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SOME SPACES OF LACUNARY CONVERGENT SEQUENCES DEFINED BY ORLICZ FUNCTIONS

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Abstract. We introduce the sequence spaces [ŵ(M)] and $[\hat{w}(M)]_\theta$ of strong almost convergence and lacunary strong almost convergence respectively defined by the Orlicz function M. We establish certain inclusion relations and show that the above spaces are same for any bounded sequences.

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1. Definitions and notations

A sequence $x \in l_\infty$, the space of bounded sequences $x = (x_k)$, is said to be almost convergent [5] to s if

$$\lim_{k\to\infty} t_{km}(x) = s$$

uniformly in m, where

$$t_{km}(x) = \frac{1}{k+1}\sum_{i=0}^{k} x_{m+i}.$$

Recently, Das and Sahoo [2] introduced the following sequence spaces using the concept of almost convergence.

$$\hat{w} = \left\{ x : \lim_n \frac{1}{n+1}\sum_{k=0}^{n} t_{km}(x-s) = 0, \text{ uniformly in } m, \text{ for some } s \right\},$$

$$[\hat{w}] = \left\{ x : \lim_n \frac{1}{n+1}\sum_{k=0}^{n} |t_{km}(x-s)| = 0, \text{ uniformly in } m, \text{ for some } s \right\}.$$

By a *lacunary sequence* $\theta = (k_r)$, $r = 0, 1, 2, \cdots$, where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers $h_r = (k_r - k_{r-1}) \to \infty$, $(r \to \infty)$. The intervals determined by θ are denoted by $I_r = (k_{r-1} - k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r.

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A sequence x is said to be *lacunary* $[\hat{w}]$*-convergent* to the value s (cf. [1]) if

$$\lim_{r} \sup_{m} \frac{1}{h_r} \sum_{k \in I_r} |t_{km}(x-s)| = 0.$$

By $[\hat{w}]_\theta$, we denote the set of all lacunary $[\hat{w}]$-convergent sequences and we write $[\hat{w}]_\theta - \lim x = s$, for $x \in [\hat{w}]_\theta$.

An *Orlicz function* is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0,\ M(x) > 0\ for\ x > 0\ and\ M(x) \to \infty$ as $x \to \infty$. If convexity of the Orlicz function M is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called a modulus function defined and discussed by Ruckle [7] and Maddox [6].

Lindenstrauss and Tzafriri [4] used the idea of Orlicz function to construct the sequence space

$$l_M = \left\{ x = w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty,\ \text{for some } \rho > 0 \right\}.$$

The space l_M with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

becomes a Banach space which is called an *Orlicz sequence space*, where w is the space of all real or complex sequences $x = x_k$. An Orlicz function M can always be represented (see Krasnoselskii and Rutitsky [3]) in the integral form $M(x) = \int_0^x q(t)dt$, where q, known as the kernel of M, is right differentiable for $t \geq 0$, $q(0) = 0$, $q(t) > 0$ for $t > 0$, q is non-decreasing, $q(t) \to \infty$ as $t \to \infty$. The space l_M that is closely related to the space l_p is an Orlicz sequence space with $M(x) = x^p, x \geq 0, 1 \leq p < \infty$.

In the present paper we introduce and examine the following sequence spaces defined by an Orlicz function.

For some $\rho > 0$,

$$\begin{aligned} [\hat{w}(M)] \quad = \quad & \Big\{ x : \frac{1}{n+1} \sum_{k=0}^{n} M\left(\frac{|t_{km}(x-s)|}{\rho}\right) \to 0, (n \to \infty), \\ & \text{uniformly in } m,\ \text{for some } s \Big\}, \end{aligned}$$

$$[\hat{w}(M)]_\theta = \Big\{ x : \sup_{m} \frac{1}{h_r} \sum_{k \in I_r} M\left(\frac{|t_{km}(x-s)|}{\rho}\right) \to 0, (r \to \infty),\ \text{for some } s \Big\},$$

where M is an Orlicz function.
If $M(x) = x$, then $[\hat{w}(M)] = [\hat{w}]$ and $[\hat{w}(M)]_\theta = [\hat{w}]_\theta$.

2.

In this section we prove some inclusion relations.

Theorem 2.1. *Let* $\theta = (k_r)$ *be a lacunary sequence with* $\liminf q_r > 1$. *Then* $[\hat{w}(M)] \subset [\hat{w}(M)]_\theta$ *and* $[\hat{w}(M)] - \lim x = [\hat{w}(M)]_\theta - \lim x$.

