ would have countable character, then it would have a neighborhood base of the form $\{[a_n, b_n] : n \in \omega\}$ and the set $\{\pi_1(a_n) : n \in \omega\}$ would be a cofinal subset of ω_1, which is impossible.

Secondly, we show $l(L) = \aleph_0$. let $L = \bigcup_{i \in I} [a_i, b_i]$ where $[a_i, b_i] \in \mathcal{B}_{AB}$ and let $\langle \omega_1, 0 \rangle \in [a_{i_0}, b_{i_0}] = [\langle \alpha, p \rangle, \langle \omega_1, q \rangle]$. Then $\alpha < \omega_1$ and consequently $L \setminus [a_{i_0}, b_{i_0}]$ is a countable set which can be covered by countably many members of the given cover.

Example 7. $e < l$. Let the set $L = (\omega_1 \times [0, \rightarrow)_{\mathbb{Q}}) \setminus \{\langle 0, 0 \rangle\}$ (that is $L = \bigcup_{\alpha < \omega_1} Q_\alpha$ where $Q_0 = \{0\} \times (0, \rightarrow)_{\mathbb{Q}}$ and $Q_\alpha = \{\alpha\} \times [0, \rightarrow)_{\mathbb{Q}}$, for $0 < \alpha < \omega_1$) be ordered lexicographically and let the sets $A', B' \subset (0, \rightarrow)_{\mathbb{Q}}$ be disjoint and dense in $(0, \rightarrow)_{\mathbb{Q}}$. Then, clearly, the sets $A = \omega_1 \times A'$ and $B = \omega_1 \times B'$ are dense in L and we observe the space $\langle L, \mathcal{O}_{AB} \rangle$. Since the open cover $L = \bigcup_{x \in L} (\leftarrow, x)_L$ does not contain a countable subcover, we have $l(L, \mathcal{O}_{AB}) = \aleph_1$.

Suppose that there exists a closed and discrete subset $D \subset L$ such that $|D| = \aleph_1$. Since $|Q_\alpha| = \aleph_0$, the set $C = \{\alpha < \omega_1 : D \cap Q_\alpha \neq \emptyset\}$ is of cardinality $\aleph_1$. Let $\langle \alpha_n : n \in \omega \rangle$ be an increasing sequence of elements of C and $\alpha = \sup\{\alpha_n : n \in \omega\}$. If U is a neighborhood of the point $\langle \alpha, 0 \rangle$ then there are $a = \langle \xi, q \rangle \in A$ and $b \in B$ such that $\langle \alpha, 0 \rangle \in [a, b] \subset U$. Clearly we have $\xi < \alpha$ so there exists $n \in \omega$ such that $\xi < \alpha_n < \alpha$. Consequently, there is $q \in \mathbb{Q}$ such that $\langle \alpha_n, q \rangle \in D \cap U$. Thus $\langle \alpha, 0 \rangle \in \overline{D} = D$. But now the point $\langle \alpha, 0 \rangle$ is not isolated in D, which is impossible, since the set D is discrete. The equality $e(L, \mathcal{O}_{AB}) = \aleph_0$ is proved.

The inequality $c < d$ will be considered later in this section. Now we compare the cardinal invariants of the standard topology and of the topology $\mathcal{O}_{AB}$.

Theorem 4. *Let $\langle L, < \rangle$ be a dense linear ordering without end points and $A, B \subset L$ disjoint dense subsets of L. Then*

a) $w(L, \mathcal{O}_<) = d(L, \mathcal{O}_<) = d(L, \mathcal{O}_{AB})$;
b) $c(L, \mathcal{O}_<) = c(L, \mathcal{O}_{AB})$;
c) $\chi(L, \mathcal{O}_<) \geq \chi(L, \mathcal{O}_{AB})$;
d) $l(L, \mathcal{O}_<) \leq l(L, \mathcal{O}_{AB})$;
e) $e(L, \mathcal{O}_<) \leq e(L, \mathcal{O}_{AB})$.

