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# ABOUT d-LINEAR CONNECTIONS COMPATIBLE WITH A CONFORMAL METRICAL STRUCTURE

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**Abstract.** Starting from the notion of conformal metrical structure in the tangent bundle, given by R. Miron and M. Anastasiei in [10], [11], we define the notion of conformal metrical d-linear connection with respect to a conformal metrical structure corresponding to the 1-forms $\omega$ and $\tilde{\omega}$ in $TM$. We determine all conformal metrical d-linear connections in the case when the nonlinear connection is arbitrary and we give important particular cases. Further, we find the transformation group of these connections. We study the role of the torsion tensor fields T and S in this theory, especially the semi-symmetric d-linear connections, and the group of transformations of semi-symmetric conformal metrical d-linear connections, having the same nonlinear connection N and its important invariants.

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## 1. Introduction

The geometry of the tangent bundle: $(TM, \pi, M)$ has been studied by V. Oproiu [13], by M. Matsumoto [8], [9], by K. Yano and S. Ishihara [15], by I. Čomić [4], [5], by Gh. Atanasiu and I. Ghinea [1], by R. Miron and M. Hashiguchi [12], by R. Miron and M. Anastasiei [10], [11], by R. Bowman [3], by D.Bao at al. [2] and many others.

Concerning the terminology and notations, we use those from [8].

Let M be a real $C^\infty$-differentiable manifold with dimension n, and $(TM, \pi, M)$ its tangent bundle.

If $(x^i)$ is a local coordinates system on a domain U of a chart on M, the induced system of coordonates on $\pi^{-1}(U)$ is: $(x^i, y^i), (i = \overline{1,n})$.

Let N be a nonlinear connection on $TM$, with the coefficients $N^i{}_j(x, y)$, $(i = \overline{1,n}, j = \overline{1,n})$.

We consider on $TM$ a metrical structure G defined by ([10], [11]):

$$(1.1) \quad G(x, y) = \tfrac{1}{2} g_{ij}(x, y) dx^i \wedge dx^j + \tfrac{1}{2} \tilde{g}_{ij}(x, y) \delta y^i \wedge \delta y^j,$$

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where $\{dx^i, \delta y^i\}, (i = \overline{1,n})$ is the dual basis of $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$, and $(g_{ij}(x,y), \tilde{g}_{ij}(x,y))$ is a pair of given d-tensor fields on $TM$, of the type (0,2), each of them being symmetric and nondegenerate.

We asociate to the lift G the Obata's operators:

$$(1.2)\quad \begin{cases} \Omega^{ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j - g_{sj}g^{ir}),\ \Omega^{*ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j + g_{sj}g^{ir}), \\ \tilde{\Omega}^{ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j - \tilde{g}_{sj}\tilde{g}^{ir}),\ \tilde{\Omega}^{*ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j + \tilde{g}_{sj}\tilde{g}^{ir}). \end{cases}$$

Obata's operators have the same properties as the ones associated with a Finsler space [12].

Let $\mathcal{S}_2(TM)$ be the set of all symmetric d-tensor fields of the type (0,2) on $TM$. As is easily shown, the relations on $\mathcal{S}_2(TM)$ defined by:

$$(1.3)\quad \begin{cases} (a_{ij} \sim b_{ij}) \Leftrightarrow \left((\exists)\,\lambda(x,y) \in \mathcal{F}(TM), a_{ij}(x,y) = e^{2\lambda(x,y)}b_{ij}(x,y)\right), \\ \left(\tilde{a}_{ij} \sim \tilde{b}_{ij}\right) \Leftrightarrow \left((\exists)\,\mu(x,y) \in \mathcal{F}(TM), \tilde{a}_{ij}(x,y) = e^{2\mu(x,y)}\tilde{b}_{ij}(x,y)\right), \end{cases}$$

are an equivalence relations on $\mathcal{S}_2(TM)$.

**Definition 1.1.** *([10], [11]) The equivalent class: $\hat{G}$ of $\mathcal{S}_2(TM)/_\sim$ to which the metrical tensor field G belongs, is called conformal metrical structure on $TM$.*

Thus:

$$(1.4)\quad \hat{G} = \{G' | G'_{ij}(x,y) = e^{2\lambda(x,y)}g_{ij}(x,y) \text{ and } \tilde{G}'_{ij}(x,y) = e^{2\mu(x,y)}\tilde{g}_{ij}(x,y)\}.$$

## 2. Conformal metrical d-linear connections on TM

**Definition 2.1.** *A d-linear connection, $D$, on $TM$, with local coefficients $D\Gamma(N) = (L^i{}_{jk}, \tilde{L}^i{}_{jk}, \tilde{C}^i{}_{jk}, C^i{}_{jk})$, for which there exist the 1-forms $\omega$ and $\tilde{\omega}$ in $TM$, $\omega = \omega_i dx^i + \dot{\omega}_i \delta y^i$, $\tilde{\omega} = \tilde{\omega}_i dx^i + \dot{\tilde{\omega}}_i \delta y^i$ such that:*

$$(2.1)\quad \begin{cases} g_{ij|k} = 2\omega_k g_{ij}, \quad g_{ij}|_k = 2\dot{\omega}_k g_{ij}, \\ \\ \tilde{g}_{ij|k} = 2\tilde{\omega}_k \tilde{g}_{ij}, \quad \tilde{g}_{ij}|_k = 2\ \dot{\tilde{\omega}}_k\ \tilde{g}_{ij}, \end{cases}$$

*where $|$ and $|$ denote the h- and v-covariant derivatives with respect to $D$, is called conformal metrical d-linear connection on $TM$, with respect to the conformal metrical structure $\hat{G}$, corresponding to the 1-forms $\omega, \tilde{\omega}$, or is said to be compatible with the conformal metrical structure $\hat{G}$ and is denoted by $D\Gamma(N, \omega, \tilde{\omega})$.*

We shall determine the set of all conformal metrical d-linear connections, with respect to $\hat{G}$.

