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COSINE OPERATOR FUNCTIONS AND HILBERT TRANSFORMS[1]

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Abstract. In this paper we consider bounded cosine operator functions and their connection to Hilbert transforms.

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1. Introduction

In this paper X denotes a complex Banach space and $B(X)$ the complex Banach algebra of all bounded linear operators on X. A function $\mathbf{C}(t)$ from $\mathbb{R} = (-\infty, +\infty)$ into $B(X)$ is a cosine operator function if $\mathbf{C}(0) = I$ (I – the identity operator on X) and if

$$\mathbf{C}(t+s) + \mathbf{C}(t-s) = 2\mathbf{C}(t)\mathbf{C}(s), \quad (t, s \in \mathbb{R}). \tag{1}$$

We will assume that $\mathbf{C}(t)$ is strongly continuous, i.e. that the vector function $\mathbf{C}(t)x$ is continuous on $\mathbb{R}$ for all $x \in X$. It is well known that there exist constants $K \geq 1$ and $\omega \geq 0$ for which

$$||\mathbf{C}(t)|| \leq K e^{\omega|t|}, \quad (t \in \mathbb{R}). \tag{2}$$

By A we will denote the infinitesimal operator of the function $\mathbf{C}(t)$, by $\mathcal{D}(A)$ its domain and by $\mathcal{R}(A)$ its rank. It is well known that $\mathcal{D}(A)$ is the set of all $x \in X$ for which

$$\lim_{t\to 0} 2\frac{\mathbf{C}(t)x - x}{t^2}$$

exists. For $x \in \mathcal{D}(A)$ we define

$$Ax = \lim_{t\to 0} 2\frac{\mathbf{C}(t)x - x}{t^2}.$$

The infinitesimal operator A of the strongly continuous cosine function $\mathbf{C}(t)$ is closed and $\overline{\mathcal{D}(A)} = X$. This operator is bounded only in the case that the

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function $\mathbf{C}(t)$ is uniformly continuous. The spectrum of this operator is, as already known, included in the set

$$\{\lambda^2 \quad |Re(\lambda) \leq \omega, \lambda \text{ - complex number}\}.$$

If $\omega = 0$ then we have to deal with a bounded cosine function:

$$||\mathbf{C}(t)|| \leq K \quad (t \in \mathbb{R}).$$

In this case the spectrum of A is contained in the half-line $(-\infty, 0]$.

In Section 2 of this paper we consider a bounded, strongly continuous cosine operator function $\mathbf{C}(t)$ for which $K = 1$. In the Section 3 we give a relationship between cosine functions and Hilbert transforms.

2. The Family F_a, $a \geq 0$

Here we consider a strongly continuous cosine operator function $\mathbf{C}(t)$ for which

$$||\mathbf{C}(t)|| \quad \leq 1, \quad (t \in \mathbb{R}).$$

The family F_a, $a \geq 0$ was introduced in [2], and a detailed investigation of its properties was given in [3] and [4]. In [3] F_a , $a \geq 0$ is defined by

$$F_a x = \lim_{\alpha \downarrow 0} F_{a,\alpha} x$$

where for $x \in X$

$$F_{a,\alpha} = \int_0^a E_{u,\alpha} x du = \frac{1}{\pi\, i} \int_0^a \left(\int_{\alpha+i0}^{\alpha+iu} [\lambda R(\lambda) + \overline{\lambda} R(\overline{\lambda})] x \, d\lambda \right) du, \quad \alpha + iy = \lambda.$$

Here $R(\lambda)$ denotes the resolvent of the operator $A : R(\lambda) = (\lambda^2 I - A)^{-1}$.

It is easy to see that for all $a \geq 0$ $||F_a|| \leq a$ and $\frac{F_a x}{a} \to x$ when $a \to +\infty$ (see [2], [3] and [4]).

Using this family we can define $f(A)$ as follows:

$$f(A) F_a x = f(-a^2) F_a x + \int_0^a ((a-u) f(-u^2))''_{uu} F_u x \, du, \quad x \in X. \tag{3}$$

where $f(-u^2)$ $(u \geq 0)$ is a twice continuously differentiable function.

The operator $f(A)$ is defined on the set of all those vectors that can be written in the form $F_a x$, $x \in X$ and $a \geq 0$. This set is dense in X.

Now we are going to prove that this definition is correct, namely that $f(A) F_a x$ does not depend on the form of the vector $F_a x$.

First let us rewrite (3) in a somewhat different form. In [3] it is proved that for $0 \le a \le b$

$$F_a F_b = F_b F_a = 2\int_0^a F_u du + (b-a)F_a. \tag{4}$$

From there, dealing sightly freely, we get that for $0 \le u \le a$

$$F_a dF_u = dF_u F_a = F_u du + (a-u)dF_u. \tag{5}$$

But if we perform the partial integration in (3) we get

$$\begin{aligned} f(A)F_a x &= -\int_0^a ((a-u)f(-u^2))' dF_u x \\ &= -\int_0^a (a-u)(f(-u^2))' dF_u x + \int_0^a f(-u^2) dF_u x \end{aligned}$$

which, together with (5), gives

$$f(A)F_a x = f(-a^2)F_a x - \int_0^a (f(-u^2))' dF_u F_a x. \tag{6}$$

Let us now assume that $F_a x = F_b y$ and let, for example, $a \le b$. Then from (4) it follows that $F_a dF_u = F_a du$ for $a \le u \le b$.