Proof. Since $\mathcal{O}_< \subset \mathcal{O}_{AB}$ we have $d(L, \mathcal{O}_<) \leq d(L, \mathcal{O}_{AB})$, $c(L, \mathcal{O}_<) \leq c(L, \mathcal{O}_{AB})$, $l(L, \mathcal{O}_<) \leq l(L, \mathcal{O}_{AB})$ and $e(L, \mathcal{O}_<) \leq e(L, \mathcal{O}_{AB})$. Clearly, the density of L implies the first equality in a).

$d(L, \mathcal{O}_<) \geq d(L, \mathcal{O}_{AB})$. If the set D is dense in $\langle L, \mathcal{O}_< \rangle$, then it intersects each open interval. Consequently, $D \cap [a, b] \neq \emptyset$, for each $[a, b] \in \mathcal{B}_{AB}$, so D is a dense set in the space $\langle L, \mathcal{O}_{AB} \rangle$.

$c(L, \mathcal{O}_<) \geq c(L, \mathcal{O}_{AB})$. If $\mathcal{C}$ is a cellular family in the space $\langle L, \mathcal{O}_{AB} \rangle$ then for each $C \in \mathcal{C}$ there are $a_C \in A$ and $b_C \in B$ such that $[a_C, b_C] \subset C$. Since L is a dense ordering, $\{(a_C, b_C) : C \in \mathcal{C}\}$ is a cellular family in $\langle L, \mathcal{O}_< \rangle$ of size $|\mathcal{C}|$, so $|\mathcal{C}| \leq c(L, \mathcal{O}_<)$.

$\chi(L, \mathcal{O}_<) \geq \chi(L, \mathcal{O}_{AB})$. Let $x \in L$, $\mathrm{cf}((\leftarrow, x)) = \kappa$ and $\mathrm{cf}((x, \rightarrow)^*) = \lambda$. Then $\chi(x, (L, \mathcal{O}_<)) = \kappa \cdot \lambda$, while the character of the point x with respect to the topology $\mathcal{O}_{AB}$ is κ (if $x \in B$), λ (if $x \in A$) or $\kappa \cdot \lambda$ (if $x \in L \setminus (A \cup B)$). □

We remark that, according to Theorem 3, for the spaces $\langle L, \mathcal{O}_< \rangle$ and $\langle L, \mathcal{O}_{AB} \rangle$ the cardinal functions *hd*, *hc*, and *hl* are equal. In the following example we show that the inequalities given in the previous theorem can be strict.

Example 8. $l(L, \mathcal{O}_<) < l(L, \mathcal{O}_{AB})$, $\chi(L, \mathcal{O}_<) > \chi(L, \mathcal{O}_{AB})$ and $l(L, \mathcal{O}_<) < e(L, \mathcal{O}_{AB})$. Let the set $L = ((\omega_1 + 1) \times [0, 1)) \setminus \{\langle 0, 0 \rangle\}$ be ordered lexicographically, let A be the set of the elements of L having the second coordinate rational and let $B = L \setminus A$. Then it is easy to prove that $l(L, \mathcal{O}_<) = \aleph_0$, while $L = \bigcup_{\alpha < \omega_1} (\leftarrow, \langle \alpha, 0 \rangle) \cup [\langle \omega_1, 0 \rangle, \rightarrow)$ is an open cover of L having no countable subcover. Considering the point $a = \langle \omega_1, 0 \rangle$ we conclude that $\chi(a, (L, \mathcal{O}_<)) = \aleph_1$, while $\chi(L, \mathcal{O}_{AB}) = \aleph_0$. Since the set $E = \{\langle \alpha, 0 \rangle : 0 < \alpha < \omega_1\}$ is closed and discrete in $\langle L, \mathcal{O}_{AB} \rangle$ we have $e(L, \mathcal{O}_{AB}) = \aleph_1 > l(L, \mathcal{O}_<)$.