Let $\overset{0}{D}\Gamma(\overset{0}{N}) = (\overset{0}{L^i}{}_{jk}, \overset{0}{\tilde{L}^i}{}_{jk}, \overset{0}{\tilde{C}^i}{}_{jk}, \overset{0}{C^i}{}_{jk})$ be the local coefficients of a fixed d-linear connection $\overset{0}{D}$ on $TM$. Then, any d-linear connection, $D$, on $TM$, with local coefficients: $D\Gamma(N) = (L^i{}_{jk}, \tilde{L}^i{}_{jk}, \tilde{C}^i{}_{jk}, C^i{}_{jk})$, can be expresed in the form:

$$(2.2)\quad \begin{cases} N^i{}_j = \overset{0}{N^i}{}_j - A^i{}_j, \\ L^i{}_{jk} = \overset{0}{L^i}{}_{jk} + A^l{}_k \overset{0}{\tilde{C}^i}{}_{jl} - B^i{}_{jk}, \\ \tilde{L}^i{}_{jk} = \overset{0}{\tilde{L}^i}{}_{jk} + A^l{}_k \overset{0}{C^i}{}_{jl} - \tilde{B}^i{}_{jk}, \\ \tilde{C}^i{}_{jk} = \overset{0}{\tilde{C}^i}{}_{jk} - \tilde{D}^i{}_{jk}, \\ C^i{}_{jk} = \overset{0}{C^i}{}_{jk} - D^i{}_{jk}, \\ A^l{}_{\overset{0}{j|k}} = 0, \end{cases}$$

where $(A^i{}_j, B^i{}_{jk}, \tilde{B}^i{}_{jk}, \tilde{D}^i{}_{jk}, D^i{}_{jk})$ are components of the difference tensor fields of $D\Gamma(N)$ from $\overset{0}{D}\ \Gamma(\overset{0}{N})$, [8] and $\overset{0}{|}$, $\overset{0}{\Big|}$ denote the h- and respective v-covariant derivatives with respect to $\overset{0}{D}$.

**Theorem 2.1.** *Let $\overset{0}{D}$ be a given d-linear connection on $TM$, with local coefficients: $\overset{0}{D}\Gamma(\overset{0}{N}) = (\overset{0}{L^i}{}_{jk}, \overset{0}{\tilde{L}^i}{}_{jk}, \overset{0}{\tilde{C}^i}{}_{jk}, \overset{0}{C^i}{}_{jk})$. Then the set of all conformal metrical d-linear connections on $TM$, with respect to $\hat{G}$, corresponding to the 1-forms $\omega$ and $\tilde{\omega}$, with local coefficients $D\Gamma(N,\omega,\tilde{\omega}) = (L^i{}_{jk}, \tilde{L}^i{}_{jk}, \tilde{C}^i{}_{jk}, C^i{}_{jk})$ is given by:*

$$(2.3)\quad \begin{cases} N^i_j = \overset{0}{N^i_j} - X^i_j, \\ L^i{}_{jk} = \overset{0}{L^i}{}_{jk} + \overset{0}{\tilde{C}^i}{}_{jm} X^m{}_k + \frac{1}{2} g^{is}(g_{\overset{0}{sj|k}} + g_{sj} \overset{0}{\Big|}_m X^m{}_k) - \delta^i{}_j \omega_k + \Omega^{ir}_{hj} X^h{}_{rk}, \\ \tilde{L}^i{}_{jk} = \overset{0}{\tilde{L}^i}{}_{jk} + \overset{0}{C^i}{}_{jm} X^m{}_k + \frac{1}{2} \tilde{g}^{is}(\tilde{g}_{\overset{0}{sj|k}} + \tilde{g}_{sj} \overset{0}{\Big|}_m X^m{}_k) - \delta^i{}_j \tilde{\omega}_k + \tilde{\Omega}^{ir}_{hj} \tilde{X}^h{}_{rk}, \\ \tilde{C}^i{}_{jk} = \overset{0}{\tilde{C}^i}{}_{jk} + \frac{1}{2} g^{is} g_{sj} \overset{0}{\Big|}_k - \delta^i{}_j \dot{\omega}_k + \Omega^{ir}_{hj} \tilde{Y}^h{}_{rk}, \\ C^i{}_{jk} = \overset{0}{C^i}{}_{jk} + \frac{1}{2} \tilde{g}^{is} \tilde{g}_{sj} \overset{0}{\Big|}_k - \delta^i{}_j \dot{\tilde{\omega}}_k + \tilde{\Omega}^{ir}_{hj} Y^h{}_{rk}, \\ X^i{}_{\overset{0}{j|k}} = 0, \end{cases}$$

*where $X^i_j$, $X^i_{jk}$, $\tilde{X}^i_{jk}$, $\tilde{Y}^i_{jk}$, $Y^i_{jk}$ are arbitrary tensor fields on $TM$, $\omega = \omega_i dx^i + \dot{\omega}_i \delta y^i$ and respective $\tilde{\omega} = \tilde{\omega}_i dx^i + \dot{\tilde{\omega}}_i \delta y^i$ are arbitrary 1-forms in $TM$, and $\overset{0}{|}$, $\overset{0}{\Big|}$ denote the h- and respective v-covariant derivatives with respect to $\overset{0}{D}$.*

**Observation 2.1.** The set of all conformal metrical d-linear connections on TM obtained in the particular case when the nonlinear connection N is fixed, support the findings of R. Miron and M. Anastasiei [10], [11].