Based on this and on (6) we have

$$\begin{aligned} f(A)F_b y &= f(-b^2)F_b y - \int_0^b (f(-u^2))' dF_u F_b y \\ &= f(-b^2)F_a x - \int_0^a (f(-u^2))' dF_u F_a x - \int_a^b (f(-u^2))' dF_u F_a x \\ &= f(-b^2)F_a x - \int_0^a (f(-u^2))' dF_u F_a x - \int_a^b (f(-u^2))' du F_a x \\ &= f(-a^2)F_a x - \int_0^a (f(-u^2))' dF_u F_a x \\ &= f(A)F_a x. \end{aligned}$$

This proves the correctness of the definition of the operator $f(A)$.

It was shown in [4] that

$$\mathbf{C}(t)F_a x = \cos at F_a x + \int_0^a ((a-u)\cos ut)'' F_u x\, du \text{ for } a \ge 0, t \in \mathbb{R}, x \in X. \tag{7}$$

From here, by differentiating twice and by putting $t = 0$, we get

$$AF_a x = -a^2 F_a x + \int_0^a (6u - 2a) F_u x \, du. \tag{8}$$

This can be also proved by putting $f(u) = u$ in (3).

Let us put $f(u) = \sqrt{u}$ in (3). We get an operator which we will denote by $\sqrt{A}$ or $A^{\frac{1}{2}}$

$$A^{\frac{1}{2}} F_a x = iaF_a x - 2i \int_0^a F_u x \, du, \quad a \geq 0, x \in X. \tag{9}$$

Since according to (4), $F_a^2 = 2\int_0^a F_u du$, we can write (9) in the form

$$A^{\frac{1}{2}} F_a x = iaF_a x - iF_a^2 x. \tag{10}$$

Let us show that the operator $A^{\frac{1}{2}}$ defined in this way can be closed. If we put in (3)

$$f(u) = \frac{1}{\lambda - \sqrt{u}}, \text{ for } u \in (-\infty, 0] \text{ and } Re(\lambda) > 0$$

then we get an operator B_λ defined by

$$B_\lambda F_a x = \frac{1}{\lambda - ia} F_a x + \int_0^a ((a-u)\frac{1}{\lambda - iu})'' F_u x \, du.$$

By dividing this by a and letting $a \to +\infty$ we obtain the operator B_λ defined on the whole space X by

$$B_\lambda x = -2 \int_0^{+\infty} \frac{F_u x}{(\lambda - iu)^3} du.$$

It is obvious that the operator B_λ is bounded and therefore it is closed. From (3) it is easy to see that for all $a \geq 0$ and for all $x \in X$

$$B_\lambda(\lambda I - A^{\frac{1}{2}}) F_a x = (\lambda I - A^{\frac{1}{2}}) B_\lambda F_a x = F_a x.$$

This means that the operator B_λ (on the set $\bigcup_{a \geq 0} F_a(X)$) is the inverse operator of the operator $\lambda I - A^{\frac{1}{2}}$. Since the operator B_λ is closed, it follows that $\lambda I - A^{\frac{1}{2}}$ and therefore $A^{\frac{1}{2}}$ can be closed. The domain of the closure of the operator $A^{\frac{1}{2}}$ consists of all those $x \in X$ for which there exists

$$\lim_{a \to +\infty} (F_a x - \frac{F_a^2 x}{a}).$$

For all such x, $A^{\frac{1}{2}}x = i \lim_{a\to+\infty}(F_a x - \frac{F_a^2 x}{a})$, where the closure of the operator$A^{\frac{1}{2}}$ is denoted by $A^{\frac{1}{2}}$ too.

From [4] it is easy to see that for all $x \in \mathcal{D}((A^{\frac{1}{2}})^2)$, $(A^{\frac{1}{2}})^2 x = Ax$. In fact, it can be shown that $A = (A^{\frac{1}{2}})^2$.

The following lemma is easy to prove.

Lemma 1. *If $x \in \mathcal{D}(A)$ then*

$$A^{\frac{1}{2}}x = \frac{2i}{\pi}\int_0^{+\infty} \frac{(I - \mathbf{C}(t))x}{t^2} dt. \tag{11}$$

Proof. It is obvious that for all $x \in \mathcal{D}(A)$ the integral on the right side of (11) exists. Particularly, it exists for $F_a x$ $(a \geq 0,\ x \in \mathcal{D}(A))$. Using the equation (7), after some calculation we get

$$\frac{2\,i}{\pi}\int_0^{+\infty} \frac{(I - \mathbf{C}(t))F_a x}{t^2} dt = iaF_a x - iF_a^2 x$$

From here, using (10) and the fact that $\frac{F_a x}{a} \to x$ if $a \to +\infty$, (11) easily follows. □

Let $x, y \in X$. It is known that there exists

$$\langle x, y\rangle \overset{def}{=} \lim_{t\downarrow 0} \frac{\|x + ty\|^2 - \|x\|^2}{2t}.$$

The following theorem was proved in [5]:

Theorem. *Suppose X is a real Banach space. Then the "Riesz representation theorem" holds: Given $\delta \in X^*$ there exists $x_\delta \in X$ such that*

$$\|x_\delta\| = \|\delta\| \text{ and } \langle x_\delta, y\rangle = \delta(y) \text{ for all } y \in X \tag{$*$}$$

if and only if X is reflexive with a Gâteaux differentiable norm.

