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REMARKS ON THE EMBEDDING OF SPACES OF DISTRIBUTIONS INTO SPACES OF COLOMBEAU GENERALIZED FUNCTIONS

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Abstract. We present some remarks about the embedding of spaces of Schwartz distributions into spaces of Colombeau generalized functions. Following ideas of M. Nedeljkov *et al.*, we recall how a good choice of compactly supported mollifiers allows to perform globally the embedding of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$. We show that this embedding is equal to the one obtained with local and sheaf arguments by M. Grosser *et al.*, this giving various equivalent techniques to embed $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$.

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1 Introduction

The question of embedding classical spaces such as $\mathrm{C}^0(\Omega)$, $\mathrm{C}^\infty(\Omega)$, $\mathcal{D}'(\Omega)$ (where Ω is an open subset of $\mathbb{R}^d$, $d \in \mathbb{N}$) into spaces of generalized functions arises naturally. The main goal of this paper is to give a complete analysis of the various techniques used in the literature to solve this question in the case of Colombeau simplified algebra ([1], [2], [5], [6]).

The embedding of $\mathrm{C}^\infty(\Omega)$ into $\mathcal{G}(\Omega)$ is classically done by the canonical map

$$\sigma : \mathrm{C}^\infty(\Omega) \to \mathcal{G}(\Omega) \quad f \to [(f_\varepsilon)_\varepsilon] \text{ with } f_\varepsilon = f \text{ for } \varepsilon \in (0,1]$$

which is an injective homomorphism of algebras. ($[(f_\varepsilon)_\varepsilon]$ denotes the class of $(f_\varepsilon)_\varepsilon$ in the factor algebra $\mathcal{G}(\Omega)$; see section 2 for a short presentation of $\mathcal{G}(\Omega)$ or [1], [2] for a complete construction.)

For the embedding of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$ the following additional assumption is required: If ι_A is the expected embedding, one wants the following diagram to be commutative:

$$(1) \qquad \begin{array}{ccc} \mathrm{C}^\infty(\Omega) & \longrightarrow & \mathcal{D}'(\Omega) \\ & \searrow\sigma & \downarrow \iota_A \\ & & \mathcal{G}(\Omega) \end{array} ,$$

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that is: $\iota_{A|\mathrm{C}^\infty(\Omega)} = \sigma$. (It is well known that one could not expect such a commutative diagram for bigger spaces containing $\mathrm{C}^\infty(\Omega)$ such as $\mathrm{C}^0(\Omega)$; see [2], [6].)

In [2], this program is fulfilled by using the sheaf properties of Colombeau algebras. Let us quote the main step of the construction for the case of the simplified model. First, an embedding ι_0 of $\mathcal{E}'(\mathbb{R}^d)$ into $\mathcal{G}(\mathbb{R}^d)$ is realized by convolution of compactly supported distributions with suitable mollifiers $(\rho_\varepsilon)_\varepsilon$ belonging to $\mathcal{S}(\mathbb{R}^d)$. In fact, this map ι_0 can be considered as an embedding of $\mathcal{E}'(\mathbb{R}^d)$ into $\mathcal{G}_C(\mathbb{R}^d)$, the subalgebra of $\mathcal{G}(\mathbb{R}^d)$ of compactly supported generalized functions, since the support of $T \in \mathcal{E}'(\mathbb{R}^d)$ is equal to the support of its image by ι_0 (Proposition 1.2.12 of [2]). The following step of the construction of ι_A is to consider for every open set $\Omega \subset \mathbb{R}^d$ an open covering $(\Omega_\lambda)_\lambda$ of Ω with relatively compact open sets and to embed $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega_\lambda)$ with the help of cutoff functions and ι_0. Using a partition of unity subordinate to $(\Omega_\lambda)_\lambda$, ι_A is constructed by "gluing the bits obtained before together". Finally, it is shown that the embedding ι_A does not depend on the choice of $(\Omega_\lambda)_\lambda$ and other material of the construction, excepted the net $(\rho_\varepsilon)_\varepsilon$.

On one hand, the mollifiers that render the diagram (1) commutative in a straightforward way are not compactly supported. On the other hand, ρ_ε cannot be convoluted with elements of $\mathcal{D}'(\Omega)$ unrestrictedly, obliging to consider first compactly supported distributions, and then sheaf arguments.

In [5], the authors give another construction which avoids this drawback. The main idea is to use compactly supported mollifiers close enough to the *ad hoc* mollifiers $(\rho_\varepsilon)_\varepsilon$ of [2]. This is done by a regular cutoff of $(\rho_\varepsilon)_\varepsilon$, this cutoff being defined with another rate of growth than the net $(\rho_\varepsilon)_\varepsilon$, let us say in $|\ln \varepsilon|$, whereas the scale of growth of $(\rho_\varepsilon)_\varepsilon$ is in $1/\varepsilon$. This permits to keep the good properties of the embedding, in particular the commutativity of the diagram (1). We present this construction in details in Section 3 for the case $\Omega = \mathbb{R}^d$.

In Section 4, we show that these embeddings are in fact equal, consequently only depending on the choice of the mollifiers $(\rho_\varepsilon)_\varepsilon$. (This dependence is well known for the simplified Colombeau algebra.) We finally turn to the case of the embedding of $\mathcal{D}'(\Omega)$ into the simplified Colombeau Algebra $\mathcal{G}(\Omega)$ where Ω is an arbitrary open subset of $\mathbb{R}^d$ (Section 5). We show that for the global construction of [5] an additional cutoff, applied to the elements of $\mathcal{D}'(\Omega)$, is needed. We also give a local version (with no cutoff on the distribution) of the construction of [5].

2 Preliminaries

2.1 The sheaf of Colombeau simplified algebras

Let C^∞ be the sheaf of complex valued smooth functions on $\mathbb{R}^d$ ($d \in \mathbb{N}$), with the usual topology of uniform convergence. For every open set Ω of $\mathbb{R}^d$, this topology can be described by the family of seminorms

$$p_{K,l}(f) = \sup_{|\alpha| \le l, K \Subset \Omega} |\partial^\alpha f(x)|,$$

where the notation $K \Subset \Omega$ means that the set K is a compact set included in Ω.