What about the inequality $c(L, \mathcal{O}_{AB}) < d(L, \mathcal{O}_{AB})$? For the answer we need the following fact.

Fact 2. *If $\langle L, < \rangle$ is a dense linear ordering without end points, then there are disjoint dense subsets $A, B \subset L$.*

Proof. An interval $(a, b) \subset L$ will be called stable if each its subinterval is of size $|(a, b)|$. Firstly, since there is no infinite decreasing sequence of cardinals, each interval in L contains a stable subinterval. Secondly, if the interval (a, b) is stable, then there are disjoint subsets $A, B \subset (a, b)$ which are dense in (a, b). (If $|(a, b)| = \kappa$ and if I_α, $\alpha < \kappa$, is an enumeration of all subintervals of (a, b), then we define $A = \{a_\alpha : \alpha < \kappa\}$ and $B = \{b_\alpha : \alpha < \kappa\}$ picking different a_α and b_α from $I_\alpha \setminus (\{a_\beta : \beta < \alpha\} \cup \{b_\beta : \beta < \alpha\}$.) By Zorn's Lemma the partial ordering $\langle \mathbb{P}, \subset \rangle$, where $\mathbb{P} = \{\mathcal{I} : \mathcal{I}$ is a disjoint family of stable intervals$\}$ has a maximal element, say $\mathcal{I}^* = \{I_j : j \in J\}$. For $j \in J$, let A_j and B_j be disjoint dense subsets of I_j. The sets $A = \bigcup_{j \in J} A_j$ and $B = \bigcup_{j \in J} B_j$ are disjoint and we prove that they are dense subsets of L. If $a, b \in L$ and $a < b$, and if I is a stable subinterval of (a, b), then, by the maximality of $\mathcal{I}^*$ we have $I \cap I_{j_0} \neq \emptyset$, for some $j_0 \in J$. Since $I \cap I_{j_0}$ is a subinterval of I_{j_0} it contains an element of A_{j_0}, thus $A \cap (a, b) \neq \emptyset$. Similarly, $B \cap (a, b) \neq \emptyset$. □

Theorem 5. *The following conditions are equivalent:*

a) There exists a dense linear ordering without end points, $\langle L, < \rangle$, having disjoint dense subsets A and B such that $c(L, \mathcal{O}_{AB}) < d(L, \mathcal{O}_{AB})$;

b) ¬ GSH (that is, there exists a linearly ordered continuum $\langle L, < \rangle$ satisfying $c(L, \mathcal{O}_<) < d(L, \mathcal{O}_<)$).

Proof. (a $\Rightarrow$ b) Let condition (a) hold. Then, by Theorem 4, there holds $c(L, \mathcal{O}_<) < d(L, \mathcal{O}_<)$, so $\langle L, < \rangle$ is a dense $d(L, \mathcal{O}_<)$-Suslin line and its Dedekind completion is a $d(L, \mathcal{O}_<)$-Suslin continuum (see [14] p. 274).

(b $\Rightarrow$ a) Let $\langle L, < \rangle$ be a κ-Suslin continuum. By Fact 2 there are disjoint dense sets $A, B \subset L$ and by Theorem 4 we have $c(L, \mathcal{O}_{AB}) < d(L, \mathcal{O}_{AB})$. □

Similarly, considering ccc linear orderings we have:

Theorem 6. *SH* $\Leftrightarrow$ *For each dense linear ordering without end points,* $\langle L, < \rangle$*, and each disjoint dense sets* $A, B \subset L$ *there holds: if* $c(L, \mathcal{O}_{AB}) = \aleph_0$ *then* $d(L, \mathcal{O}_{AB}) = \aleph_0$*.*

It is well known that SH is independent even of ZFC+CH (Namely, Jensen showed the consistency of ZFC+GCH+SH, while $\neg$SH follows from $V = L$.)

Now a few words on metrizability.