Furthermore x_δ is unique (and the mapping $\delta \to x_\delta$ is continuous from the norm topology on $X^*$to the weak topology on X) if and only if X is also strictly convex. In addition to that, the mapping $\delta \to x_\delta$ is also continuous from the norm topology on $X^*$to the norm on X if and only if X is also weakly uniformly convex.

From now on, let X be a complex Banach space. Let

$$(x, y) \overset{def}{=} \langle x, y\rangle - i\,\langle x, iy\rangle$$

Then a similar theorem holds for X. Here the role of the function $\langle x, y\rangle$ is taken by the function (x, y). Specifically, if X is a reflexive, strictly convex space with a Gâteaux differentiable norm, then $|(x,y)| \le \|x\| \cdot \|y\|$ and (x,y) is a continuous linear functional relative to y (for fixed $x \in X$). If we denote by φ the transformation $\delta \to x_\delta$ we can introduce a new operation "$\overset{*}{+}$" in X by

$$x \overset{*}{+} y = \varphi(\varphi^{-1}(x) + \varphi^{-1}(y)).$$

Then, the function (x,y) is antilinear in X relative to the operation "$\overset{*}{+}$". We will denote (if necessary) the space X provided with the operation "$\overset{*}{+}$" by $(X, \overset{*}{+}\,)$, and by $(X,+)$ we will denote the space X provided with the operation "+".

Lemma 2. *Suppose that X is a reflexive, strictly convex Banach space with a Gâteaux differentiable norm and let A be the infinitesimal operator of the bounded cosine function $\mathbf{C}(t)$, ($\|\mathbf{C}(t)\| \le 1$). Then the point 0 is not contained in the residual spectrum of the operator A.*

Proof. Suppose that the set $A(\mathcal{D}(A))$ is not dense in X. Then there exists $y \in X$, $y \ne 0$ such that

$$\langle y, Ax\rangle = 0, \quad x \in \mathcal{D}(A),$$

and we have (because of (8))

$$-a^2 \langle y, F_a x\rangle + \int_0^a (6u - 2a) \langle y, F_u x\rangle \, du = 0, \quad x \in X, a \ge 0.$$

If we put $\langle y, F_u x\rangle = \psi(u)$ we have $a^2\psi(a) = \int_0^a (6u-2a)\psi(u)du$.

Since $\psi(u)$ is a continuous function, it follows from here that ψ has derivatives of all orders (for $a > 0$). From the last equality we get $\psi''(a) = 0$. It follows from here that $\psi(a) = \alpha a + \beta$.

Since $\psi(0) = 0$, we have $\beta = 0$, and since $\frac{\psi(a)}{a} = \left\langle y, \frac{F_a x}{a}\right\rangle \to \langle y, x\rangle$ if $a \to +\infty$, we get $\alpha = \langle y, x\rangle$. It means that $\langle y, F_a x\rangle = \langle y, x\rangle a$ for all $x \in X$ and $a \ge 0$. This implies

$$\langle y, F_a y\rangle = a \|y\|^2$$

and further

$$a \|y\|^2 \le \|y\| \cdot \|F_a y\| \le a \|y\|^2 \text{, i.e. } \|F_a y\| = a \|y\| .$$

We see that $\langle y, F_a y\rangle = \|y\| \cdot \|F_a y\|$.

Since X is strictly convex, this gives $F_a y = ay$.

From this and from (8) we get $y \in \mathcal{D}(A)$ and $Ay = 0$.

So, the number 0 is a point of the point spectrum of the operator A. The lemma is proved. □

For the dual cosine function $\mathbf{C}^*(t)$ of the function $\mathbf{C}(t)$ we have a similar result.

We still assume that X is a reflexive, strictly convex Banach space with a Gâteaux differentiable norm. It is known that in a reflexive space X the dual semigroup $T^*(t)$ of a $\mathbf{C}_0-$semigroup $T(t)$ is also a $\mathbf{C}_0$-semigroup. It is true for the cosine operator functions too. In our case we can get it in the following way:

First, we are going to prove two lemmas.

Lemma 3. *Let X be the Banach space with the same properties as in Lemma 2. Then the set*

$$X_1^* \stackrel{def}{=} \{F_a^* \, f \mid \; a \geq 0, f \in X^*\}$$

is dense in X^*.

