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ON THE ULAM PROBLEM FOR EULER QUADRATIC MAPPINGS

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Abstract. In 1940 and in 1968 S. M. Ulam proposed *the general problem:"When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?".* In 1941 D. H. Hyers solved this stability problem for linear mappings. In 1951 D. G. Bourgin was the second author to treat the same problem for additive mappings. According to P. M. Gruber (1978) this kind of stability problems are of particular interest in *probability theory* and in the case of *functional equations* of different types. In 1982-2002 we solved the above Ulam problem for linear and non-linear mappings and established analogous stability problems even on restricted domains. Besides, we applied some of our recent results to the asymptotic behavior of functional equations of different types. In this paper we investigate the *Euler quadratic mappings* $Q : X \to Y$, satisfying the functional equation

$$\begin{aligned}&Q(x_0 - x_1) + Q(x_1 - x_2) + Q(x_2 - x_3) + Q(x_3 - x_0)\\&= Q(x_0 - x_2) + Q(x_1 - x_3) + Q(x_0 - x_1 + x_2 - x_3)\end{aligned}$$

and then solve the corresponding Ulam stability problem.

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1. EULER QUADRATIC EQUATION

Definition 1. *Let X be a normed linear space and let Y be a complete normed linear space. Then a mapping $Q : X \to Y$, is called Euler quadratic, if the functional equation*

$$\begin{aligned}&Q(x_0 - x_1) + Q(x_1 - x_2) + Q(x_2 - x_3) + Q(x_3 - x_0)\\&= Q(x_0 - x_2) + Q(x_1 - x_3) + Q(x_0 - x_1 + x_2 - x_3)\end{aligned} \tag{1}$$

holds for all $(x_0, x_1, x_2, x_3) \in X^4$ ([16]-[22]).

Note that Q is called *Euler quadratic* because the following quadratic vector identity

$$|x_0 - x_1|^2 + |x_1 - x_2|^2 + |x_2 - x_3|^2 + |x_3 - x_0|^2 =$$
$$= |x_0 - x_2|^2 + |x_1 - x_3|^2 + |x_0 - x_1 + x_2 - x_3|^2$$

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holds for all real vectors x_0, x_1, x_2, x_3, whose geometric interpretation leads to the Euler theorem on quadrilaterals $A_1A_2A_3A_4$ with position vectors x_0, x_1,x_2, x_3 of vertices A_1, A_2, A_3, A_4, respectively, and because the functional equation

$$Q(2^n x) = (2^n)^2 Q(x) \tag{2}$$

holds for all $x \in X$ and all $n \in N$ ([21]).

In fact, substitution of $x_0 = x_1 = x_2 = x_3 = 0$ in equation(1) yields that

$$Q(0) = 0 \tag{1a}$$

Lemma 1. *Let $Q : X \to Y$ be an Euler quadratic mapping satisfying equation (1). Then Q is* **an even mapping***; that is the equation*

$$Q(-x) = Q(x) \tag{3}$$

holds for all $x \in X$.

Proof. Substituting $x_0 = x_1 = x_2 = 0$ and $x_3 = x$ in the equation (1) and employing (1a) one gets that equation

$$2Q(0) + Q(-x) + Q(x) = Q(0) + Q(-x) + Q(-x),$$

or

$$Q(-x) + Q(x) = 2Q(-x),$$

or the required equation (3), completing the proof of Lemma 1. □

Lemma 2. *Let $Q : X \to Y$ be an Euler quadratic mapping satisfying equation (1). Then Q satisfies the equation*

$$Q(x) = 2^{-2n} Q(2^n x) \tag{2a}$$

for all $x \in X$ and all $n \in N$.

Proof. Substituting $x_0 = x$, x_1= 0, $x_2 = x$, $x_3 = 0$ in equation (1) and employing equations (1a) and (3) one gets the equation

$$2Q(x) + 2Q(-x) = 2Q(0) + Q(2x),$$

or

$$4Q(x) = Q(2x),$$

or

$$Q(x) = 2^{-2} Q(2x) \tag{4}$$

for all $x \in X$.

Then induction on $n \in N$ with $x \to 2^{n-1}x$ in the equation (4) yields equation (2a). In fact, the equation (4) with $x \to 2^{n-1}x$ yield that the functional equation

$$Q(2^{n-1}x) = 2^{-2} Q(2^n x) \tag{4a}$$

holds for all $x \in X$.