Let us set

$$\begin{aligned}
\mathcal{F}\left(\mathrm{C}^{\infty}\left(\Omega\right)\right) &= \Big\{\left(f_{\varepsilon}\right)_{\varepsilon} \in \mathrm{C}^{\infty}\left(\Omega\right)^{(0,1]} \mid \forall l \in \mathbb{N},\ \forall K \Subset \Omega,\ \exists q \in \mathbb{N}, \\
&\qquad p_{K,l}\left(f_{\varepsilon}\right) = \mathrm{o}\left(\varepsilon^{-q}\right) \text{ for } \varepsilon \to 0\Big\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}\left(\mathrm{C}^{\infty}\left(\Omega\right)\right) &= \Big\{\left(f_{\varepsilon}\right)_{\varepsilon} \in \mathrm{C}^{\infty}\left(\Omega\right)^{(0,1]} \mid \forall l \in \mathbb{N},\ \forall K \Subset \Omega,\ \forall p \in \mathbb{N}, \\
&\qquad p_{K,l}\left(f_{\varepsilon}\right) = \mathrm{o}\left(\varepsilon^{p}\right) \text{ for } \varepsilon \to 0\Big\}.
\end{aligned}$$

Lemma 1. *[3] and [4]*
i. The functor $\mathcal{F} : \Omega \to \mathcal{F}\left(\mathrm{C}^{\infty}\left(\Omega\right)\right)$ defines a sheaf of subalgebras of the sheaf $\left(\mathrm{C}^{\infty}\right)^{(0,1]}$.
ii. The functor $\mathcal{N} : \Omega \to \mathcal{N}\left(\mathrm{C}^{\infty}\left(\Omega\right)\right)$ defines a sheaf of ideals of the sheaf $\mathcal{F}$.

We shall not prove in detail this lemma but quote the two mains arguments:
i. For each open subset Ω of X, the family of seminorms $\left(p_{K,l}\right)$ related to Ω is compatible with the algebraic structure of $\mathrm{C}^{\infty}\left(\Omega\right)$; In particular:

$$\forall l \in \mathbb{N},\ \forall K \Subset \Omega,\ \exists C \in \mathbb{R}_{+}^{*},\ \forall\left(f,g\right) \in \left(\mathrm{C}^{\infty}\left(\Omega\right)\right)^{2}\ p_{K,l}\left(fg\right) \leq C p_{K,l}\left(f\right) p_{K,l}\left(g\right),$$

ii. For two open subsets $\Omega_1 \subset \Omega_2$ of $\mathbb{R}^d$, the family of seminorms $\left(p_{K,l}\right)$ related to Ω_1 is included in the family of seminorms related to Ω_2 and

$$\forall l \in \mathbb{N},\ \forall K \Subset \Omega_1,\ \ \forall f \in \mathrm{C}^{\infty}\left(\Omega_2\right),\ \ p_{K,l}\left(f_{|\Omega_1}\right) = p_{K,l}\left(f\right).$$

Definition 2. *The sheaf of factor algebras*

$$\mathcal{G}\left(\mathrm{C}^{\infty}\left(\cdot\right)\right) = \mathcal{F}\left(\mathrm{C}^{\infty}\left(\cdot\right)\right)/\mathcal{N}\left(\mathrm{C}^{\infty}\left(\cdot\right)\right)$$

is called the sheaf of Colombeau simplified algebras.

The sheaf $\mathcal{G}$ turns to be a sheaf of differential algebras and a sheaf of modules on the factor ring $\overline{\mathbb{C}} = \mathcal{F}\left(\mathbb{C}\right)/\mathcal{N}\left(\mathbb{C}\right)$ with

$$\mathcal{F}\left(\mathbb{K}\right) = \left\{\left(r_{\varepsilon}\right)_{\varepsilon} \in \mathbb{K}^{(0,1]} \mid \exists q \in \mathbb{N},\ \left|r_{\varepsilon}\right| = \mathrm{o}\left(\varepsilon^{-q}\right) \text{ for } \varepsilon \to 0\right\},$$

$$\mathcal{N}\left(\mathbb{K}\right) = \left\{\left(r_{\varepsilon}\right)_{\varepsilon} \in \mathbb{K}^{(0,1]} \mid \forall p \in \mathbb{N},\ \left|r_{\varepsilon}\right| = \mathrm{o}\left(\varepsilon^{p}\right) \text{ for } \varepsilon \to 0\right\},$$

with $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R},\ \mathbb{R}_{+}$.

Notation 3. *In the sequel we shall note, as usual, $\mathcal{G}\left(\Omega\right)$ instead of $\mathcal{G}\left(\mathrm{C}^{\infty}\left(\Omega\right)\right)$ the algebra of generalized functions on Ω.*

2.2 Local structure of distributions

To fix notations, we recall here two classical results on the local structure of distributions, which are going to be used in the sequel. We refer the reader to [7] Chapter 3, specially Theorems XXI and XXVI, for proofs and details. Let Ω be an open subset of $\mathbb{R}^d$ ($d \in \mathbb{N}$).

Theorem 4. *For all $T \in \mathcal{D}'(\Omega)$ and all Ω' open subset of $\mathbb{R}^d$ with $\overline{\Omega'} \Subset \Omega$, there exists $f \in \mathrm{C}^0(\mathbb{R}^d)$ whose support is contained in an arbitrary neighborhood of $\overline{\Omega'}$, $\alpha \in \mathbb{N}^d$ such that $T_{|\Omega'} = \partial^\alpha f$.*

Theorem 5. *For all $T \in \mathcal{E}'(\Omega)$, there exists an integer $r \geq 0$, a finite family $(f_\alpha)_{0 \leq |\alpha| \leq r}$ ($\alpha \in \mathbb{N}^d$) with each $f_\alpha \in \mathrm{C}^0(\mathbb{R}^d)$ having its support contained in the same arbitrary neighborhood of the support of T, such that $T = \sum_{0 \leq |\alpha| \leq r} \partial^\alpha f_\alpha$.*

3 Embedding of $\mathcal{D}'(\mathbb{R}^d)$ into $\mathcal{G}(\mathbb{R}^d)$

3.1 Construction of the mollifiers

Take $\rho \in \mathcal{S}(\mathbb{R}^d)$ even such that

$$(2) \qquad \int \rho(x)\, \mathrm{d}x = 1, \qquad \int x^m \rho(x)\, \mathrm{d}x = 0 \text{ for all } m \in \mathbb{N}^d \backslash \{0\},$$

and $\chi \in \mathcal{D}(\mathbb{R}^d)$ such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $\overline{B(0,1)}$ and $\chi \equiv 0$ on $\mathbb{R}^d \backslash B(0,2)$. Define

$$\forall \varepsilon \in (0,1],\ \forall x \in \mathbb{R}^d, \quad \rho_\varepsilon(x) = \frac{1}{\varepsilon^d}\rho\left(\frac{x}{\varepsilon}\right),$$

and

$$\forall \varepsilon \in (0,1),\ \forall x \in \mathbb{R}^d, \quad \theta_\varepsilon(x) = \rho_\varepsilon(x)\,\chi(|\ln \varepsilon|\, x)\ ;\ \theta_1(x) = 1.$$

Remark 6. *The nets $(\rho_\varepsilon)_\varepsilon$ and $(\theta_\varepsilon)_\varepsilon$ defined above belong to $\mathcal{F}(\mathrm{C}^\infty(\mathbb{R}^d))$.*