Proof. Let us assume that X_1^* is not dense in X^*. Then, because of reflexivity of the space X, there exists $x_0 \in X$, $x_0 \neq 0$ for which $(F_a^* f)(x_0) = 0$, $a \geq 0$, $f \in X^*$.

It means that $f(F_a x_0) = 0$, $a \geq 0$, $f \in X^*$.

From here and from $\frac{F_a x_0}{a} \to x_0$ if $a \to +\infty$ it follows $f(x_0) = 0 (\forall f \in X^*)$, i.e. $x_0 = 0$, which is a contradiction. It proves that X_1^* is dense in X^*. □

Lemma 4. *For all sufficiently small $h > 0$, the following holds*

$$\|F_{a+h} - F_{a-h}\| \leq \frac{4}{\pi}(a+e)h \; |\ln h| \, .$$

Proof. According to [2], we have

$$F_a = \frac{2}{\pi} \int\limits_0^{+\infty} \frac{\sin^2 at}{t^2} \mathbf{C}(2t) dt$$

and further

$$\|F_{a+h} - F_{a-h}\| \leq \frac{4}{\pi} \int\limits_0^1 \frac{|\sin at \sin ht|}{t^2} dt + \frac{4}{\pi} \int\limits_1^{+\infty} \frac{|\sin at \sin ht|}{t^2} dt$$

(here we used $\|\mathbf{C}(t)\| \leq 1$).

From here, for all $\alpha \in (0,1)$ we obtain

$$\|F_{a+h} - F_{a-h}\| \leq \frac{4ah}{\pi} + \frac{4h^\alpha}{\pi} \int\limits_1^{+\infty} \frac{dt}{t^2} = \frac{4ah}{\pi} + \frac{4h^\alpha}{\pi(1-\alpha)} \quad .$$

For all $h \in (0, e^{-1})$ the function $\frac{h^\alpha}{1-\alpha}$ has a minimum $eh\,|\ln h|$ at the point $\alpha = 1 + \frac{1}{\ln h} \in (0,1)$ and from the last inequality we get

$$\|F_{a+h} - F_{a-h}\| \leq \frac{4}{\pi}(a+e)h \, |\ln h| \, . \qquad \square$$

From Lemma 4 it immediately follows that

$$\left\|F_{a+h}^{*}-F_{a-h}^{*}\right\| \leq \frac{4}{\pi}(a+e)h\,|\ln h|$$

(for small $h>0$), and the function $F_a^*, a \geq 0$ is uniformly continuous.

We can identify the space X^* with $(X, \overset{*}{+})$ and we will do so in the sequel.

Now, from (7) we have, for $x \in X$ and $a \geq 0$

$$\mathbf{C}^*(t)F_a^*x = \cos at\, F_a^*x + \int_0^{a}{}^{*}((a-u)\cos ut)'' F_u^* x dt$$

where $\int_0^a{}^*$ means that the integration was performed relative to the operation "$\overset{*}{+}$".

Now, the uniform continuity of the function F_a^*, the boundedness of the function $\mathbf{C}^*(t)$ and Lemma 3 show that the function $\mathbf{C}^*(t)$ is strongly continuous.

It is not difficult to see that $X = (X, \overset{*}{+})$ is strictly convex. Namely, let us put

$$\langle x, y\rangle^* = \lim_{t\downarrow 0} \frac{\|x \overset{*}{+} ty\|^2 - \|x\|^2}{2t} = \|x\| \lim_{t\downarrow 0} \frac{\|x \overset{*}{+} ty\| - \|x\|}{t}.$$

Then for $\|x\| = 1$ we have

$$\langle x, y\rangle^* = \lim_{t\downarrow 0} \frac{\|x \overset{*}{+} ty\| - \|x\|}{t} \geq \lim_{t\downarrow 0} \frac{\langle x \overset{*}{+} ty, x\rangle - 1}{t} = \langle y, x\rangle\,.$$

If we put here $-y$ instead of y we get $\langle x, y\rangle^* \leq \langle y, x\rangle$ and further $\langle x, y\rangle^* = \langle y, x\rangle$. Now we see that from $\left|\langle x, y\rangle^*\right| = \|x\| \cdot \|y\|$, because of strict convexity of the space X (see [5]), follows that $y = \lambda x, \lambda \in \mathbb{R}$. But this is equivalent to the strict convexity of the space X^*.

From the equality $\langle x, y\rangle^* = \langle y, x\rangle$ it follows that the space $X^* = (X, \overset{*}{+})$ has a Gâteaux differentiable norm (see [5], Theorem 2).

We have seen that the space X^* and the function $\mathbf{C}^*(t)$ satisfy all conditions of Lemma 2, so the number 0 is not a point of the residual spectrum of the infinitesimal operator A^* of the cosine function $\mathbf{C}^*(t)$.

It is obvious now that the point 0 belongs to the point spectrum of the operator A iff 0 belongs to the point spectrum of the operator A^*.

For the linear operator $A : X \to X$ which is defined on a dense set in X we say that it is Hermitian if the operator A is the infinitesimal generator of the group of isometries $\left\|e^{i\,tA}x\right\| = \|x\|$ for all $x \in X$ and $t \in \mathbb{R}$.