Moreover by induction hypothesis with $n \to n-1$ in equation (2a) one gets that

$$Q(x) = 2^{-2(n-1)} Q(2^{n-1}x) \tag{4b}$$

holds for all $x \in X$.

Thus functional equations (4a)-(4b) imply

$$Q(x) = 2^{-2(n-1)} 2^{-2} Q(2^n x),$$

or

$$Q(x) = 2^{-2n} Q(2^n x),$$

for all $x \in X$ and all $n \in N$, completing the proof of the required functional equation (2a) and hence the proof of Lemma 2. □

2. EULER QUADRATIC INEQUALITY

Definition 2. *Let X be a normed linear space and let Y be a real complete normed linear space. Then a mapping $f : X \to Y$, is called approximately Euler quadratic, if the new Euler quadratic functional inequality*

$$\begin{aligned} &\|f(x_0 - x_1) + f(x_1 - x_2) + f(x_2 - x_3) + f(x_3 - x_0) \\ &- [f(x_0 - x_2) + f(x_1 - x_3) + f(x_0 - x_1 + x_2 - x_3)]\| \le c, \end{aligned} \tag{1)'$$

holds for all $(x_0, x_1, x_2, x_3) \in X^4$ with a constant c (independent of x_0, x_1,x_2, x_3) ≥ 0.

Definition 3. *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition that there exists a constant c (independent of $x \in X$) ≥ 0. Then an Euler quadratic mapping $Q : X \to Y$, is said that exists near an approximately Euler quadratic mapping $f : X \to Y$, if the following inequality*

$$\|f(x) - Q(x)\| \le c, \tag{1)''$$

holds for all $x \in X$.

Theorem 1. *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition the above-mentioned mappings Q, f and the three definitions. Then the limit*

$$Q(x) = \lim_{n\to\infty} 2^{-2n} f(2^n x) \tag{5}$$

exists *for all $x \in X$ and all $n \in N$ and $Q : X \to Y$ is the* **unique** *Euler quadratic mapping near the approximately Euler quadratic mapping $f : X \to Y$.*

Proof of Existence in Theorem. Substitution of $x_i = 0$ $(i = 0, 1, 2, 3)$ in inequality $(1)'$ yields

$$\|4f(0) - 3f(0)\| \leq c,$$

or

$$\|f(0)\| \leq c \tag{1a$'$}$$

Lemma 3. *Let $f : X \to Y$ be an approximately Euler quadratic mapping satisfying inequality (1)$'$. Then f is* **an approximately even mapping***; that is the inequality*

$$\|f(-x) - f(x)\| \leq c \tag{3$'$}$$

holds for all $x \in X$ with constant c (independent of $x \in X$) ≥ 0.

Proof. Substitution of $x_0 = x_1 = x_2 = 0$, $x_3 = x$ in inequality $(1)'$ one gets that the inequality

$$\|2f(0) + f(-x) + f(x) - [f(0) + 2f(-x)]\| \leq c,$$

or

$$\|f(-x) - f(x) - f(0)\| \leq c \tag{3$''$}$$

holds for all $x \in X$.

Similarly, substituting $x_0 = x_1 = x_2 = x$, $x_3 = 0$ in inequality $(1)'$ we establish

$$\|f(-x) - f(x) + f(0)\| \leq c \tag{3$'''$}$$

Note that the substitution of x with $-x$ in inequality $(3)''$ also yields inequality $(3)'''$ for all $x \in X$.

Thus employing Ineqs. $(3)''$- $(3)'''$ and triangle inequality one finds that

$$2\,\|f(-x) - f(x)\| \leq \|f(-x) - f(x) - f(0)\| + \|f(-x) - f(x) + f(0)\|$$

$$\leq c + c = 2c,$$

or

$$\|f(-x) - f(x)\| \leq c$$

holds for all $x \in X$, completing the proof of Lemma 3. □

Lemma 4. *Let $f : X \to Y$ be an approximately Euler quadratic mapping satisfying inequality $(1)'$. Then f satisfies the inequality*

$$\left\|f(x) - 2^{-2n} f(2^n x)\right\| \leq (1 - 2^{-2n})c, \tag{2a$'$}$$

for all $x \in X$ and all $n \in N$ with constant c (independent of $x \in X$) ≥ 0.