Let us verify this result for $(\theta_\varepsilon)_\varepsilon$ and $d = 1$. Fixing $\alpha \in \mathbb{N}$, we have

$$\begin{aligned} \forall x \in \mathbb{R}, \quad \partial^\alpha \theta_\varepsilon(x) &= \sum_{\beta=0}^{\alpha} \mathrm{C}_\alpha^\beta \partial^\beta \rho_\varepsilon(x)\, \partial^{\alpha-\beta}(\chi(x\,|\ln \varepsilon|)) \\ &= \sum_{\beta=0}^{\alpha} \mathrm{C}_\alpha^\beta \varepsilon^{-1-\beta} |\ln \varepsilon|^{\alpha-\beta} \rho^{(\beta)}\left(\frac{x}{\varepsilon}\right) \chi^{(\alpha-\beta)}(x\,|\ln \varepsilon|). \end{aligned}$$

For all $\beta \in \{0, \ldots, \alpha\}$, we have $\varepsilon^{-1-\beta}|\ln \varepsilon|^{\alpha-\beta} = \mathrm{o}\left(\varepsilon^{-2-\alpha}\right)$ for $\varepsilon \to 0$. As $\rho^{(n)}$ and $\chi^{(n)}$ are bounded, there exists $C(\alpha)$ such that

$$\forall x \in \mathbb{R}, \quad |\partial^\alpha(\theta_\varepsilon(x))| \leq C(\alpha)\, \varepsilon^{-2-\alpha}.$$

Our claim follows from this last inequality.

Lemma 7. *With the previous notations, the following properties hold*

$$(\theta_\varepsilon)_\varepsilon - (\rho_\varepsilon)_\varepsilon \in \mathcal{N}\left(\mathrm{C}^\infty\left(\mathbb{R}^d\right)\right), \tag{3}$$

$$\forall k \in \mathbb{N}, \quad \int \theta_\varepsilon(x)\,\mathrm{d}x = 1 + \mathrm{o}\left(\varepsilon^k\right) \quad \textit{for } \varepsilon \to 0, \tag{4}$$

$$\forall k \in \mathbb{N}, \ \ \forall m \in \mathbb{N}^d \backslash \{0\}, \quad \int x^m \theta_\varepsilon(x)\,\mathrm{d}x = \mathrm{o}\left(\varepsilon^k\right) \quad \textit{for } \varepsilon \to 0. \tag{5}$$

In other words, we have

$$\left(\int \theta_\varepsilon(x)\,\mathrm{d}x - 1\right)_\varepsilon \in \mathcal{N}(\mathbb{R}) \qquad \forall m \in \mathbb{N}^d \backslash \{0\}, \ \left(\int x^m \theta_\varepsilon(x)\,\mathrm{d}x\right)_\varepsilon \in \mathcal{N}(\mathbb{R}).$$

Proof. We consider the case $d = 1$ in order to simplify notations.
First assertion.- We have, for all $x \in \mathbb{R}$ and $\varepsilon \in (0,1)$,

$$|\rho_\varepsilon(x) - \theta_\varepsilon(x)| \leq \frac{1}{\varepsilon}\left|\rho\left(\frac{x}{\varepsilon}\right)\right|(1 - \chi(x\,|\ln\varepsilon|)) \leq \frac{1}{\varepsilon}\left|\rho\left(\frac{x}{\varepsilon}\right)\right|. \tag{6}$$

Since ρ belongs to $\mathcal{S}(\mathbb{R})$, for all integers $k > 0$ there exists a constant $C(k)$ such that

$$\forall x \in \mathbb{R}, \quad |\rho(x)| \leq \frac{C(k)}{(1+|x|)^k}.$$

Then, for all $x \in \mathbb{R}$ with $|x| \geq 1/|\ln\varepsilon|$,

$$\frac{1}{\varepsilon}\left|\rho\left(\frac{x}{\varepsilon}\right)\right| \leq \frac{C(k)}{(\varepsilon+|x|)^k}\varepsilon^{k-1} \leq C(k)\,|\ln\varepsilon|^k\,\varepsilon^{k-1} = \mathrm{o}\left(\varepsilon^{k-2}\right). \tag{7}$$

According to remark 6, $(\rho_\varepsilon - \theta_\varepsilon)_\varepsilon \in \mathcal{F}(\mathrm{C}^\infty(\mathbb{R}))$. Then we can conclude, without estimating the derivatives, that $(\rho_\varepsilon - \theta_\varepsilon)_\varepsilon \in \mathcal{N}(\mathrm{C}^\infty(\mathbb{R}))$ by using Theorem 1.2.3. of [2].
Second and third assertions.- According to the definition of ρ, we find that

$$\int \rho_\varepsilon(x)\,\mathrm{d}x = 1\,; \qquad \forall m \in \mathbb{N}\backslash\{0\}, \ \int x^m \rho_\varepsilon(x)\,\mathrm{d}x = 0.$$

So, it suffices to show that, for all $m \in \mathbb{N}$ and $k \in \mathbb{N}$,

$$\Delta_{m,\varepsilon}(x) = \int x^m\left(\rho_\varepsilon(x) - \theta_\varepsilon(x)\right)\mathrm{d}x = \mathrm{o}\left(\varepsilon^k\right) \text{ for } \varepsilon \to 0,$$

to complete our proof. Let fix $m \in \mathbb{N}$. We have

$$\Delta_{m,\varepsilon}(x) = \underbrace{\int_{-\infty}^{1/|\ln\varepsilon|} x^m\left(\rho_\varepsilon(x) - \theta_\varepsilon(x)\right)\mathrm{d}x}_{(*)} + \underbrace{\int_{1/|\ln\varepsilon|}^{+\infty} x^m\left(\rho_\varepsilon(x) - \theta_\varepsilon(x)\right)\mathrm{d}x}_{(**)}.$$

Let us find an estimate of $(**)$. According to relations (6) and (7), we have

$$\forall x > 0,\ \ \forall k \in \mathbb{N},\ \ x^m \left|\rho_\varepsilon (x) - \theta_\varepsilon (x)\right| \leq \frac{1}{\varepsilon} x^m \left|\rho\left(\frac{x}{\varepsilon}\right)\right| \leq C(k)\, \varepsilon^{k-1} x^{m-k}.$$

Therefore, by choosing $k \geq m+2$,

$$\begin{aligned}\left|\int_{1/|\ln\varepsilon|}^{+\infty} x^m \left(\rho_\varepsilon (x) - \theta_\varepsilon (x)\right) \mathrm{d}x\right| &\leq C(k)\, \varepsilon^{k-1} \int_{1/|\ln\varepsilon|}^{+\infty} x^{m-k}\, \mathrm{d}x \\ &\leq \frac{C(k)}{k-1-m} \varepsilon^{k-1} \left|\ln\varepsilon\right|^{k-m-1} \\ &= \mathrm{o}\left(\varepsilon^{k-2}\right),\ \text{for } \varepsilon \to 0.\end{aligned}$$

With a similar estimate for $(*)$, we obtain our claim. □

3.2 Construction of the embedding ι_S

Proposition 8. *With notations of Lemma 7, the map*

$$\iota_S : \mathcal{D}'\left(\mathbb{R}^d\right) \to \mathcal{G}\left(\mathbb{R}^d\right) \quad T \mapsto \left(T * \theta_\varepsilon\right)_\varepsilon + \mathcal{N}\left(\mathrm{C}^\infty\left(\mathbb{R}^d\right)\right)$$

is an injective homomorphism of vector spaces. Moreover $\iota_{S|\mathrm{C}^\infty(\mathbb{R}^d)} = \sigma$.