From Lemma 2 we get the following theorem:

Theorem 5. *Let the space X satisfy the same conditions as in Lemma 2 and let $A : X \to X$ be a Hermitian operator. Then the residual spectrum of A is empty.*

Proof. Let $\lambda_0 \in \mathbb{R}$ be any number. It is obvious that the operator $A - \lambda_0 I$ is Hermitian too. It follows that $-(A-\lambda_0 I)^2$ is the infinitesimal generator of the bounded cosine function $\mathbf{C}(t)$:

$$\mathbf{C}(t) = \frac{e^{it(A-\lambda_0 I)} + e^{-it(A-\lambda_0 I)}}{2}, \|\mathbf{C}(t)\| \leq 1.$$

According to Lemma 2 the number 0 is not a point of the residual spectrum of the operator $(A-\lambda_0 I)^2$. But then the number λ_0 is not a point of the residual spectrum of the operator A.

The theorem is proved. □

Let us assume that the space $(X, +)$ has the same properties as in Lemma 2 and let L be some closed subspace of X. Then for all $x \in X$ we can find a sequence $u_n \in L$ for which

$$\|x - u_n\| \to d$$

where d is the distance of the vector x from the subspace L. The sequence $\{ u_n \}$ is obviously bounded and there exists a subsequence of that sequence which weakly converges to some $x_0 \in L$. Because of this we assume that $\{ u_n \}$ weakly converges to x_0. Then we have

$$\langle x - x_0, x - u_n \rangle \to \|x - x_0\|^2 .$$

Because of $\|x - x_0\| \cdot \|x - u_n\| \geq \langle x - x_0, x - u_n\rangle$ and because of $\|x - u_n\| \to d$ we have $\|x - x_0\| = d$ and from here

$$\frac{\|x - x_0 + tu\|^2 - \|x - x_0\|^2}{2t} \geq 0 \text{ for all } u \in L \text{ and } t > 0.$$

Hence, for all $u \in L$ we have $\langle x - x_0, u\rangle = \lim\limits_{t \downarrow 0} \frac{\|x-x_0+tu\|^2 - \|x-x_0\|^2}{2t} \geq 0$.

This immediately gives $\langle x - x_0, u\rangle = 0 (\forall u \in L)$.

It will be denoted as follows $(x - x_0) \perp u$ $(\forall u \in L)$ and we will say that $x - x_0$ is orthogonal to L. We can formulate these considerations into the next theorem:

Theorem 6. *Let X be a (complex) reflexive and strictly convex Banach space that has a Gâteaux differentiable norm. For every closed linear subspace L of the space $(X, +)$ there exists a subspace L^* of the space $(X, \overset{*}{+})$ such that*

$$X = L \oplus L^*$$

(i.e. every $x \in X$ can be written in a unique way in the form $x = l + l^*, l \in L, l^* \in L^*$ and $\langle l^*, l\rangle = 0$).

Obviously, we proved the following theorem too:

Theorem 6'. *Let X be the same as in Theorem 6 and let L^* be a closed subspace of the space $X^* = (X, \overset{*}{+})$. Then there exists a closed subspace L of the space $(X, +)$ such that*

$$X = L^* \overset{*}{\oplus} L$$

(i.e. every $x \in X$ can be written in a unique way in the form $x = l^* \overset{*}{+} l$, $l^* \in L^*$, $l \in L$ and $\langle l^*, l \rangle = 0$).

Let us go back to the function $\mathbf{C}(t)$ and its generator A. Let the number 0 be a point of the point spectrum of the operator A. Then, as we have already seen, this number is a point of the point spectrum of the operator A^* too. By L we will denote the null-subspace of the operator $A$and by M^* the null-subspace of the operator A^*. For $m^* \in M^*$ one has $A^*m^* = 0$, and therefore

$$\langle A^*m^*, x\rangle = 0 \qquad (\forall x \in \mathcal{D}(A)).$$

It means that

$$\langle m^*, Ax\rangle = 0 \qquad (\forall x \in \mathcal{D}(A)).$$

But from the proof of Lemma 2 we see that this relation implies $Am^* = 0$, so we see that $M^* \subseteq L$. Obviously, it is also $L \subseteq M^*$, and therefore $M^* = L$. The subspace L is linear in $(X, +)$ and M^* is linear in $(X, \overset{*}{+})$. Therefore $L = M^*$ is linear in $(X, +)$ and $(X, \overset{*}{+})$.

Let $L^*$and M be such that

$$X = L \oplus L^*\,,\ X = L \overset{*}{\oplus} M \ (= M^* \overset{*}{\oplus} M)$$

(see Theorem 6 and Theorem 6').

Let us prove that :

1) $X = L + M$, $L \cap M = \{0\}$ i.e. X is the direct sum of the subspaces L and M.
2) The subspace M is invariant relative to all operators $\mathbf{C}(t)$, $t \in \mathbb{R}$.