Proof. Let $\liminf q_r > 1$. Then there exists $\delta > 0$ such that $q_r > 1 + \delta$ and hence

$$\frac{h_r}{k_r} = 1 - \frac{k_{r-1}}{k_r} > 1 - \frac{1}{1+\delta} = \frac{\delta}{1+\delta}.$$

Therefore,

$$\begin{aligned} \frac{1}{k_r}\sum_{i=1}^{k_r} M\left(\frac{|t_{im}(x-s)|}{\rho}\right) &\geq \frac{1}{k_r}\sum_{i\in I_r} M\left(\frac{|t_{im}(x-s)|}{\rho}\right) \\ &\geq \frac{\delta}{1+\delta}\frac{1}{h_r}\sum_{i\in I_r} M\left(\frac{|t_{im}(x-s)|}{\rho}\right), \end{aligned}$$

and if $x \in [\hat{w}(M)]$ with $[\hat{w}(M)] - \lim x = s$, then it follows that $x \in [\hat{w}(M)]_\theta$ with $[\hat{w}(M)]_\theta - \lim x = s$.

This completes the proof of the theorem. □

Theorem 2.2. *Let* $\theta = (k_r)$ *be a lacunary sequence with* $\limsup q_r < \infty$. *Then* $[\hat{w}(M)]_\theta \subset [\hat{w}(M)]$ *and* $[\hat{w}(M)]_\theta - \lim x = [\hat{w}(M)] - \lim x$.

Proof. Let $x \in [\hat{w}(M)]_\theta$ with $[\hat{w}(M)]_\theta - \lim x = s$. Then for $\varepsilon > 0$, there exists j_0 such that for every $j \geq j_0$ and all m,

$$g_{jm} = \frac{1}{h_j}\sum_{i\in I_j} M\left(\frac{|t_{im}(x-s)|}{\rho}\right) < \varepsilon,$$

that is, we can find some positive constant C, such that

$$g_{jm} < C \tag{1}$$

for all j and m. $\limsup q_r < \infty$ (given) implies that there exists some positive number K such that

$$q_r < K \text{ for all } r \geq 1. \tag{2}$$

Therefore, for $k_{r-1} < n \leq k_r$, we have by (1) and (2)

$$\begin{aligned} \frac{1}{n+1}\sum_{i=0}^{n} M\left(\frac{|t_{im}(x-s)|}{\rho}\right) &\leq \frac{1}{k_{r-1}}\sum_{i=0}^{k_r} M\left(\frac{|t_{im}(x-s)|}{\rho}\right) \\ &= \frac{1}{k_{r-1}}\sum_{j=0}^{r}\sum_{i\in I_j} M\left(\frac{|t_{im}(x-s)|}{\rho}\right) \end{aligned}$$

$$= \frac{1}{k_{r-1}}\left[\sum_{j=0}^{j_0}\sum_{j=j_0+1}^{r}\right]\sum_{i\in I_j} M\left(\frac{|t_{im}(x-s)|}{\rho}\right)$$
$$\leq \frac{1}{k_{r-1}}\left(\sup_{l\leq p\leq j_0} g_{pm}\right)k_{j_0}+\varepsilon\left(k_r-k_{j_0}\right)\frac{1}{k_{r-1}}$$
$$\leq C\frac{k_{j_0}}{k_{r-1}}+\varepsilon K.$$

Since $k_{r-1}\to\infty$ as $r\to\infty$, we get $x\in[\hat{w}(M)]$ with $[\hat{w}(M)]-\lim x = s$.

This completes the proof of the theorem. □

Theorem 2.3. *Let* $1<\liminf q_r\leq\limsup q_r<\infty$. *Then* $[\hat{w}(M)]=[\hat{w}(M)]_\theta$.

Proof. It follows from Theorem 2.1 and Theorem 2.2. □

Theorem 2.4. *Let* $x\in[\hat{w}(M)]\cap[\hat{w}(M)]_\theta$. *Then* $[\hat{w}(M)]-\lim x=[\hat{w}(M)]_\theta-\lim x$ *and* $[\hat{w}(M)]_\theta-\lim x$ *is unique for any lacunary sequence* $\theta=(k_r)$.