Proof. We have first to show that for all $T \in \mathcal{D}'\left(\mathbb{R}^d\right)$, $\left(T * \theta_\varepsilon\right)_\varepsilon \in \mathcal{F}\left(\mathrm{C}^\infty(\Omega)\right)$. (This allows us to define the map ι_S.) Let us fix a compact set K. Consider Ω open subset of $\mathbb{R}^d$ such that $K \subset \Omega \subset \overline{\Omega} \Subset \mathbb{R}^d$. Let us recall that

$$\forall y \in \mathbb{R}^d, \quad T * \theta_\varepsilon(y) = \langle T, \{x \mapsto \theta_\varepsilon (y-x)\}\rangle$$

For $y \in K$ and $x \in \mathbb{R}^d$, we have

$$\theta_\varepsilon (y-x) \neq 0 \Rightarrow y - x \in B(0, \frac{2}{|\ln\varepsilon|}) \Rightarrow x \in B(y, \frac{2}{|\ln\varepsilon|}) \Rightarrow x \in \Omega, \tag{8}$$

for ε small enough.

Then, the function $x \mapsto \theta_\varepsilon (y-x)$ belongs to $\mathcal{D}(\Omega)$ and

$$\langle T, \theta_\varepsilon (y - \cdot)\rangle = \left\langle T_{|\Omega}, \theta_\varepsilon (y-\cdot)\right\rangle.$$

Using Theorem 4, we can write $T_{|\Omega} = \partial_x^\alpha f$ where f is a compactly supported continuous function. Then $T * \theta_\varepsilon = f * \partial^\alpha \theta_\varepsilon$ and

$$\forall y \in K, \quad \left(T * \theta_\varepsilon\right)(y) = \int_\Omega f(y-x)\, \partial^\alpha \theta_\varepsilon (x)\, \mathrm{d}x.$$

According to remark 6, there exists $m(\alpha) \in \mathbb{N}$ such that

$$\forall x \in \mathbb{R}^d, \quad \left|\partial^\alpha \theta_\varepsilon (x)\right| \leq C\varepsilon^{-m(\alpha)}.$$

We get

$$\forall y \in K, \quad |(T * \theta_\varepsilon)(y)| \leq C \sup_{\xi \in \overline{\Omega}} |f(\xi)| \operatorname{vol}\left(\overline{\Omega}\right) \varepsilon^{-m(\alpha)},$$

and $\sup_{y \in K} |(T * \theta_\varepsilon)(y)| = \mathrm{O}\left(\varepsilon^{-m(\alpha)}\right)$ for $\varepsilon \to 0$.

Since $\partial^\beta (f * \partial^\alpha \theta_\varepsilon) = f * \partial^{\alpha+\beta} \theta_\varepsilon$, the same arguments apply to derivatives and the claim follows.

Let us now prove that ι is injective, *i.e.*

$$(T * \theta_\varepsilon)_\varepsilon \in \mathcal{N}\left(\mathrm{C}^\infty\left(\mathbb{R}^d\right)\right) \Rightarrow T = 0.$$

Indeed, taking $\varphi \in \mathcal{D}\left(\mathbb{R}^d\right)$ we have $\langle T * \theta_\varepsilon, \varphi \rangle \to \langle T, \varphi \rangle$, since $T * \theta_\varepsilon \to T$ in $\mathcal{D}'$. But, $T * \theta_\varepsilon \to 0$ uniformly on $\operatorname{supp} \varphi$ since $T * \theta_\varepsilon \in \mathcal{N}\left(\mathrm{C}^\infty\left(\mathbb{R}^d\right)\right)$. Then $\langle T * \theta_\varepsilon, \varphi \rangle \to 0$ and $\langle T, \varphi \rangle = 0$.

We shall prove the last assertion in the case $d = 1$, the general case only differs by more complicate algebraic expressions.

Let f be in $\mathrm{C}^\infty(\mathbb{R})$ and set $\Delta = \iota_S(f) - \sigma(f)$. One representative of Δ is given by

$$\Delta_\varepsilon : \mathbb{R} \to \mathcal{F}\left(\mathrm{C}^\infty(\mathbb{R})\right) \qquad y \mapsto (f * \theta_\varepsilon)(y) - f(y) = \int f(y-x)\theta_\varepsilon(x)\,\mathrm{d}x - f(y).$$

Fix K a compact set of $\mathbb{R}$. Writing $\int \theta_\varepsilon(x)\,\mathrm{d}x = 1 + N_\varepsilon$ with $(N_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R})$, we get

$$\Delta_\varepsilon(y) = \int \left(f(y-x) - f(y)\right)\theta_\varepsilon(x)\,\mathrm{d}x + N_\varepsilon\, f(y).$$

The integration is performed on the compact set $\operatorname{supp} \theta_\varepsilon \subset [-2/|\ln \varepsilon|, 2/|\ln \varepsilon|]$.

Let k be a positive integer. Taylor's formula gives

$$f(y-x) - f(y) = \sum_{i=1}^{k} \frac{(-x)^i}{i!} f^{(i)}(y) + \frac{(-x)^k}{k!} \int_0^1 f^{(k+1)}(y - ux)(1-u)^k\,\mathrm{d}u$$

and

$$\begin{aligned}
\Delta_\varepsilon(y) \quad = \quad & \underbrace{\sum_{i=1}^{k} \frac{(-1)^i}{i!} f^{(i)}(y) \int_{-2/|\ln \varepsilon|}^{2/|\ln \varepsilon|} x^i\, \theta_\varepsilon(x)\,\mathrm{d}x}_{P_\varepsilon(k,y)} \\
+ \quad & \underbrace{\int_{-2/|\ln \varepsilon|}^{2/|\ln \varepsilon|} \frac{(-x)^k}{k!} \int_0^1 f^{(k+1)}(y-ux)(1-u)^k\,\mathrm{d}u\, \theta_\varepsilon(x)\,\mathrm{d}x}_{R_\varepsilon(k,y)} + \mathcal{N}_\varepsilon f(y).
\end{aligned}$$

According to Lemma 7, we have $\left(\int x^i \theta_\varepsilon(x)\,\mathrm{d}x\right)_\varepsilon \in \mathcal{N}(\mathbb{R})$ and consequently

$$(P_\varepsilon(k,y))_\varepsilon \in \mathcal{N}(\mathbb{R}).$$

Using the definition of θ_ε, we have

$$= R_\varepsilon(k,y) =$$

$$\frac{1}{\varepsilon}\int_{-2/|\ln\varepsilon|}^{2/|\ln\varepsilon|}\frac{(-x)^k}{k!}\int_0^1 f^{(k+1)}(y-ux)(1-u)^k\,\mathrm{d}u\,\rho\left(\frac{x}{\varepsilon}\right)\chi\left(x\left|\ln\varepsilon\right|\right)\mathrm{d}x.$$

Setting $v = x/\varepsilon$, we get

$$R_\varepsilon(k,y) =$$

$$= \varepsilon^{k+1}\int_{-2/(\varepsilon|\ln\varepsilon|)}^{2/(\varepsilon|\ln\varepsilon|)}\frac{(-v)^k}{k!}\int_0^1 f^{(k+1)}(y-\varepsilon uv)(1-u)^k\,\mathrm{d}u\,\rho(v)\,\chi\left(\varepsilon\left|\ln\varepsilon\right|v\right)\mathrm{d}v.$$

For $(u,v)\in[0,1]\times[-2/(\varepsilon|\ln\varepsilon|),2/(\varepsilon|\ln\varepsilon|)]$, we have $y-\varepsilon uv\in[y-1,y+1]$ for ε small enough. Then, for $y\in K$, $y-\varepsilon uv$ lies in a compact set K' for (u,v) in the domain of integration.