If $l \in L \cap M$ then (because of $L = M^*$) $l \in M^* \cap M$ and we have $l = 0$ (because of $\langle l, l\rangle = 0$). It means that $L \cap M = \{0\}$.

Let y be a vector that is orthogonal to $L + M$ (i.e. $\langle y, l + m\rangle = 0$ for all $l \in L$ and $m \in M$). Then $y \in M^* \cap L^* = L \cap L^*$ and therefore $y = 0$.

Now let us prove that $L + M$ is a closed subspace of $(X, +)$. Let $x_n \to x_0$ and $x_n \in L + M$. Then $x_n = l_n + m_n$, ($l_n \in L$, $m_n \in M$).

From here (because of $l_n \in M^*$, $X = M^* \overset{*}{\oplus} M$) it follows that $\langle l_n, x_n\rangle = \|l_n\|^2$.

Hence we have $\|l_n\| \le \|x_n\|$.

The sequence $\{ x_n \}$ is bounded, so the sequence $\{ l_n \}$ is bounded too. Therefore the sequence $\{ m_n \}$ is also bounded. Because of this we can assume that the sequences $\{ l_n \}$ and $\{ m_n \}$ are weakly convergent, i.e. $m_n \to m_0$, $l_n \to l_0$ weakly. From $l_n \in L$, $m_n \in M$ and from the previous relations it follows that $l_0 \in L$, $m_0 \in M$. Moreover $x_n \to x_0$ weakly and we have $x_0 = l_0 + m_0 \in L + M$. This proves our statement 1).

Now we take $x \in M$. For any $l \in M^*(= L)$ we have $\mathbf{C}^*(t)l = l$ and therefore $\langle \mathbf{C}^*(t)l, x\rangle = 0$. This means that $\langle l, \mathbf{C}(t)x\rangle = 0$ for all $l \in M^*$ and $t \in \mathbb{R}$. Hence $\mathbf{C}(t)x \in M$, so the statement 2) is also proved.

From these considerations we get the following theorem:

Theorem 7. *Let X be a Banach space with the same properties as in Theorem 6 and let $\mathbf{C}(t)$ be a bounded cosine operator function ($\|\mathbf{C}(t)\| \le 1$). If the point 0 belongs to the point spectrum of the infinitesimal generator of the function $\mathbf{C}(t)$, then X is a direct sum of the subspaces L and M of the space $(X, +)$ for which we have:*

α) $\mathbf{C}(t)x = x \qquad (\forall x \in L\,,\, \forall t \in \mathbb{R})$.

β) *The subspace M is invariant relative to all operators $\mathbf{C}(t)$, $t \in \mathbb{R}$.*

γ) *If we consider the function $\mathbf{C}(t)$ and its generator A on M then the point 0 does not belong to the point spectrum of the operator A.*

It is evident that a similar theorem is true for the dual function $\mathbf{C}^*(t)$ too.

3. Hilbert Transforms

Let $U(t)$ be a strongly continuous group of bounded operators on X. For $0 < \varepsilon < N$ and $x \in X$ we put

$$H_{\varepsilon,N}x = \frac{1}{\pi} \int\limits_{\varepsilon \le |t| \le N} \frac{U(t)x}{t} dt \tag{12}$$

If $\lim\limits_{\varepsilon\to 0,\, N\to+\infty} H_{\varepsilon,N}x$ exists then we will call it the Hilbert transform of the element x and denote it by Hx, i.e. $Hx = \lim\limits_{\varepsilon\to 0,\, N\to+\infty} H_{\varepsilon,N}x$ (see [6]).

If we put $\mathbf{C}(t) = \frac{1}{2}\left[U(t) + U(-t)\right]$, $t \in \mathbb{R}$, then $\mathbf{C}(t)$ is a strongly continuous cosine operator function with the infinitesimal generator A^2, where A is the infinitesimal generator of the group $U(t)$.

If $U(t)$ is a bounded group, i.e. if there exists a constant K with

$$\|U(t)\| \le K \qquad (\forall t \in \mathbb{R}) \tag{13}$$

then $\mathbf{C}(t)$ is bounded on $\mathbb{R}$, i.e. $\|\mathbf{C}(t)\| \le K$, $(\forall t \in \mathbb{R})$.

In the sequel we assume that $K = 1$. From here it easily follows that $U(t)$ is an isometry for all $t \in \mathbb{R}$:

$$\|U(t)x\| = \|x\| \qquad (t \in \mathrm{R} \quad , \quad x \in X).$$

In that case the generator A of the group $U(t)$ has the form $A = iB$, where B is a Hermitian linear operator. For the cosine function $\mathbf{C}(t)$ we have $\|\mathbf{C}(t)\| \le 1$, $t \in \mathbb{R}$.