Proof. Substituting $x_0 = x$, $x_1 = 0$, $x_2 = x$, $x_3 = 0$ in inequality $(1)'$ one gets that the inequality

$$\|2f(x) + 2f(-x) - [2f(0) + f(2x)]\| \le c,$$

or

$$\|f(2x) - 2f(-x) - 2f(x) + 2f(0)\| \le c \tag{$3a$}''$$

holds for all $x \in X$.

inequality $(3)''$ yields the functional inequality

$$\|2f(-x) - 2f(x) - 2f(0)\| \le 2c \tag{$3b$}''$$

for all $x \in X$.

Applying inequlities $(3a)''$-$(3b)''$ and triangle inequality we find that *the inequality*

$$\begin{aligned} 4\left\|f(x) - 2^{-2}f(2x)\right\| &= \|-4f(x) + f(2x)\| = \\ &= \|[f(2x) - 2f(-x) - 2f(x) + 2f(0)] + [2f(-x) - 2f(x) - 2f(0)]\| \\ &\le \|f(2x) - 2f(-x) - 2f(x) + 2f(0)\| + 2\|f(-x) - f(x) - f(0)\| \\ &\le \quad c + 2c = 3c \end{aligned}$$

or

$$\left\|f(x) - 2^{-2}f(2x)\right\| \le \frac{3}{4}c,$$

or

$$\left\|f(x) - 2^{-2}f(2x)\right\| \le (1 - 2^{-2})c \tag{4}'$$

holds for all $x \in X$ with constant c (independent of x) ≥ 0.

Replacing now x with $2x$ in inequality $(4)'$ one concludes that

$$\left\|f(2x) - 2^{-2}f(2^2x)\right\| \le (1 - 2^{-2})c,$$

or

(6) $$\left\|2^{-2}f(2x) - 2^{-4}f(2^2x)\right\| \le (2^{-2} - 2^{-4})c$$

holds for all $x \in X$.

Functional inequalities $(4)'$ - (6) and the triangle inequality yield

$$\begin{aligned} \left\|f(x) - 2^{-4}f(2^2x)\right\| &\le \left\|f(x) - 2^{-2}f(2x)\right\| + \left\|2^{-2}f(2x) - 2^{-4}f(2^2x)\right\| \\ &\le \left[(1 - 2^{-2}) + (2^{-2} - 2^{-4})\right] c, \end{aligned}$$

or that the functional inequality

$$\left\|f(x) - 2^{-4}f(2^2x)\right\| \le (1 - 2^{-4})c, \tag{6a}$$

holds for all $x \in X$.

Similarly, by induction on $n \in N$ with $x \to 2^{n-1}x$ in inequality $(4)'$ claim that $(2a)'$ holds for all $x \in X$ and all $n \in N$ with constant c (independent of $x \in X$)≥ 0.

In fact, inequality $(4)'$ with $x \to 2^{n-1}x$ yield the functional inequality

$$\left\|f(2^{n-1}x) - 2^{-2}f(2^n x)\right\| \le (1 - 2^{-2})c,$$

or that the functional inequality

$$\left\|2^{-2(n-1)}f(2^{n-1}x) - 2^{-2n}f(2^n x)\right\| \le (2^{-2(n-1)} - 2^{-2n})c, \tag{7a}$$

holds for all $x \in X$.

Moreover, by induction the hypothesis with $n \to n-1$ in inequality $(2a)'$ one gets that

$$\left\|f(x) - 2^{-2(n-1)}f(2^{n-1}x)\right\| \le (1 - 2^{-2(n-1)})c, \tag{7b}$$

holds for all $x \in X$.

Thus functional inequalities (7a)-(7b) and the triangle inequality imply

$$\begin{aligned}\left\|f(x) - 2^{-2n}f(2^n x)\right\| &\le \left\|f(x) - 2^{-2(n-1)}f(2^{n-1}x)\right\| + \\ &+ \left\|2^{-2(n-1)}f(2^{n-1}x) - 2^{-2n}f(2^n x)\right\| \\ &\le \left[(1 - 2^{-2(n-1)}) + (2^{-2(n-1)} - 2^{-2n})\right] c\end{aligned}$$

or

$$\left\|f(x) - 2^{-2n}f(2^n x)\right\| \le (1 - 2^{-2n})c,$$

completing the proof of the required functional inequality $(2a)'$, and thus the proof of Lemma 4. □

Lemma 5. *Let $f : X \to Y$ be an approximately Euler quadratic mapping satisfying inequality* $(1)'$. *Then the sequence*

$$\left\{2^{-2n}f(2^n x)\right\} \tag{8}$$

converges.