It follows

$$\begin{aligned}|R_\varepsilon(k,y)| &\leq \frac{\varepsilon^k}{k!}\sup_{\xi\in K'}\left|f^{(k+1)}(\xi)\right|\int_{-2/(\varepsilon|\ln\varepsilon|)}^{2/(\varepsilon|\ln\varepsilon|)}|v|^{k+1}\,|\rho(v)|\,\mathrm{d}v\\ &\leq \frac{\varepsilon^k}{k!}\sup_{\xi\in K'}\left|f^{(k+1)}(\xi)\right|\int_{-\infty}^{+\infty}|v|^{k+1}\,|\rho(v)|\,\mathrm{d}v\leq C\varepsilon^k\ \ (C>0).\end{aligned}$$

The constant C depends only on the integer k, the compact sets K and K', ρ and f.

Finally, for all $k>0$

$$\sup_{y\in K}\Delta_\varepsilon(y) = \mathrm{o}\left(\varepsilon^k\right)\ \text{ for } \varepsilon\to 0.$$

As $(\Delta_\varepsilon)_\varepsilon\in\mathcal{F}\left(\mathrm{C}^\infty\left(\mathbb{R}^d\right)\right)$ and $\sup_{y\in K}\Delta_\varepsilon(y)=\mathrm{o}\left(\varepsilon^k\right)$ for all $k>0$ and $K\Subset\mathbb{R}$, we can conclude without estimating the derivatives that $(\Delta_\varepsilon)_\varepsilon\in\mathcal{N}\left(\mathrm{C}^\infty\left(\mathbb{R}^d\right)\right)$ by using Theorem 1.2.3. of [2]. □

4 Comparison of i_A and i_S

As mentioned in the introduction, the embedding $\iota_A:\mathcal{D}'\left(\mathbb{R}^d\right)\to\mathcal{G}\left(\mathbb{R}^d\right)$ constructed in [2] depends on the choice of the chosen net $\rho\in\mathcal{S}\left(\mathbb{R}^d\right)$. This dependence is a well known fact for the simplified Colombeau algebra. Of course, ι_S depends also on the choice of $\rho\in\mathcal{S}\left(\mathbb{R}^d\right)$, but not on the choice of χ. Moreover:

Proposition 9. *For the same choice of ρ, we have: $\iota_A=\iota_S$.*

The proof is carried out in the two following subsections.

4.1 Embedding of $\mathcal{E}'\left(\mathbb{R}^d\right)$ into $\mathcal{G}\left(\mathbb{R}^d\right)$

In [2], the embedding of $\mathcal{E}'\left(\mathbb{R}^d\right)$ into $\mathcal{G}\left(\mathbb{R}^d\right)$ is realized with the map

$$\iota_0 : \mathcal{E}'\left(\mathbb{R}^d\right) \to \mathcal{G}\left(\mathbb{R}^d\right) \qquad T \mapsto \left(T * \rho_\varepsilon\right)_\varepsilon + \mathcal{N}\left(\mathrm{C}^\infty\left(\mathbb{R}^d\right)\right).$$

We compare here with $\iota_{S|\mathcal{E}'(\mathbb{R}^d)}$. Let us fix $T \in \mathcal{E}'\left(\mathbb{R}^d\right)$. We have to estimate $\left(T * \rho_\varepsilon\right)_\varepsilon - \left(T * \theta_\varepsilon\right)_\varepsilon$.

Using Theorem 5, we can write $T = \sum_{finite} \partial^\alpha f_\alpha$, each f_α having a compact support. We only need to obtain estimation for one summand, and we shall consider that $T = \partial^\alpha f$. Setting $\Delta_\varepsilon = T * \theta_\varepsilon - T * \rho_\varepsilon$ we have

$$\forall y \in \mathbb{R}^d, \quad \Delta_\varepsilon\left(y\right) = \int f(y-x)\left(\partial^\alpha \theta_\varepsilon\left(x\right) - \partial^\alpha \rho_\varepsilon\left(x\right)\right) \mathrm{d}x.$$

Then

$$\begin{aligned} \left|\Delta_\varepsilon\left(y\right)\right| &\leq C \int \left|\partial^\alpha \theta_\varepsilon\left(x\right) - \partial^\alpha \rho_\varepsilon\left(x\right)\right| \mathrm{d}x \qquad \text{with } C = \sup_{\xi \in \mathbb{R}^d} \left|f\left(\xi\right)\right| \\ &\leq C \int_{\mathbb{R}^d \setminus B(0,1/\left|\ln \varepsilon\right|)} \left|\partial^\alpha \theta_\varepsilon\left(x\right) - \partial^\alpha \rho_\varepsilon\left(x\right)\right| \mathrm{d}x, \end{aligned}$$

since $\partial^\alpha \theta_\varepsilon = \partial^\alpha \rho_\varepsilon$ on $B(0, 1/\left|\ln \varepsilon\right|)$.

To simplify notations, we suppose $d = 1$ and $\alpha = 1$. We have

$$\theta_\varepsilon'\left(x\right) - \rho_\varepsilon'\left(x\right) = \varepsilon^{-1}\left|\ln \varepsilon\right| \chi'\left(x\left|\ln \varepsilon\right|\right) \rho\left(\varepsilon^{-1}x\right) + \varepsilon^{-2}\rho'\left(\varepsilon^{-1}x\right)\left(\chi\left(x\left|\ln \varepsilon\right|\right) - 1\right).$$

Since $\rho \in \mathcal{S}\left(\mathbb{R}\right)$, for all $k \in \mathbb{N}$, with $k \geq 2$, there exists $C(k) \in \mathbb{R}_+$ such that

$$\left|\rho^{(i)}\left(x\right)\right| \leq \frac{C(k)}{1 + \left|x\right|^k} \quad (\text{for } i = 0 \text{ and } i = 1).$$

Then, for all x with $\left|x\right| \geq 1/\left|\ln \varepsilon\right|$, we get

$$\left|\rho^{(i)}\left(\varepsilon^{-1}x\right)\right| \leq C(k)\,\varepsilon^k \frac{1}{\varepsilon^k + \left|x\right|^k} \leq C(k)\,\varepsilon^k \left|x\right|^{-k}.$$