**Particular cases:**

**1.** If $X^i_j = X^i_{jk} = \tilde{X}^i_{jk} = \tilde{Y}^i_{jk} = Y^i_{jk} = 0$ in theorem 2.1. we have:

**Theorem 2.2.** *Let* $\overset{0}{D}$ *be a given d-linear connection on* $TM$*, with local coefficients:* $\overset{0}{D}\Gamma(\overset{0}{N}) = (\overset{0}{L^i}_{jk}, \overset{0}{\tilde{L}^i}_{jk}, \overset{0}{\tilde{C}^i}_{jk}, \overset{0}{C^i}_{jk})$. *Then the following d-linear connection* $K$*, with local coefficients:* $K\Gamma(N,\omega,\tilde{\omega}) = (L^i{}_{jk}, \tilde{L}^i{}_{jk}, \tilde{C}^i{}_{jk}, C^i{}_{jk})$*, given by (2.4), is a conformal metrical d-linear connection with respect to* $\hat{G}$*, corresponding to the 1-forms* $\omega$ *and* $\tilde{\omega}$*:*

$$(2.4)\quad \begin{cases} L^i{}_{jk} = \overset{0}{L^i}_{jk} + \frac{1}{2}g^{is}g_{sj\overset{0}{|}k} - \delta^i_j\omega_k, \\ \tilde{L}^i{}_{jk} = \overset{0}{\tilde{L}^i}_{jk} + \frac{1}{2}\tilde{g}^{is}\tilde{g}_{sj\overset{0}{|}k} - \delta^i_j\tilde{\omega}_k, \\ \tilde{C}^i{}_{jk} = \overset{0}{\tilde{C}^i}_{jk} + \frac{1}{2}g^{is}g_{sj}\overset{0}{|}_k - \delta^i_j\dot{\omega}_k, \\ C^i{}_{jk} = \overset{0}{C^i}_{jk} + \frac{1}{2}\tilde{g}^{is}\tilde{g}_{sj}\overset{0}{|}_k - \delta^i_j\dot{\tilde{\omega}}_k, \end{cases}$$

*where* $\overset{0}{|}, \overset{0}{|}$ *denote the h- and respective v-covariant derivatives with respect to the given d-linear connection* $\overset{0}{D}$ *and* $\omega = \omega_i dx^i + \dot{\omega}_i \delta y^i$ *and respective* $\tilde{\omega} = \tilde{\omega}_i dx^i + \dot{\tilde{\omega}}_i \delta y^i$ *are two given 1-forms in* $TM$*. As sugested in [7], we shall call the d-linear connection* $K\Gamma(N,\omega,\tilde{\omega})$ *the Kawaguchi conformal metrical d-linear connection, derived from* $\overset{0}{D}$*.*

As an example of conformal metrical d-linear connection with respect to $\hat{G}$ is W, with the local coefficients: $W\Gamma(\overset{W}{N},\omega,\tilde{\omega}) = (\overset{W}{L^i}_{jk}, \overset{W}{\tilde{L}^i}_{jk}, \overset{W}{\tilde{C}^i}_{jk}, \overset{W}{C^i}_{jk})$ given by:

$$(2.5)\quad \begin{cases} \overset{W}{L^i}_{jk} = \frac{1}{2}g^{ir}\left(\frac{\delta g_{jr}}{\delta x^k} + \frac{\delta g_{kr}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^r}\right) + (g_{jk}g^{ir}\omega_r - \delta^i_j\omega_k - \omega_j\delta^i_k), \\ \overset{W}{\tilde{L}^i}_{jk} = \frac{1}{2}\tilde{g}^{ir}\left(\frac{\delta \tilde{g}_{jr}}{\delta x^k} + \frac{\delta \tilde{g}_{kr}}{\delta x^j} - \frac{\delta \tilde{g}_{jk}}{\delta x^r}\right) + (\tilde{g}_{jk}\tilde{g}^{ir}\tilde{\omega}_r - \delta^i_j\tilde{\omega}_k - \tilde{\omega}_j\delta^i_k), \\ \overset{W}{\tilde{C}^i}_{jk} = \frac{1}{2}g^{ir}\left(\frac{\partial g_{jr}}{\partial y^k} + \frac{\partial g_{kr}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^r}\right) + (g_{jk}g^{ir}\dot{\omega}_r - \delta^i_j\dot{\omega}_k - \dot{\omega}_j\delta^i_k), \\ \overset{W}{C^i}_{jk} = \frac{1}{2}\tilde{g}^{ir}\left(\frac{\partial \tilde{g}_{jr}}{\partial y^k} + \frac{\partial \tilde{g}_{kr}}{\partial y^j} - \frac{\partial \tilde{g}_{jk}}{\partial y^r}\right) + (\tilde{g}_{jk}\tilde{g}^{ir}\dot{\tilde{\omega}}_r - \delta^i_j\dot{\tilde{\omega}}_k - \dot{\tilde{\omega}}_j\delta^i_k), \end{cases}$$

where $\omega$ and $\tilde{\omega}$ two 1-forms given on TM.

**2.** If we take a metrical d-linear connection as $\overset{0}{D}$ in Theorem 2.2, then (2.4) becomes:

$$(2.6)\quad \begin{cases} L^i{}_{jk} = \overset{0}{L^i}_{jk} - \delta^i_j\omega_k, \\ \tilde{L}^i{}_{jk} = \overset{0}{\tilde{L}^i}_{jk} - \delta^i_j\tilde{\omega}_k, \\ \tilde{C}^i{}_{jk} = \overset{0}{\tilde{C}^i}_{jk} - \delta^i_j\dot{\omega}_k, \\ C^i{}_{jk} = \overset{0}{C^i}_{jk} - \delta^i_j\dot{\tilde{\omega}}_k. \end{cases}$$

**3.** If we take a conformal metrical d-linear connection with respect to $\hat{G}$, e.g. W as $\overset{0}{D}$ in Theorem 2.1, we have:

**Theorem 2.3.** *Let $\overset{0}{D}$ be a given conformal metrical d-linear connection on $TM$, with local coefficients: $\overset{0}{D}\Gamma(\overset{0}{N},\omega,\tilde{\omega}) = (\overset{0}{L}{}^{i}{}_{jk}, \overset{0}{\tilde{L}}{}^{i}{}_{jk}, \overset{0}{\tilde{C}}{}^{i}{}_{jk}, \overset{0}{C}{}^{i}{}_{jk})$. The set of all conformal metrical d-linear connections on $TM$, with respect to $\hat{G}$, corresponding to the 1-forms $\omega$ and $\tilde{\omega}$, with local coefficients: $D\Gamma(N,\omega,\tilde{\omega}) = (L^{i}{}_{jk}, \tilde{L}^{i}{}_{jk}, \tilde{C}^{i}{}_{jk}, C^{i}{}_{jk})$ is given by:*

$$(2.7)\quad \begin{cases} N^{i}_{j} = \overset{0}{N}{}^{i}_{j} - X^{i}{}_{j}, \\ L^{i}{}_{jk} = \overset{0}{L}{}^{i}{}_{jk} + (\overset{0}{\tilde{C}}{}^{i}{}_{jm} + \delta^{i}_{j}\dot{\omega}_{m})X^{m}{}_{k} + + \Omega^{ir}_{hj}X^{h}{}_{rk}, \\ \tilde{L}^{i}{}_{jk} = \overset{0}{\tilde{L}}{}^{i}{}_{jk} + (\overset{0}{C}{}^{i}{}_{jm} + \delta^{i}_{j}\dot{\tilde{\omega}}_{m})X^{m}{}_{k} + \tilde{\Omega}^{ir}_{hj}\tilde{X}^{h}{}_{rk}, \\ \tilde{C}^{i}{}_{jk} = \overset{0}{\tilde{C}}{}^{i}{}_{jk} + \Omega^{ir}_{hj}\tilde{Y}^{h}{}_{rk}, \\ C^{i}{}_{jk} = \overset{0}{C}{}^{i}{}_{jk} + \tilde{\Omega}^{ir}_{hj}Y^{h}{}_{rk}, \\ \\ X^{i}{}_{j\overset{0}{|}k} = 0, \end{cases}$$

*where $X^{i}_{j}$, $X^{i}_{jk}$, $\tilde{X}^{i}_{jk}$, $\tilde{Y}^{i}_{jk}$, $Y^{i}_{jk}$ are arbitrary tensor fields on $TM$, $\omega = \omega_{i}dx^{i} + \dot{\omega}_{i}\delta y^{i}$, and respective $\tilde{\omega} = \tilde{\omega}_{i}dx^{i} + \dot{\tilde{\omega}}_{i}\delta y^{i}$ are two arbitrary 1-forms in $TM$ and $\overset{0}{|}$, $\overset{0}{\Big|}$ denote h- and respective v-covariant derivatives with respect to $\overset{0}{D}$.*

**4.** If we take $X^{i}{}_{j} = 0$ in Theorem 2.3. we obtain:

**Theorem 2.4.** *([10], [11]) Let $\overset{0}{D}$ be a given conformal metrical d-linear connection on $TM$, with local coefficients: $\overset{0}{D}\Gamma(\overset{0}{N},\omega,\tilde{\omega}) = (\overset{0}{L}{}^{i}{}_{jk}, \overset{0}{\tilde{L}}{}^{i}{}_{jk}, \overset{0}{\tilde{C}}{}^{i}{}_{jk}, \overset{0}{C}{}^{i}{}_{jk})$. The set of all conformal metrical d-linear connections on $TM$, with respect to $\hat{G}$, having the same nonlinear connection $\overset{0}{N}$, corresponding to the 1-forms $\omega$ and $\tilde{\omega}$, with local coefficients: $D\Gamma(\overset{0}{N},\omega,\tilde{\omega}) = (L^{i}{}_{jk}, \tilde{L}^{i}{}_{jk}, \tilde{C}^{i}{}_{jk}, C^{i}{}_{jk})$ is given by:*

$$(2.8)\quad \begin{cases} L^{i}{}_{jk} = \overset{0}{L}{}^{i}{}_{jk} + \Omega^{ir}_{hj}X^{h}{}_{rk}, \\ \tilde{L}^{i}{}_{jk} = \overset{0}{\tilde{L}}{}^{i}{}_{jk} + \tilde{\Omega}^{ir}_{hj}\tilde{X}^{h}{}_{rk}, \\ \tilde{C}^{i}{}_{jk} = \overset{0}{\tilde{C}}{}^{i}{}_{jk} + \Omega^{ir}_{hj}\tilde{Y}^{h}{}_{rk}, \\ C^{i}{}_{jk} = \overset{0}{C}{}^{i}{}_{jk} + \tilde{\Omega}^{ir}_{hj}Y^{h}{}_{rk}, \end{cases}$$

*where $X^{i}_{j}$, $X^{i}_{jk}$, $\tilde{X}^{i}_{jk}$, $\tilde{Y}^{i}_{jk}$, $Y^{i}_{jk}$ are arbitrary tensor fields on $TM$.*

## 3. Some special classes of conformal metrical d-linear connections

We shall try to replace the arbitrary tensor fields $X^i{}_{jk}, Y^i_{jk}$ in Theorem 2.4. by the torsion tensor fields $T_{(0)}{}^i{}_{jk}, S^i{}_{jk}$.