Proof. Let $x\in[\hat{w}(M)]\cap[\hat{w}(M)]_\theta$ and $[\hat{w}(M)]-\lim x=s$, $[\hat{w}(M)]_\theta-\lim x=s'$.

Suppose $s\neq s'$. We see that

$$M\left(\frac{|s-s'|}{\rho}\right) \leq \frac{1}{h_r}\sum_{i\in I_r} M\left(\frac{|t_{im}(x-s)|}{\rho}\right)$$
$$+ \frac{1}{h_r}\sum_{i\in I_r} M\left(\frac{|t_{im}(x-s')|}{\rho}\right),\ \text{for each } m$$
$$\leq \limsup_{r}\sup_{m}\frac{1}{h_r}\sum_{i\in I_r} M\left(\frac{|t_{im}(x-s)|}{\rho}\right)+0.$$

Hence there exists r_0, such that, for $r>r_0$,

$$\frac{1}{h_r}\sum_{i\in I_r} M\left(\frac{|t_{im}(x-s)|}{\rho}\right) > \frac{1}{2}M\left(\frac{|s-s'|}{\rho}\right).$$

Since $[\hat{w}(M)]-\lim x=s$, it follows that $0\geq\limsup(\frac{h_r}{k_r})M\left(\frac{|s-s'|}{\rho}\right)\geq \liminf(\frac{h_r}{k_r})M\left(\frac{|s-s'|}{\rho}\right)\geq 0$ and so, $\lim q_r=1$. Hence by Theorem 2.2, $[\hat{w}(M)]_\theta\subset[\hat{w}(M)]_\theta$ and $[\hat{w}(M)]_\theta-\lim x=s'=s=[\hat{w}(M)]-\lim x$.

Further,

$$\frac{1}{n+1}\sum_{i=0}^{n} M\left(\frac{|t_{im}(x-s)|}{\rho}\right)+\frac{1}{n+1}\sum_{i=0}^{n} M\left(\frac{|t_{im}(x-s')|}{\rho}\right)\geq M\left(\frac{|s-s'|}{\rho}\right)\geq 0$$

and taking the limit on both sides as $n\to\infty$, we have $M\left(\frac{|s-s'|}{\rho}\right)=0$ i.e., $s=s'$ for any Orlicz function M.

This completes the proof. □

3.

In this section, we show that for any bounded sequences, the sequence spaces $[\hat{w}(M)]$ and $[\hat{w}(M)]_\theta$ are the same.

Before we prove our main result, we prove a lemma.

Lemma 3.1. *Suppose for a given $\varepsilon > 0$, there exists n_0 and m_0, such that*

(3) $\frac{1}{n}\sum_{k=0}^{n-1} M\left(\frac{|t_{km}(x-s)|}{\rho}\right) < \varepsilon$ *for all* $n \geq n_0, m \geq m_0$ *Then,* $x \in [\hat{w}(M)]$.

Proof. Let $\varepsilon > 0$ be given. Choose n_0', m_0 such that

$$\frac{1}{n}\sum_{k=0}^{n-1} M\left(\frac{|t_{km}(x-s)|}{\rho}\right) < \frac{\varepsilon}{4}$$

for $n \geq n_0'$, $m \geq m_0$.

It is enough to show that there exists n_0'' such that for $n \geq n_0''$, $0 \leq m \leq m_0$, we have

$$\frac{1}{n}\sum_{k=0}^{n-1} M\left(\frac{|t_{km}(x-s)|}{\rho}\right) < \varepsilon.$$

Since m_0 is fixed, put $\sum_{k=0}^{m_0-1} \frac{1}{k} \sum_{j=0}^{m_0-1} M\left(\frac{|(x_j-s)|}{\rho}\right) = B$.

Now, let $0 \leq m \leq m_0$ and $n > m_0$, then

$$\begin{aligned}
\frac{1}{n}\sum_{k=0}^{n-1} M\left(\frac{|t_{km}(x-s)|}{\rho}\right) &\leq \frac{1}{n}\sum_{k=0}^{m_0-1}\frac{1}{k}\sum_{j=0}^{m_0-1} M\left(\frac{|(x_j-s)|}{\rho}\right) \\
&\quad + \frac{1}{n}\sum_{k=0}^{m_0-1}\left|\frac{1}{k}\sum_{j=m_0}^{m+k-1} M\left(\frac{(x_j-s)}{\rho}\right)\right| \\
&\quad + \frac{1}{n}\sum_{k=m_0}^{n-1}\frac{1}{k}\sum_{j=m}^{m+k-1} M\left(\frac{|(x_j-s)|}{\rho}\right) \qquad (4) \\
&\leq \frac{B}{n} + \frac{1}{n}\sum_{k=0}^{m_0-1}\left|\frac{1}{k}\sum_{j=m_0}^{m_0+(k+m-m_0)-1} M\left(\frac{(x_j-s)}{\rho}\right)\right| \\
&\quad + \frac{1}{n}\sum_{k=m_0}^{n-1}\left|\frac{1}{k}\sum_{j=m}^{m+k-1} M\left(\frac{(x_j-s)}{\rho}\right)\right|.
\end{aligned}$$