Since $\left|\ln \varepsilon\right| \leq \varepsilon^{-1}$ for $\varepsilon \in (0,1]$ and $\left|\chi\left(x\left|\ln \varepsilon\right|\right) - 1\right| \leq 1$ for all $x \in \mathbb{R}$, we get

$$\begin{gathered} \left|\theta_\varepsilon'\left(x\right) - \rho_\varepsilon'\left(x\right)\right| \leq \\ \leq \left|x\right|^{-k}\left(\varepsilon^{k-1}\left|\ln \varepsilon\right| \sup_{\xi \in \mathbb{R}}\left|\chi'\left(\xi\right)\right| \left|x\right|^{-k} C(k) \left|x\right|^{-k} + \varepsilon^{k-2}\, C(k) \left|x\right|^{-k}\right) \leq \\ \leq \varepsilon^{k-2}\, C(k) \left(\sup_{\xi \in \mathbb{R}}\left|\chi'\left(\xi\right)\right| + 1\right) \left|x\right|^{-k}. \end{gathered}$$

Then, we get a constant $C' = C'\left(k, \chi, f\right) > 0$ such that

$$\left|\Delta_\varepsilon\left(y\right)\right| \leq 2\,\varepsilon^{k-2} C' \int_{1/\left|\ln \varepsilon\right|}^{+\infty} \left|x\right|^{-k} \mathrm{d}x = \frac{2C'}{k-1}\varepsilon^{k-2}\left|\ln \varepsilon\right|^{k-1}.$$

Finally, we have $\sup_{y\in\mathbb{R}} |\Delta_\varepsilon(y)| = \mathrm{o}\left(\varepsilon^k\right)$ for all $k \in \mathbb{N}$.

As $(\Delta_\varepsilon)_\varepsilon \in \mathcal{F}\left(\mathrm{C}^\infty\left(\mathbb{R}^d\right)\right)$, we finally conclude that $(\Delta_\varepsilon)_\varepsilon \in \mathcal{N}\left(\mathrm{C}^\infty\left(\mathbb{R}^d\right)\right)$ by using Theorem 1.2.3. of [2]. Then:

Lemma 10. *For the same choice of ρ, we have: $\iota_0 = \iota_{S|\mathcal{E}'(\mathbb{R}^d)}$.*

4.2 Embedding of $\mathcal{D}'\left(\mathbb{R}^d\right)$ into $\mathcal{G}\left(\mathbb{R}^d\right)$

Notation 11. *In this subsection we shall note $\mathbb{N}_m = \{1,\ldots,m\}$ for all $m \in \mathbb{N}\backslash\{0\}$.*

Let us recall briefly the construction of [2]. Fix some locally finite open covering $(\Omega_\lambda)_{\lambda\in\Lambda}$ with $\overline{\Omega}_\lambda \Subset \mathbb{R}^d$ and a family $(\psi_\lambda)_{\lambda\in\Lambda} \in \mathcal{D}\left(\mathbb{R}^d\right)^\Lambda$ with $0 \leq \psi_\lambda \leq 1$ and $\psi_\lambda \equiv 1$ on a neighborhood of $\overline{\Omega}_\lambda$. For each λ define

$$\iota_\lambda : \mathcal{D}'\left(\mathbb{R}^d\right) \to \mathcal{G}(\Omega_\lambda)$$

$$T \mapsto \iota_\lambda(T) = \iota_0(\psi_\lambda T)_{|\Omega_\lambda} = \left((\psi_\lambda T * \rho_\varepsilon)_{|\Omega_\lambda}\right)_\varepsilon + \mathcal{N}(\mathrm{C}^\infty(\Omega_\lambda)).$$

The family $(\iota_\lambda)_{\lambda\in\Lambda}$ is coherent and, by sheaf arguments, there exists a unique $\iota_A : \mathcal{D}'\left(\mathbb{R}^d\right) \to \mathcal{G}\left(\mathbb{R}^d\right)$ such that

$$\forall\lambda \in \Lambda, \quad \iota_{A|\Omega_\lambda} = \iota_\lambda.$$

Moreover, an explicit expression of ι_A can be given: Let $(\chi_j)_{j\in\mathbb{N}}$ be a smooth partition of unity subordinate to $(\Omega_\lambda)_{\lambda\in\Lambda}$. We have

$$\forall T \in \mathcal{D}'\left(\mathbb{R}^d\right), \quad \iota_A(T) = \left(\sum_{j=1}^{+\infty} \chi_j\left(\left(\psi_{\lambda(j)}T\right) * \rho_\varepsilon\right)\right)_\varepsilon + \mathcal{N}\left(\mathrm{C}^\infty\left(\mathbb{R}^d\right)\right).$$

Let us compare ι_A and ι_S. Using sheaf properties, we only need to verify that

$$\forall\lambda \in \Lambda, \ \ \iota_{s\,|\Omega_\lambda} = \iota_{A\,|\Omega_\lambda} \quad (= \iota_\lambda).$$

For a fixed $\lambda \in \Lambda$ and $T \in \mathcal{D}'\left(\mathbb{R}^d\right)$, we have $\iota_\lambda(T) = \iota_0(\psi_\lambda T)_{|\Omega_\lambda}$ and

$$\iota_{A\,|\Omega_\lambda} - \iota_{s\,|\Omega_\lambda}(T) = \iota_0(\psi_\lambda T) - \iota_s(\psi_\lambda T) + \iota_s(\psi_\lambda T) - \iota_s(T).$$

(We omit the restriction symbol in the right-hand side.)

As $\psi_\lambda T \in \mathcal{E}'\left(\mathbb{R}^d\right)$, we have $\iota_0(\psi_\lambda T) = \iota_s(\psi_\lambda T)$ according to Lemma 10. It remains to show that $\iota_s(\psi_\lambda T) = \iota_s(T)$, that is to compare $((\psi_\lambda T) * \theta_\varepsilon)_\varepsilon$ and $(T * \theta_\varepsilon)_\varepsilon$. Let us recall that

$$\forall y \in \Omega_\lambda, \ \ ((\psi_\lambda T) * \theta_\varepsilon)_\varepsilon(y) - (T * \theta_\varepsilon)_\varepsilon(y) = \langle \psi_\lambda T - T, \{x \mapsto \theta_\varepsilon(y-x)\}\rangle,$$

for ε small enough.