We put:

$$(3.1)\quad \begin{cases} T_{(0)}{}^{*i}{}_{jk} = \frac{1}{2}g^{ir}(g_{rh}T_{(0)}{}^h{}_{jk} - g_{jh}T_{(0)}{}^h{}_{rk} + g_{kh}T_{(0)}{}^h{}_{jr}), \\ S^{*i}{}_{jk} = \frac{1}{2}\tilde{g}^{ir}(\tilde{g}_{rh}S^h{}_{jk} - \tilde{g}_{jh}S^h{}_{rk} + \tilde{g}_{kh}S^h{}_{jr}). \end{cases}$$

**Theorem 3.1.** *Let $T_{(0)}{}^i{}_{jk}$ and $S^i{}_{jk}$ be two given alternate tensor fields, of the type (1,2) and let $\omega, \tilde{\omega}$ be two given 1-forms in $TM$. Then there exists a unique conformal metrical d-linear connection with respect to $\hat{G}$, corresponding to the 1-forms $\omega$ and $\tilde{\omega}$, with local coefficients: $D\Gamma(\overset{W}{N}, \omega, \tilde{\omega}) = (L^i{}_{jk}, \tilde{L}^i{}_{jk}, \tilde{C}^i{}_{jk}, L^i{}_{jk})$, having $T_{(0)}{}^i{}_{jk}$ and $S^i{}_{jk}$ as torsion tensor fields. It is given by:*

$$(3.2)\quad \begin{cases} L^i{}_{jk} = \overset{W}{L^i}{}_{jk} + T_{(0)}{}^{*i}{}_{jk}, \\ \tilde{L}^i{}_{jk} = \overset{W}{\tilde{L}^i}{}_{jk}, \\ \tilde{C}^i{}_{jk} = \overset{W}{\tilde{C}^i}{}_{jk}, \\ C^i{}_{jk} = \overset{W}{C^i}{}_{jk} + S^{*i}{}_{jk}, \end{cases}$$

*where $W\Gamma(\overset{W}{N}, \omega, \tilde{\omega}) = \left(\overset{W}{L^i}{}_{jk}, \overset{W}{\tilde{L}^i}{}_{jk}, \overset{W}{\tilde{C}^i}{}_{jk}, \overset{W}{C^i}{}_{jk}\right)$ are the local coefficients of the conformal metrical d-linear connection given in (2.5).*

**Definition 3.1.** *A conformal metrical d-linear connection on $TM$, with local coefficients: $D\Gamma(N, \omega, \tilde{\omega}) = (L^i{}_{jk}, \tilde{L}^i{}_{jk}, \tilde{C}^i{}_{jk}, L^i{}_{jk})$, is called semi-symmetric conformal metrical d-linear connection if the torsion tensor fields $T_{(0)}{}^i{}_{jk}$ and $S^i{}_{jk}$ have the form:*

$$(3.3)\quad \begin{cases} T_{(0)}{}^i{}_{jk} = \frac{1}{n-1}(T_{(0)j}\delta^i_k - T_{(0)k}\delta^i_j), \\ S^i{}_{jk} = \frac{1}{n-1}(S_j\delta^i_k - S_k\delta^i_j), \end{cases}$$

*where $T_{(0)j} = T_{(0)}{}^i{}_{ji}$ and $S_j = S^i{}_{ji}$*

**Observation 3.1.** The conformal metrical d-linear connection $W$, with local coefficients: $W\Gamma(\overset{W}{N}, \omega, \tilde{\omega}) = \left(\overset{W}{L^i}{}_{jk}, \overset{W}{\tilde{L}^i}{}_{jk}, \overset{W}{\tilde{C}^i}{}_{jk}, \overset{W}{C^i}{}_{jk}\right)$, given in (2.5), is considered as the semi-symmetric conformal metrical d-linear connection, with the vanishing h- and v-torsion vector fields.

Putting:

$$(3.4)\quad \sigma_j = \frac{1}{n-1}T_{(0)j}, \dot{\tilde{\sigma}}_j = \frac{1}{n-1}S_j,$$

then (3.1) become:

$$(3.5)\quad T_{(0)}{}^{*i}{}_{jk} = 2\Omega^{ri}_{jk}\sigma_r, S^{*i}{}_{jk} = 2\tilde{\Omega}^{ir}_{kj}\dot{\tilde{\sigma}}_r.$$

Using the Theorem 3.1. and the relations (3.5) we have:

**Theorem 3.2.** *The set of all semi-symmetric conformal metrical d-linear connections with respect to* $\hat{G}$ *, corresponding to the 1-forms* $\omega$ *and* $\tilde{\omega}$ *in* $TM$, *with local coefficients:* $D\Gamma(\overset{W}{N},\omega,\tilde{\omega},\sigma,\tilde{\sigma}) = (L^i{}_{jk}, \tilde{L}^i{}_{jk}, \tilde{C}^i{}_{jk}, L^i{}_{jk})$, *is given by:*

$$(3.6)\quad \begin{cases} L^i{}_{jk} = \overset{W}{L^i}{}_{jk} + 2\Omega^{ri}_{jk}\sigma_r, \\ \tilde{L}^i{}_{jk} = \overset{W}{\tilde{L}^i}{}_{jk}, \\ \tilde{C}^i{}_{jk} = \overset{W}{\tilde{C}^i}{}_{jk}, \\ C^i{}_{jk} = \overset{W}{C^i}{}_{jk} + 2\tilde{\Omega}^{ri}_{jk}\dot{\tilde{\sigma}}_r, \end{cases}$$

*where* $W\Gamma(\overset{W}{N},\omega,\tilde{\omega}) = \left(\overset{W}{L^i}{}_{jk}, \overset{W}{\tilde{L}^i}{}_{jk}, \overset{W}{\tilde{C}^i}{}_{jk}, \overset{W}{C^i}{}_{jk}\right)$ *are the local coefficients of an arbitrary semi-symmetric conformal metrical d-linear connection, W, given in (2.5) and* $\sigma = \sigma_i dx^i + \dot{\sigma}_i \delta y^i, \tilde{\sigma} = \tilde{\sigma}_i dx^i + \dot{\tilde{\sigma}}_i\ \delta y^i$ *are two arbitrary 1-forms in* $TM$.