Let $k - m_0 > n_0'$. Then, for $0 \leq m \leq m_0$, we have $k + m - m_0 \geq n_0'$. Then from (3)

$$(5) \qquad \frac{1}{m_0}\sum_{k=0}^{m_0-1}\left|\frac{1}{k+m-m_0}\sum_{j=m_0}^{m_0+(k+m-m_0)-1} M\left(\frac{(x_j-s)}{\rho}\right)\right| < \frac{\varepsilon}{4}.$$

From (4) and (5),

$$\frac{1}{n}\sum_{k=0}^{n-1} M\left(\frac{|t_{km}(x-s)|}{\rho}\right) \leq \frac{B}{n} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon;$$

for a sufficiently large n. Hence the result. $\square$

Theorem 3.1. *For every lacunary sequence $\theta = (k_r)$, we have $[\hat{w}(M)]_\theta \cap l_\infty = [\hat{w}(M)]$.*

Proof. Let x $\in [\hat{w}(M)]_\theta \cap l_\infty$. For $\varepsilon > 0$, then there exist r_0 and q_0, such that

$$\frac{1}{h_r}\sum_{k=0}^{h_r-1} M\left(\frac{|t_{kq}(x-s)|}{\rho}\right) < \frac{\varepsilon}{2} \tag{6}$$

for $r \geq r_0$ and $q \geq q_0$, $q = k_{r-1} + 1 + i,\ i \geq 0$.

Now, let $n \geq h_r$, m be an integer greater than or equal to 1. Then,

$$\begin{aligned} \frac{1}{n}\sum_{k=0}^{n-1} M\left(\frac{|t_{kq}(x-s)|}{\rho}\right) &\leq \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{k}\sum_{k=0}^{m-1}\left|\sum_{j=q+\mu h_r}^{q+(\mu+1)h_r-1} M\left(\frac{(x_j-s)}{\rho}\right)\right| \\ &\quad + \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{k}\sum_{j=q+mh_r}^{q+k-1} M\left(\frac{(x_j-s)}{\rho}\right) \\ &\leq \frac{1}{n}\sum_{\mu=0}^{m-1}\sum_{k=\mu h_r}^{(\mu+1)h_r-1}\frac{1}{k}\left|\sum_{j=q}^{q+k-1} M\left(\frac{(x_j-s)}{\rho}\right)\right| \\ &\quad + \frac{1}{n}\sum_{k=mh_r}^{n-1}\frac{1}{k}\sum_{j=q}^{q+k-1} M\left(\frac{|(x_j-s)|}{\rho}\right) \end{aligned} \tag{7}$$

Since $x \in l_\infty$, for all j, $M\left(\frac{|(x_j-s)|}{\rho}\right) < B$.

So from (6) and (7), we have

$$\frac{1}{n}\sum_{k=0}^{n-1} M\left(\frac{|t_{kq}(x-s)|}{\rho}\right) \leq \frac{1}{n} m h_r \frac{\varepsilon}{2} + \frac{Bh_r}{n}.$$

For $\frac{h_r}{n} \leq 1$, $\frac{Bh_r}{n}$ can be made less than $\frac{\varepsilon}{2}$ by taking n sufficiently large and since $\frac{mh_r}{n} \leq 1$, then

$$\frac{1}{n}\sum_{k=0}^{n-1} M\left(\frac{|t_{kq}(x-s)|}{\rho}\right) < \varepsilon$$

for $r \geq r_0$, $q \geq q_0$. Hence by Lemma 3.1, $[\hat{w}(M)]_\theta \cap l_\infty \subset [\hat{w}(M)]$. It is trivial that $[\hat{w}(M)] \subset [\hat{w}(M)]_\theta \cap l_\infty$. Hence the result. $\square$

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