Let us consider K a compact set included in Ω_λ. According to relation (8), we have $\operatorname{supp}\theta_\varepsilon(y-\cdot) \subset B(y, 2/|\ln\varepsilon|)$. Using the fact that Ω_λ is open, we obtain that

$$\forall y \in K, \quad \exists\varepsilon_y \in (0,1], \quad B(y, 2/|\ln\varepsilon|) \subset \Omega_\lambda.$$

The family $(B(y, 1/|\ln \varepsilon_y|))_{y \in K}$ is an open covering of K from which we can extract a finite one, $(B(y_l, 1/|\ln \varepsilon_l|))_{1 \leq l \leq n}$ (with $\varepsilon_l = \varepsilon_{y_l}$). Put

$$\varepsilon_K = \min_{1 \leq l \leq n} \varepsilon_l.$$

For $y \in K$, there exists $l \in \mathbb{N}_n$ such that $y \in B(y_l, 1/|\ln \varepsilon_l|)$. Then, for $\varepsilon \leq \varepsilon_K^2$, we have

$$*supp\theta_\varepsilon (y - \cdot) \subset B(y, 2/|\ln \varepsilon|) \subset B(y, 1/|\ln \varepsilon_K|) \subset B(y_l, 2/|\ln \varepsilon_l|) \subset \Omega_\lambda,$$

since $d(y, y_l) < 1/|\ln \varepsilon_l|$.

For all $y \in K$, $\theta_\varepsilon (y - \cdot) \in \mathcal{D}(\Omega_\lambda)$ for $\varepsilon \in \left(0, \varepsilon_K^2\right]$. Since $T_{|\Omega_\lambda} = (\psi_\lambda T)_{|\Omega_\lambda}$, we finally obtain

$$\forall y \in K,\ \forall \varepsilon \in \left(0, \varepsilon_K^2\right],\ \ \langle \psi_\lambda T - T, \{x \mapsto \theta_\varepsilon (y - x)\}\rangle = 0,$$

showing that $((\psi_\lambda T - T) * \theta_\varepsilon)$ lies in $\mathcal{N}\left(\mathrm{C}^\infty\left(\mathbb{R}^d\right)\right)$.

5 Embedding of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$

All embeddings of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$ considered in the literature are based on convolution of distributions by C^∞ functions. This product is possible under additional assumptions, in particular about supports. Let us consider both constructions compared in this paper.

For the construction of [2], the local construction with cutoff techniques applied to the elements of $\mathcal{D}'(\Omega)$ is needed to obtain a well defined product of convolution between elements of $\mathcal{E}'\left(\mathbb{R}^d\right)$ and $\mathcal{S}\left(\mathbb{R}^d\right)$. Note that the cutoff is fixed once for all, and in particular does not depend on ε.

The construction of [5] allows a "global" embedding of $\mathcal{D}'\left(\mathbb{R}^d\right)$ into $\mathcal{G}\left(\mathbb{R}^d\right)$ since the convolution of elements of $\mathcal{D}'\left(\mathbb{R}^d\right)$ with $(\theta_\varepsilon)_\varepsilon \in \left(\mathcal{D}\left(\mathbb{R}^d\right)\right)^{(0,1]}$ is well defined. But, for the case of an open subset $\Omega \subsetneq \mathbb{R}^d$, previous arguments show that for $y \in \Omega$, the functions $\{x \mapsto \theta_\varepsilon (y - x)\}$ belong to $\mathcal{D}(\Omega)$ for ε smaller than some ε_y depending on y. This does not allow the definition of the net $(T * \theta_\varepsilon)_\varepsilon$ for $T \in \mathcal{D}'(\Omega)$ not compactly supported. To overcome this difficulty, a net of cutoffs $(\kappa_\varepsilon) \in \left(\mathcal{D}\left(\mathbb{R}^d\right)\right)^{(0,1]}$ such that $\kappa_\varepsilon T \to T$ in $\mathcal{D}'(\Omega)$ is considered, giving a well defined convolution of elements of $\mathcal{E}'\left(\mathbb{R}^d\right)$ with elements of $\mathcal{D}\left(\mathbb{R}^d\right)$. We present this construction below with small changes and another construction combining local techniques and compactly supported mollifiers of [5].

5.1 Embedding using cutoff arguments

Let us fix $\Omega \subset \mathbb{R}^d$ an open subset and set, for all $\varepsilon \in (0, 1]$,

$$K_\varepsilon = \left\{x \in \Omega \;\middle|\; d(x, \mathbb{R}^d \setminus \Omega) \geq \varepsilon \text{ and } d(x, 0) \leq 1/\varepsilon \right\}.$$

Consider $(\kappa_\varepsilon) \in \left(\mathcal{D}\left(\mathbb{R}^d\right)\right)^{(0,1]}$ such that

$$\forall \varepsilon \in (0, 1],\quad 0 \leq \kappa_\varepsilon \leq 1,\ \ \kappa_\varepsilon \equiv 1 \text{ on } K_\varepsilon.$$

(Such a net $(\kappa_\varepsilon)_\varepsilon$ is obtained, for example, by convolution of the characteristic function of $K_{\varepsilon/2}$ with a net of mollifiers $(\varphi_\varepsilon) \in \left(\mathcal{D}\left(\mathbb{R}^d\right)\right)^{(0,1]}$ with support decreasing rapidly enough to $\{0\}$.)

Proposition 12. *With notations of Lemma 7, the map*

$$\iota_S : \mathcal{D}'(\Omega) \to \mathcal{G}(\Omega) \quad T \mapsto ((\kappa_\varepsilon T) * \theta_\varepsilon)_\varepsilon + \mathcal{N}(\mathrm{C}^\infty(\Omega)) \tag{9}$$

is an injective homomorphism of vector spaces. Moreover $\iota_{S|\mathrm{C}^\infty(\mathbb{R}^d)} = \sigma$.

We shall not give a complete proof since it is a slight adaptation of the proof of proposition 8. We just quote here the main point. As seen above, many estimates have to be done on compact sets. Let K be a compact set included in Ω and Ω' an open set such that $K \subset \Omega' \subset \overline{\Omega}' \Subset \Omega$. There exists $\varepsilon_0 \in (0,1]$ such that

$$\forall \varepsilon \in (0, \varepsilon_0], \quad \Omega' \subset K_\varepsilon.$$

On the one hand this, implies that we have $(\kappa_\varepsilon T)_{|\Omega'} = (\kappa_{\varepsilon_0} T)_{|\Omega'} = T_{|\Omega'}$, for all $T \in \mathcal{D}'(\Omega)$ and $\varepsilon \in (0, \varepsilon_0]$. On the other hand, we already noticed that for $y \in K$, the functions $\{x \mapsto \theta_\varepsilon(y - x)\}$ belongs to $\mathcal{D}'(\Omega')$ for all $\varepsilon \in \left(0, \varepsilon_K^2\right]$, ε_K only depending on K.

Thus a representative of $\iota_S(T)$ is given, for all $y \in K$, by the convolution of an element of $\mathcal{E}'\left(\mathbb{R}^d\right)$ with an element of $\mathcal{D}(\Omega)$, this being valid for ε smaller than $\min\left(\varepsilon_0, \varepsilon_K^2\right)$ only depending on K. Proof of propositions 8 and 9 can now be adapted using this remark.

Remark 13. *For the presentation of the construction of [5] we chose to consider first the case* $\Omega = \mathbb{R}^d$. *In fact, we can unify the construction and consider for all* Ω *(included in* $\mathbb{R}^d$*) the embedding defined by (9). In the case* $\Omega = \mathbb{R}^d$*, the cutoff functions* κ_ε *are equal to one on the closed ball* $\overline{B(0, 1/\varepsilon)}$.