## 4. The group of transformations of conformal metrical d-linear connections

We study the transformations $D\Gamma(N,\omega,\tilde{\omega}) \to \bar{D}\Gamma(\bar{N},\omega',\tilde{\omega}')$ of the conformal metrical d-linear connections with respect to $\hat{G}$.

If we replace $\overset{0}{D}\ \Gamma(\overset{0}{N})$ and $D\Gamma(N,\omega,\tilde{\omega})$ in Theorem 2.1 by $D\Gamma(N,\omega,\omega')$ and $\bar{D}\Gamma(\bar{N},\omega',\tilde{\omega}')$, respectively, two conformal metrical d-linear connections, we obtain:

**Theorem 4.1.** *Two conformal metrical d-linear connections with respect to* $\hat{G}$, $D$ *and* $\bar{D}$, *with local coefficients:* $D\Gamma(N,\omega,\tilde{\omega}) = (L^i{}_{jk}, \tilde{L}^i{}_{jk}, \tilde{C}^i{}_{jk}, C^i{}_{jk})$, *and* $\bar{D}\Gamma(\bar{N},\omega',\tilde{\omega}') = (\bar{L}^i{}_{jk}, \bar{\tilde{L}}^i{}_{jk}, \bar{\tilde{C}}^i{}_{jk}, \bar{C}^i{}_{jk})$, *respectively, are related as follows:*

$$(4.1)\quad \begin{cases} \bar{N}^j_i = N^j_i - X^j_i, \\ \bar{L}^i{}_{jk} = L^i{}_{jk} + \tilde{C}^i{}_{jl}X^l{}_k - \delta^i_j p_k + \delta^i_j \dot{\omega}_s X^s{}_k + \Omega^{ir}_{sj}X^s{}_{rk}, \\ \bar{\tilde{L}}^i{}_{jk} = \tilde{L}^i{}_{jk} + C^i{}_{jl}X^l{}_k - \delta^i_j \tilde{p}_k + \delta^i_j \dot{\tilde{\omega}}_s X^s{}_k + \tilde{\Omega}^{ir}_{sj}X^s{}_{rk}, \\ \bar{\tilde{C}}^i{}_{jk} = \tilde{C}^i{}_{jk} - \delta^i_j \dot{p}_k + \Omega^{ir}_{sj}\tilde{Y}^s{}_{rk}, \\ \bar{C}^i{}_{jk} = C^i{}_{jk} - \delta^i_j \dot{\tilde{p}}_k + \tilde{\Omega}^{ir}_{sj}Y^s{}_{rk}, \\ X^j_{i|k} = 0, \end{cases}$$

*where* $p = \omega' - \omega, \tilde{p} = \tilde{\omega}' - \tilde{\omega}, \omega = \omega_i dx^i + \dot{\omega}_i \delta y^i, \tilde{\omega} = \tilde{\omega}_i dx^i + \dot{\tilde{\omega}}_i \delta y^i$ , $\omega' = \omega'_i dx^i + \dot{\omega}'_i \delta y^i$ *and* $\tilde{\omega}' = \tilde{\omega}'_i dx^i + \dot{\tilde{\omega}}'_i \delta y^i$, *are given 1-forms in* $TM$ *and* $X^i_k$, $X^i{}_{jk}$, $\tilde{X}^i{}_{jk}$, $\tilde{Y}^i{}_{jk}$, $Y^i{}_{jk}$ *are arbitrary tensor fields on* $TM$.

Conversely, given the tensor fields $X^i_k, X^i{}_{jk}, \tilde{X}^i{}_{jk}, \tilde{Y}^i{}_{jk}, Y^i{}_{jk}$ and two given 1-forms $p$ and $\tilde{p}$ respectively $\left(p = p_i dx^i + \dot{p}_i \delta y^i, \tilde{p} = \tilde{p}_i dx^i + \dot{\tilde{p}}_i\ \delta y^i\right)$ the above

(4.1) is thought to be a transformation of a conformal metrical $d$-linear connection $D\Gamma(N,\omega,\tilde{\omega})$ to a conformal metrical $d$-linear connection $\bar{D}\Gamma(\bar{N},\omega',\tilde{\omega}') = \bar{D}\Gamma(\bar{N},\omega+p,\tilde{\omega}+\tilde{p})$.

We shall denote this transformation by: $t(X^i{}_k, X^i{}_{jk}, \tilde{X}^i{}_{jk}, \tilde{Y}^i{}_{jk}, Y^i{}_{jk}, p, \tilde{p})$. Thus we have:

**Theorem 4.2.** *The set $\mathcal{C}$ of all transformations $t(X^i{}_k, X^i{}_{jk}, \tilde{X}^i{}_{jk}, \tilde{Y}^i{}_{jk}, Y^i{}_{jk}, p, \tilde{p})$ given by (4.1), is a transformations group of the set of all conformal metrical d-linear connections with respect to $\hat{G}$, on $TM$, together with the mapping product:*

$t(X'^i{}_k, X'^i{}_{jk}, \tilde{X}'^i{}_{jk}, \tilde{Y}'^i{}_{jk}, Y'^i{}_{jk}, p', \tilde{p}') \circ t(X^i{}_k, X^i{}_{jk}, \tilde{X}^i{}_{jk}, \tilde{Y}^i{}_{jk}, Y^i{}_{jk}, p, \tilde{p}) =$
$= (X^i{}_k + X'^i{}_k, X^i{}_{jk} + X'^i{}_{jk} + Y^i_{jm} X'^m{}_k, \tilde{X}^i{}_{jk} + \tilde{X}'^i{}_{jk} + \tilde{Y}^i_{jm} X'^m{}_k, \tilde{Y}^i{}_{jk} + \tilde{Y}'^i{}_{jk}, Y^i{}_{jk} + Y'^i{}_{jk}, p+p', \tilde{p}+\tilde{p}')$.