It is obvious that (12) can be written in the form

$$H_{\varepsilon,N}x = \frac{1}{\pi}\int_{\varepsilon}^{N} \frac{U(t)x - U(-t)x}{t}dt. \tag{14}$$

It is well known that

$$U(t)x = x + A\int_0^t U(s)x\,ds, \quad U(-t)x = x - A\int_0^t U(-s)x\,ds.$$

We can write (14) in the form

$$H_{\varepsilon,N}x = \frac{2}{\pi}\int_{\varepsilon}^{N} \frac{AS(t)x}{t}dt = \frac{2}{\pi}A\int_{\varepsilon}^{N} \frac{S(t)x}{t}dt$$

where $S(t) = \int_0^t \mathbf{C}(s)ds$.

For $x \in \mathcal{D}(A^2)$ we have

$$\begin{aligned} H_{\varepsilon,N}Ax &= \frac{2}{\pi}\int_{\varepsilon}^{N} \frac{S(t)A^2x}{t}dt = \frac{2}{\pi}\frac{\int_0^t S(s)A^2x}{t}ds\,\Big|_{\varepsilon}^{N} + \int_{\varepsilon}^{N}\frac{dt}{t^2}\int_0^t S(s)A^2x ds = \\ &= \frac{2}{\pi}\,\frac{\mathbf{C}(t)x - x}{t}\Big|_{\varepsilon}^{N} + \frac{2}{\pi}\int_{\varepsilon}^{N}\frac{\mathbf{C}(t)x - x}{t^2}dt = \\ &= \frac{2}{\pi}\left[\frac{\mathbf{C}(N)x - x}{N} - \frac{\mathbf{C}(\varepsilon)x - x}{\varepsilon} + \int_{\varepsilon}^{N}\frac{\mathbf{C}(t)x - x}{t^2}dt\right]. \end{aligned}$$

From the boundedness of the function $\mathbf{C}(t)$ it follows

$$\frac{\mathbf{C}(N)x - x}{N} \to 0 \qquad \text{if } N \to +\infty$$

and from $x \in \mathcal{D}(A^2)$ it follows

$$\frac{\mathbf{C}(\varepsilon)x - x}{\varepsilon} = \varepsilon\frac{\mathbf{C}(\varepsilon)x - x}{\varepsilon^2} \to 0 \text{ if } \varepsilon \to 0.$$

In this way we get

$$\lim_{\varepsilon\to 0,\, N\to+\infty} H_{\varepsilon,N}Ax = \frac{2}{\pi}\int_0^{+\infty}\frac{\mathbf{C}(t)x - x}{t^2}dt.$$

Therefore

$$HAx = \frac{2}{\pi}\int_0^{+\infty} \frac{\mathbf{C}(t)x - x}{t^2}dt, \quad x \in \mathcal{D}(A^2). \tag{15}$$

Now we are going to apply Lemma 1 to our present situation (in which $A^2 = -B^2$ is the infinitesimal generator of the function $\mathbf{C}(t)$).

If we denote by A_+ a closed extension of the operator A_0 which is defined by

$$A_0 F_a = aF_a x - F_a^2 x,$$

then (15) becomes

$$HAx = -A_+ x, \quad x \in \mathcal{D}(A^2). \tag{16}$$

We know that $A_+^2 = B^2$ for $x \in \mathcal{D}(A^2)$. Besides, we have

$$\begin{aligned} \langle F_a x, A_+ F_a x\rangle &= a\left\|F_a x\right\|^2 - \left\langle F_a x, F_a^2 x\right\rangle \geq a\left\|F_a x\right\|^2 - \left\|F_a x\right\| \cdot \left\|F_a^2 x\right\| \geq \\ &\geq a\left\|F_a x\right\|^2 - \left\|F_a x\right\|^2 \cdot \left\|F_a\right\| \geq a\left\|F_a x\right\|^2 - a\left\|F_a x\right\|^2 = 0 \end{aligned}$$

The set $\bigcup\limits_{a\geq 0} F_a(X)$ is the core of the operator A_+, and from that we have

$$\langle x, A_+ x\rangle \geq 0 \qquad (x \in \mathcal{D}(A_+))$$

Because of that we will call the operator A_+ the positive square root of $B^2 = -A^2$.

Now, let us apply Theorem 7 to our present situation.

We have $U(t)x + U(-t)x = 2x$ for $x \in L$, $t \in \mathbb{R}$.

From here, because of strict convexity of the space X and because of isometry of the operators $U(t)$ and $U(-t)$, it follows that $U(t)x = U(-t)x = x$.

Using the fact that $\langle Vy, Vx\rangle = \langle y, x\rangle$ for all isometric operators V, which is obvious from the definition of $\langle x, y\rangle$, we have $\langle y, U(t)x\rangle = \langle U(t)y, U(t)x\rangle = \langle y, x\rangle = 0$ for $y \in L$, $x \in M$ and $t \in \mathbb{R}$.

We see that $U(t)x \in M$ for all $x \in M$ and $t \in \mathbb{R}$.

So, $U_1(t)$ $(t \in \mathbb{R})$ is a group of isometric operators on M, where $U_1(t)$ $(t \in \mathbb{R})$ denotes the restriction of the operator $U(t)$ $(t \in \mathbb{R})$ on M.