Theorem 7. *If* $|A| = |B| = \aleph_0$*, then the space* $\langle L, \mathcal{O}_{AB} \rangle$ *is metrizable. If the space* $\langle L, \mathcal{O}_{AB} \rangle$ *is metrizable, then* $|A| = |B|$*.*

Proof. If $|A| = |B| = \aleph_0$, then $w(L) = \aleph_0$ and by the Urison Metrization Theorem, (see [3] Theorem 4.2.9), every second countable Tychonov space is metrizable.

If L is a metrizable space, then $w(L) = d(L)$, so, by Theorem 3, we have $\max\{|A|, |B|\} = \min\{|A|, |B|\}$. □

In the following examples we will show that the reversed implications need not to be valid.

Example 9. For the space defined in Example 3 we have $|A| = |B| = \mathfrak{c}$, so $w(L) = \mathfrak{c}$. But this space is separable, so it is not metrizable.

Example 10. The space $Q(\kappa)$ defined in Example 5 can be represented as a sum of spaces $\{\alpha\} \times \mathbb{Q}$, where $\alpha \in \kappa$. These spaces are metrizable (Theorem 7), hence $Q(\kappa)$, as the sum of metrizable spaces, is metrizable, while $|A| = |B| = \kappa$.

We note that some properties of the spaces of the form $\langle L, \mathcal{O}_{AB} \rangle$ when $L = \mathbb{R}$ can be found in [6].

Acknowledgment

Research is supported by the MNŽS (Project 1768: Forcing, Model Theory and Set-theoretic Topology).

References

[1] Alexandroff, P., Uryson, P., Mémoire sur les espaces topologiques compacts. Amsterdam: Verh. Akad. Wetensch. 14 1929.

[2] Čech, E., Topological spaces, (Revised edition by Z. Frolk and M. Katětov. Scientific editor, V. Pták. Editor of the English translation, C. O. Junge), Prague: Publishing House of the Czechoslovak Academy of Sciences; London, New York, Sydney: Interscience Publishers John Wiley & Sons 1966.

[3] Engelking, R., General Topology. Berlin: Heldermann Verlag 1989.

[4] Hodel, R., Cardinal functions I. In: Handbook of Set-Theoretic Topology (K. Kunen, J. E. Vaughan, eds.), pp. 1-61. Amsterdam: North-Holland 1984.

[5] Juhász, I., Cardinal functions in topology. Amsterdam: Mathematisch Centrum, 1971.

[6] Kurilić, M. S., Pavlović, A., A consequence of the Proper Forcing Axiom in topology. Publ. Math. Debrecen 64/1-2 (2004), 15-20.

[7] Lutzer, D., On generalized ordered spaces. Dissertationes Math. Rozprawy Mat. 89 (1971), 32pp.

[8] Mansfield, M. J., Some generalizations of full normality. Trans. Amer. Math. Soc. 86 (1957), 489-505.

[9] Meyer, P. R., Wilson, R. G., Cardinal functions on topological spaces satisfying order conditions. Houston J. Math. 5 (1979), No. 1, 107-118.

[10] Skula, L., Hereditary m-separability of an ordered space. Časopis Pěst. Mat. 90 (1965), 451-455.

[11] Sorgenfrey, R., On the topological product of paracompact spaces, Bull. Amer. Math. Soc. 53 (1947), 631-632.

[12] Steen, L. A., A direct proof that a linearly ordered space is hereditarily collectionwise normal. Proc. Amer. Math. Soc. 24 (1970), 727-728.

[13] Todorčević, S., Aronszajn orderings. Djuro Kurepa memorial volume, Publ. Inst. Math. (Beograd) (N.S.) 57(71)(1995), 29-46.

[14] Todorčević, S., Trees and Linearly Ordered Sets. In: Handbook of Set-Theoretic Topology, (K. Kunen, J. E. Vaughan, eds.), pp. 235-293. Amsterdam: North-Holland 1984.

Received by the editors May 30, 2005