Proof. Note that from the functional inequality $(2a)'$ and the completeness of Y, one proves that the above-mentioned sequence (8) is *a Cauchy sequence.*

In fact, if $i > j > 0$, then

$$\left\|2^{-2i}f(2^i x) - 2^{-2j}f(2^j x)\right\| = 2^{-2j}\left\|2^{-2(i-j)}f(2^i x) - f(2^j x)\right\|, \tag{9}$$

holds for all $x \in X$, and all $i, j \in N$.

Setting $h = 2^j x$ in (9) and employing the functional inequality $(2a)'$ one concludes that

$$\left\|2^{-2i}f(2^i x) - 2^{-2j}f(2^j x)\right\| = 2^{-2j}\left\|2^{-2(i-j)}f(2^{i-j}h) - f(h)\right\|$$

$$\leq 2^{-2j}(1 - 2^{-2(i-j)})c,$$

or

$$\left\| 2^{-2i} f(2^i x) - 2^{-2j} f(2^j x) \right\| \leq (2^{-2j} - 2^{-2i})c < 2^{-2j}c,$$

or

$$\lim_{j\to\infty} \left\| 2^{-2i} f(2^i x) - 2^{-2j} f(2^j x) \right\| = 0, \tag{9a}$$

which yields that the sequence (8) is *a Cauchy sequence*, and thus the proof of Lemma 5 is complete. □

Lemma 6. *Let $f : X \to Y$ be an approximately Euler quadratic mapping satisfying inequality* $(1)'$. *Assume in addition a mapping $Q : X \to Y$ given by the above formula (5). Then $Q = Q(x)$ is* **a well-defined mapping** *and that Q is an Euler quadratic mapping in X.*

Proof. Employing Lemma 5 and formula (5), one gets that Q is *a well-defined mapping.* This means that the limit (5) exists for all $x \in X$.

In addition, let us prove that Q *satisfies the* functional equation (1) for all vectors $(x_0, x_1, x_2, x_3) \in X^4$. In fact, it is clear from $(1)'$ and the limit (5) that the following inequality

$$\begin{aligned} &2^{-2n} \| f(2^n x_0 - 2^n x_1) + f(2^n x_1 - 2^n x_2) + f(2^n x_2 - 2^n x_3) + f(2^n x_3 - 2^n x_0) \\ &- [f(2^n x_0 - 2^n x_2) + f(2^n x_1 - 2^n x_3) + f(2^n x_0 - 2^n x_1 + 2^n x_2 - 2^n x_3)] \| \\ &\leq 2^{-2n} c, \end{aligned}$$
(10)

holds for all $(x_0, x_1, x_2, x_3) \in X^4$ and all $n \in N$.

Therefore from inequality (10) one gets

$$\begin{aligned} &\Big\| \mathop{lim}_{n\to\infty} 2^{-2n} f\left[2^n(x_0 - x_1)\right] + \mathop{lim}_{n\to\infty} 2^{-2n} f\left[2^n(x_1 - x_2)\right] + \\ &+ \mathop{lim}_{n\to\infty} 2^{-2n} f\left[2^n(x_2 - x_3)\right] + \mathop{lim}_{n\to\infty} 2^{-2n} f\left[2^n(x_3 - x_0)\right] - \\ &- \Big\{ \mathop{lim}_{n\to\infty} 2^{-2n} f\left[2^n(x_0 - x_2)\right] + \mathop{lim}_{n\to\infty} 2^{-2n} f\left[2^n(x_1 - x_3)\right] + \\ &+ \mathop{lim}_{n\to\infty} 2^{-2n} f\left[2^n(x_0 - x_1 + x_2 - x_3)\right] \Big\} \Big\| \leq \left(\mathop{lim}_{n\to\infty} 2^{-2n} \right) c = 0, \end{aligned}$$

or

$$\begin{aligned} &\| Q(x_0 - x_1) + Q(x_1 - x_2) + Q(x_2 - x_3) + Q(x_3 - x_0) \\ &- [Q(x_0 - x_2) + Q(x_1 - x_3) \ + Q(x_0 - x_1 + x_2 - x_3)] \| = 0 \end{aligned} \tag{10a}$$

or mapping Q satisfies the quadratic equation (1) for all $(x_0, x_1, x_2, x_3) \in X$. Thus Q is an Euler quadratic mapping, completing the proof of Lemma 6.