5.2 Embedding using local arguments

Let us fix an open subset Ω of $\mathbb{R}^d$. Recall that relation (8) implies that

$$\forall y \in \Omega, \quad \exists \varepsilon_y \in (0,1], \quad \forall \varepsilon \in (0, \varepsilon_y], \quad \operatorname{supp} \theta_\varepsilon(y - \cdot) \subset B(y, 2/|\ln \varepsilon|) \subset \Omega,$$

and consequently that $\theta_\varepsilon(y - \cdot) \in \mathcal{D}(\Omega)$ for $\varepsilon \in (0, \varepsilon_y]$. We consider here a local construction to overcome the fact that ε_y depends on y.

Let Ω' be an open relatively compact subset of Ω. As in subsection 4.2, there exists $\varepsilon_{\Omega'}$ such that, for all $\varepsilon \le \varepsilon_{\Omega'}^2$ and $y \in \Omega$, $\operatorname{supp} \theta_\varepsilon(y - \cdot) \subset \Omega$ and $\theta_\varepsilon(y - \cdot) \in \mathcal{D}(\Omega)$. For $T \in \mathcal{D}'(\Omega)$, define, for all $y \in \Omega'$,

$$T_\varepsilon(y) = \langle T, \theta_\varepsilon(y - \cdot) \rangle \text{ for } \varepsilon \in \left(0, \varepsilon_{\Omega'}^2\right], \quad T_\varepsilon(y) = T_{\varepsilon_{\Omega'}^2}(y) \quad \text{for } \varepsilon \in \left(\varepsilon_{\Omega'}^2, 1\right]. \tag{10}$$

Lemma 14. *The map*

$$\iota_{\Omega'} : \mathcal{D}'(\Omega) \to \mathcal{G}(\Omega') \quad T \mapsto T_\varepsilon(y) + \mathcal{N}(\mathrm{C}^\infty(\Omega'))$$

is an injective homomorphism of vector spaces.

Proof. The proof is very similar to that of proposition 8. □

Consider now a locally finite open covering of $(\Omega_\lambda)_{\lambda\in\Lambda}$ with $\overline{\Omega}_\lambda \Subset \Omega$ and set $\iota_\lambda = \iota_{\Omega_\lambda}$ for $\lambda \in \Lambda$.

Lemma 15. *The family $(\iota_\lambda)_{\lambda\in\Lambda}$ is coherent.*

Proof. Let us take $(\lambda, \mu) \in \Lambda^2$ with $\Omega_\lambda \cap \Omega_\mu \neq \varnothing$. We have

$$\iota_{\lambda\,|\Omega_\lambda\cap\Omega_\mu} = \iota_{\mu\,|\Omega_\lambda\cap\Omega_\mu}$$

since, for all T in $\mathcal{D}'(\Omega)$, representatives of $\iota_\lambda(T)$ and $\iota_\mu(T)$, written in the form (10), are equal for $\varepsilon \leq \min\left(\varepsilon^2_{\Omega_\lambda}, \varepsilon^2_{\Omega\mu}\right)$. □

By sheaf properties of $\mathcal{G}(\Omega)$ there exists a unique $\iota'_S : \mathcal{D}'(\Omega) \to \mathcal{G}(\Omega)$ such that $\iota'_{S\,|\Omega_\lambda} = \iota_\lambda$ for all $\lambda \in \Lambda$. Moreover, we can give an explicit formula: If $(\Psi_\lambda)_{\lambda\in\Lambda}$ is a partition of unity subordinate to $(\Omega_\lambda)_{\lambda\in\Lambda}$, we have

$$\forall T \in \mathcal{D}'(\Omega), \quad \iota'_S(T) = \textstyle\sum_{\lambda\in\Lambda} \Psi_\lambda \iota_\lambda(T).$$

This map ι'_S realizes an embedding which does not depend on the particular choice of $(\Omega_\lambda)_{\lambda\in\Lambda}$, the proof thereof is left to the reader.

Remark 16. *One may think that it is regrettable to come back here to local arguments, whereas they are avoided with cutoff technique. This is partially true but the advantage of compactly supported mollifiers remains: The convolution with any distribution is possible. This renders the local arguments very simple.*

5.3 Final remarks

Let Ω be an open subset of $\mathbb{R}^d$.

Proposition 17. *For the same choice of ρ, we have: $\iota_A = \iota_S = \iota'_S$.*

With notations of previous sections, we only have to prove the equality on each open set Ω_λ, where $(\Omega_\lambda)_{\lambda\in\Lambda}$ is a covering of Ω with relatively compact open sets. As seen before, we shall have $(\kappa_\varepsilon T)_{|\Omega'_\lambda} = T_{|\Omega'_\lambda}$ and $\theta_\varepsilon(y - \cdot) \in \mathcal{D}(\Omega'_\lambda)$, for all $y \in \Omega_\lambda$ and ε small enough. (Ω'_λ is an open subset relatively compact such that $\Omega_\lambda \subset \Omega'_\lambda \subset \Omega_\lambda$.) This remark leads to our result, since we obtain for $T \in \mathcal{D}(\Omega')$ representatives for $\iota_S(T)$ and $\iota'_S(T)$ equal for ε small enough.

Remark 18.
i. Let $\mathcal{B}^\infty\left(\mathbb{R}^d\right)$ be the subset of elements of $\mathcal{S}^\infty\left(\mathbb{R}^d\right)$ satisfying (2). We saw that there exists fundamentally one class of embeddings $(\iota_\rho)_{\rho\in\mathcal{B}^\infty(\mathbb{R}^d)}$ of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$ which renders the diagram (1) commutative. For a fixed $\rho \in \mathcal{B}^\infty\left(\mathbb{R}^d\right)$, ι_ρ can be described globally using the techniques of [5] or locally using either the techniques of [2] or of subsection 5.2 of this paper. This enlarges the possibilities when questions of embeddings arise in a mathematical problem.

ii. As mentioned in the introduction, ι_0 can be considered as an embedding of $\mathcal{E}'\left(\mathbb{R}^d\right)$ into $\mathcal{G}_C\left(\mathbb{R}^d\right)$. One has the following commutative diagram

$$\begin{array}{ccccc}
\mathcal{D} & \xrightarrow{c} & \mathcal{E}' & \xrightarrow{\iota_0=\iota_S} & \mathcal{G}_C \\
\downarrow_c & & \downarrow_c & & \downarrow_i \\
\mathcal{E}=\mathrm{C}^\infty\left(\Omega\right) & \xrightarrow{c} & \mathcal{D}' & \xrightarrow{\iota_A=\iota_S=\iota_S'} & \mathcal{G}
\end{array}$$

where $\xrightarrow{c}$ denotes the classical continuous embeddings, and i the canonical embedding of $\mathcal{G}_C$ into $\mathcal{G}$.

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