We inquire about the subgroup of transformations of the semi-symmetric conformal metrical d-linear connections.

Let $\overset{W}{N}$ be a given nonlinear connection. Then any semi-symmetric conformal metrical d-linear connection with local coefficients: $\bar{D}\Gamma(\overset{W}{N},\omega',\tilde{\omega}',\sigma',\tilde{\sigma}')=$ $(\bar{L}^i{}_{jk}, \bar{\tilde{L}}^i{}_{jk}, \bar{\tilde{C}}^i{}_{jk}, \bar{C}^i{}_{jk})$, with respect to $\hat{G}$ is given by (3.2) with (3.5). Paying attention to (2.5) we have:

**Theorem 4.3.** *Two semi-symmetric conformal metrical d-linear connections with respect to $\hat{G}$, with local coefficients: $D\Gamma(\overset{W}{N},\omega,\tilde{\omega},\sigma,\tilde{\sigma}) = (L^i{}_{jk}, \tilde{L}^i{}_{jk}, \tilde{C}^i{}_{jk}, C^i{}_{jk})$, and $\bar{D}\Gamma(\overset{W}{N},\omega',\tilde{\omega}',\sigma',\tilde{\sigma}') = (\bar{L}^i{}_{jk}, \bar{\tilde{L}}^i{}_{jk}, \bar{\tilde{C}}^i{}_{jk}, \bar{C}^i{}_{jk})$, respectively, are related as follows:*

$$(4.2)\quad \begin{cases} \bar{L}^i{}_{jk} = L^i{}_{jk} - \delta^i_j p_k + 2\Omega^{ir}_{kj} q_r, \\ \bar{\tilde{L}}^i{}_{jk} = \tilde{L}^i{}_{jk} - \delta^i_j \tilde{p}_k, \\ \bar{\tilde{C}}^i{}_{jk} = \tilde{C}^i{}_{jk} - \delta^i_j \dot{p}_k, \\ \bar{C}^i{}_{jk} = C^i{}_{jk} - \delta^i_j \dot{\tilde{p}}_k + 2\tilde{\Omega}^{ir}_{kj} \dot{\tilde{q}}_r, \end{cases}$$

*where $p = \omega' - \omega, \tilde{p} = \tilde{\omega}' - \tilde{\omega}, q = \sigma' - \sigma - p, \tilde{q} = \tilde{\sigma}' - \tilde{\sigma} - \tilde{p}$, $p = p_i dx^i + \dot{p}_i \delta y^i$, $\tilde{p} = \tilde{p}_i dx^i + \dot{\tilde{p}}_i \delta y^i$, $q = q_i dx^i + \dot{q}_i \delta y^i$ and $\tilde{q} = \tilde{q}_i dx^i + \dot{\tilde{q}}_i \delta y^i$.*

Conversely, given the 1-forms $p, \tilde{p}, q, \tilde{q}$ in $TM$, the above (4.2) is thought to be a transformation of a semi-symmetric conformal metrical d-linear connection $D$, with local coefficients: $D\Gamma(\overset{W}{N},\omega,\tilde{\omega},\sigma,\tilde{\sigma}) = (L^i{}_{jk}, \tilde{L}^i{}_{jk}, \tilde{C}^i{}_{jk}, C^i{}_{jk})$, to a semi-symmetric conformal metrical d-linear connection $\bar{D}$, with local coefficients: $\bar{D}\Gamma(\overset{W}{N},\omega+p,\tilde{\omega}+\tilde{p},\sigma+p+q,\tilde{\sigma}+\tilde{p}+\tilde{q}) = (\bar{L}^i{}_{jk}, \bar{\tilde{L}}^i{}_{jk}, \bar{\tilde{C}}^i{}_{jk}, \bar{C}^i{}_{jk})$,

We shall denote this transformation by: $t(p,\tilde{p},q,\tilde{q})$.

Thus we have:

**Theorem 4.4.** *The set $\mathcal{C}^s_N$ of all transformations $t(p,\tilde{p},q,\tilde{q})$, given by (4.2), is a transformations group of the set of all semi-symmetric conformal metrical*

*d-linear connections with respect to $\hat{G}$, together with the mapping product:*
$t(p,\tilde{p},q,\tilde{q}) \circ t(p',\tilde{p}',q',\tilde{q}') = t(p+p',\tilde{p}+\tilde{p}',q+q',\tilde{q}+\tilde{q}')$.

*This group $\mathcal{C}^s_N$ is an Abelian subgroup of $\mathcal{C}$ and acts on the set of all semi-symmetric conformal metrical d-linear connections, having the same nonlinear connection N, transitively.*

The transformation: $t(p,\tilde{p},q,\tilde{q}) : D\Gamma(N,\omega,\tilde{\omega},\sigma,\tilde{\sigma}) \to \bar{D}\Gamma(N,\omega+p,\tilde{\omega}+\tilde{p}, \sigma+p+q, \tilde{\sigma}+\tilde{p}+\tilde{q})$ given by (4.2) is expressed by the product of the following two transformations:

$$(4.3) \quad \begin{cases} \bar{L}^i{}_{jk} = L^i{}_{jk} - \delta^i_j p_k, \\ \bar{\tilde{L}}^i{}_{jk} = \tilde{L}^i{}_{jk} - \delta^i_j \tilde{p}_k, \\ \bar{\tilde{C}}^i{}_{jk} = \tilde{C}^i{}_{jk} - \delta^i_j \dot{p}_k, \\ \bar{C}^i{}_{jk} = C^i{}_{jk} - \delta^i_j \dot{\tilde{p}}_k, \end{cases}$$

$$(4.4) \quad \begin{cases} \bar{L}^i{}_{jk} = L^i{}_{jk} + 2\Omega^{ir}_{kj} q_r, \\ \bar{\tilde{L}}^i{}_{jk} = \tilde{L}^i{}_{jk}, \\ \bar{\tilde{C}}^i{}_{jk} = \tilde{C}^i{}_{jk}, \\ \bar{C}^i{}_{jk} = C^i{}_{jk} + 2\tilde{\Omega}^{ir}_{kj} \dot{\tilde{q}}_r, \end{cases}$$