Now, from Theorem 7, we can conclude that the rank $\mathcal{R}(A_1^2(M))$ of the operator A_1^2 is dense in M, where A_1 is the infinitesimal generator of the group $U_1(t)$.

Now, we see that the operator H is defined at least on the set $L \oplus A_1^2(M)$, which is dense in X.

So, we get the following theorem.

Theorem 8. *Let X be a reflexive, strictly convex space with a Gâteaux differentiable norm. Then the above Hilbert transform is defined on some set which is dense in X. It can be extended to a linear bounded operator on the whole X iff there exists a constant P for which*

$$\|A_+ x\| \leq P\,\|Ax\| \qquad (\forall x \in \mathcal{D}(A^2)).$$

Moreover, for $x \in \mathcal{D}(A^2)$ *we have* $HAx = -A_+x$*, where* A_+ *is the positive square root of* $-A^2$*.*

Suppose now that the operator B is positive, i.e. that $\langle x, Bx\rangle \geq 0$ for all $x \in \mathcal{D}(A^2)$. We claim that $A_+x = Bx$ for all $x \in \mathcal{D}(A^2)$. Let us take any $\varepsilon > 0$. We have

$$\langle x, (B + \varepsilon I)x\rangle \geq \varepsilon \|x\|^2 \quad (x \in \mathcal{D}(A^2)). \tag{17}$$

Let $\mathbf{C}_\varepsilon(t)$ denote the cosine operator function defined by

$$\mathbf{C}_\varepsilon(t) = \frac{e^{i\varepsilon t}U(t) + e^{-i\varepsilon t}U(-t)}{2} \quad .$$

Then $\|\mathbf{C}_\varepsilon(t)\| \leq 1$, $t \in \mathbb{R}$. The infinitesimal generator of the function $\mathbf{C}_\varepsilon(t)$ is $-(B + \varepsilon I)^2$. Let $F_{a,\varepsilon}$, $a \geq 0$ be the family of operators that corresponds to the function $\mathbf{C}_\varepsilon(t)$ in the same way the family F_a corresponds to the function $\mathbf{C}(t)$, and let $(B + \varepsilon I)_+$ be the positive square root of $(B + \varepsilon I)^2$:

$$(B + \varepsilon I)_+ F_{a,\varepsilon}x = aF_{a,\varepsilon}x - F^2_{a,\varepsilon}x.$$

It is clear that $B + \varepsilon I$ commutes with $F_{a,\varepsilon}$ and therefore with $(B + \varepsilon I)_+$.

Now from $(B + \varepsilon I)^2 - (B + \varepsilon I)^2_+ = 0$ it follows that $\langle y, (B + \varepsilon I)y\rangle + \langle y, (B + \varepsilon I)_+ y\rangle = 0$, where $y = (B + \varepsilon I)F_{a,\varepsilon}x - (B + \varepsilon I)_+F_{a,\varepsilon}x$.

Since $\langle y, (B + \varepsilon I)y\rangle \geq 0$ and $\langle y, (B + \varepsilon I)_+y\rangle \geq 0$, we conclude that $\langle y, (B + \varepsilon I)y\rangle = 0$, which together with (17), gives $y = 0$.

Therefore $(B + \varepsilon I)F_{a,\varepsilon}x = (B + \varepsilon I)_+F_{a,\varepsilon}x$.

It is easy to see that $F_{a,\varepsilon} \to F_a$ uniformly, if $\varepsilon \to 0$, and that for all $x \in \mathcal{D}(A^2)$

$$(B + \varepsilon I)x \to Bx \text{ and } (B + \varepsilon I)_+x \to A_+x \text{ if } \varepsilon \to 0.$$

From the last equality it now follows $BF_ax = A_+F_ax, \quad (x \in X)$.

From this equality, because of the fact that A and A_+ commute with F_a we conclude that $Bx = A_+x$ for all $x \in \mathcal{D}(A^2)$.

In this way, using Theorem 8, we get the following theorem:

Theorem 9. *Let* X *satisfy the conditions of Theorem 8 and let the infinitesimal generator* A *have the form* $A = iB$ *where* B *is positive and let the point 0 not belong to the point spectrum of the operator* A*. Then the Hilbert transform is defined on the whole* X *and it is equal to* iI*.*

Let us now consider the case when $X = \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space. Then the operator A can be written in the form $A = i(B_+ - B_-)$, where B_+ and B_- are positive operators and $B_+B_- = B_-B_+ = 0$ on $\mathcal{D}(A)$.

Let $E_\pm$ denote the orthogonal projection of the space X onto the closed linear span of the range of the operator $B_\pm$. Then $E_+E_- = E_-E_+ = 0$.

From Theorem 9 (applied to $E_\pm(\mathcal{H})$) we immediately get the following theorem:

Theorem 10. *Let $\mathcal{H}$ be a Hilbert space and let $U(t)$ be a group of isometric operators on $\mathcal{H}$, and let the point 0 not belong to the point spectrum of the operator A. Then the Hilbert transform is defined on the whole $\mathcal{H}$ and*

$$Hx = i(E_+x - E_-x) \qquad (\forall x \in \mathcal{H}).$$

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