It is clear now from Lemmas 1-6 and especially from inequality $(2a)'$, $n \to \infty$, and formula (5) that inequality $(1)''$ holds in X. Thus *the proof of existence in this Theorem* is complete.

Proof of Uniqueness in Theorem. Let $Q' : X \to Y$ be another Euler quadratic mapping satisfying the new quadratic functional equation (1), such that the inequality

$$\|f(x) - Q'(x)\| \leq c, \tag{1a}''$$

holds for all$x \in X$. If there exists an Euler quadratic mapping $Q : X \to Y$ satisfying the new quadratic functional equation (1), then

$$Q(x) = Q'(x) \tag{11}$$

holds for all$x \in X$.

To prove the *uniqueness* one employs equation (2a) for Q and Q', as well, so that

$$Q'(x) = 2^{-2n}Q'(2^n x), \tag{2a}''$$

holds for all $x \in X$, and all $n \in N$. Moreover, the triangle inequality and inequalities $(1)''$-$(1a)''$ yield

$$\|Q(2^n x) - Q'(2^n x)\| \leq \|Q(2^n x) - f(2^n x)\| + \|f(2^n x) - Q'(2^n x)\|$$

$$\leq c + c = 2c,$$

or

$$\|Q(2^n x) - Q'(2^n x)\| \leq 2c, \tag{12}$$

for all $x \in X$, and all $n \in N$. Then from equations (2a)-$(2a)''$, and inequality (12), one proves that

$$\|Q(x) - Q'(x)\| = \left\|2^{-2n}Q(2^n x) - 2^{-2n}Q'(2^n x)\right\| =$$

$$= 2^{-2n}\|Q(2^n x) - Q'(2^n x)\| \leq 2(2^{-2n})c, \tag{12a}$$

holds for all $x \in X$ and all $n \in N$. Therefore, from the above inequality (12a), and $n \to \infty$, one establishes

$$\lim_{n\to\infty} \|Q(x) - Q'(x)\| \leq 2\left(\lim_{n\to\infty} 2^{-2n}\right)c = 0,$$

or

$$\|Q(x) - Q'(x)\| = 0,$$

or

$$Q(x) = Q'(x),$$

for all $x \in X$, completing the proof of *uniqueness* and thus the *stability* of this Theorem ([1]-[15]) and ([23]-[28]).

Example 1 *Take $f : R \to R$ be a real function such that $f(x) = lx^2 + k$, l = real constant ($\neq 0$), k = constant : $|k| \leq c$, in order that f satisfies inequality (1).*

Moreover, there exists a unique Euler quadratic mapping $Q : R \to R$ such that from the limit (5) one gets

$$Q(x) = \lim_{n\to\infty} 2^{-2n} f(2^n x) = \lim_{n\to\infty} 2^{-2n} \left[l(2^n x)^2 + k \right] = lx^2.$$

Finally let us prove that inequality $(1)''$ holds. In fact, the above condition on $k : |k| \le c$, implies

$$\|f(x) - Q(x)\| = \left\| \left(lx^2 + k\right) - lx^2 \right\| = |k| \le c,$$

satisfying inequality $(1)''$ *, because from inequality* $(1)'$ *one gets that*

$$\Big\| \left[l(x_0 - x_1)^2 + k \right] + \left[l(x_1 - x_2)^2 + k \right] + \left[l(x_2 - x_3)^2 + k \right] + \left[l(x_3 - x_0)^2 + k \right]$$

$$- \left\{ \left[l(x_0 - x_2)^2 + k \right] + \left[l(x_1 - x_3)^2 + k \right] + \left[l(x_0 - x_1 + x_2 - x_3)^2 + k \right] \right\} \Big\| \le c$$

or

$$|k + k + k + k - [k + k + k]| = |k| \le c.$$

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