**Definition 4.1.** *The transformation $t : D\Gamma(N) \to \bar{D}\Gamma(N)$, of d-linear connections on $TM$, defined by (4.3), is called co-parallel transformation on $TM$, where $p$ and $\tilde{p}$ are two given 1-forms in $TM$.*

**Theorem 4.5.** *The set $\mathcal{C}^p_N$ of all co-parallel transformations t, given by (4.3), is an Abelian group together with the mapping product.*

**Definition 4.2.** *The transformation $t : D\Gamma(N) \to \bar{D}\Gamma(N)$ of d-linear connections, given by (4.4), is called Miron transformation by M. Hashiguchi [6], for Finsler spaces.*

**Theorem 4.6.** *The set $\mathcal{C}^m_N$ of all Miron transformations t, given by (4.4), is a transformations group, together with the mapping product.*

Thus we have:

**Theorem 4.7.** *The group $\mathcal{C}^s_N$ of all transformations $t(p,\tilde{p},q,\tilde{q})$, given by (4.2), is the direct product of the group $\mathcal{C}^p_N$ of all co-parallel transformations and the group $\mathcal{C}^m_N$ of all Miron transformations.*

It is noted that the invariants of the group $\mathcal{C}^s_N$, will be the invariants of each of these subgroups and reciprocally.

It is directly shown that by a co-parallel transformation (4.3) the curvature tensor fields $R_{(0)j}{}^i{}_{kl}$ and $S_{(1)j}{}^i{}_{kl}$ are transformed as follows:

$$(4.5) \quad \begin{cases} \bar{R}_{(0)j}{}^i{}_{kl} = R_{(0)j}{}^i{}_{kl} - \delta^i_j p_{kl}, \\ \bar{S}_{(1)j}{}^i{}_{kl} = S_{(1)j}{}^i{}_{kl} - \delta^i_j \dot{\tilde{p}}_{kl}. \end{cases}$$

where $p_{kl}, \dot{\tilde{p}}_{kl}$ are the components of $dp$ and $d\tilde{p}$, expressed with respect to $D$

Eliminating $p_{kl}, \dot{\tilde{p}}_{kl}$ from (4.5) we have:

(4.6) $\bar{R}_{(0)}{}^{*}{}_{j}{}^{i}{}_{kl} = R_{(0)}{}^{*}{}_{j}{}^{i}{}_{kl}, \bar{S}_{(1)}{}^{*}{}_{j}{}^{i}{}_{kl} = S_{(1)}{}^{*}{}_{j}{}^{i}{}_{kl}$,

where:

(4.7) $\begin{cases} R_{(0)}{}^{*}{}_{j}{}^{i}{}_{kl} = R_{(0)j}{}^{i}{}_{kl} - \frac{1}{n}\delta^i_j R_{(0)s}{}^{s}{}_{kl}, \\ S_{(1)}{}^{*}{}_{j}{}^{i}{}_{kl} = S_{(1)j}{}^{i}{}_{kl} - \frac{1}{n}\delta^i_j S_{(1)s}{}^{s}{}_{kl}. \end{cases}$

Thus we have:

**Theorem 4.8.** *The tensor fields $R_{(0)}{}^{*}{}_{j}{}^{i}{}_{kl}$ and $S_{(1)}{}^{*}{}_{j}{}^{i}{}_{kl}$, given by (4.7), are invariants of the group $\mathcal{C}^p_N$.*

Also, we can obtain:

**Theorem 4.9.** *The following tensor field $C^{*i}{}_{jk}$, given by (4.8), is an invariant of the group $\mathcal{C}^p_N$ :*

(4.8) $C^{*i}{}_{jk} = C^i{}_{jk} - \frac{1}{n}\delta^i_j C^s{}_{sk}$.

In our previous paper [14], starting from the tensor fields:

(4.9) $\begin{cases} \mathcal{K}_{(0)j}{}^{i}{}_{kl} = R_{(0)j}{}^{i}{}_{kl} - \tilde{C}^i{}_{jm} R^m{}_{kl}, \\ \mathcal{K}_{(1)j}{}^{i}{}_{kl} = R_{(1)j}{}^{i}{}_{kl} - C^i{}_{jm} R^m{}_{kl}, \end{cases}$

we obtained the following important invariants of the group of semi-symmetric metrical d-linear connections, having the same nonlinear connection N, $\overset{ms}{\mathcal{T}}_N$, for n>2:

(4.10) $\begin{cases} H_{(0)j}{}^{i}{}_{kl} = \mathcal{K}_{(0)j}{}^{i}{}_{kl} + \frac{2}{n-2}\mathcal{A}_{kl}\left\{\Omega^{ir}_{kj}\left(\mathcal{K}_{(0)rl} - \frac{\mathcal{K}_{(0)}g_{rl}}{2(n-1)}\right)\right\}, \\ H_{(1)j}{}^{i}{}_{kl} = \mathcal{K}_{(1)j}{}^{i}{}_{kl}, \\ M_{(0)j}{}^{i}{}_{kl} = S_{(0)j}{}^{i}{}_{kl}, \\ M_{(1)j}{}^{i}{}_{kl} = S_{(1)j}{}^{i}{}_{kl} + \frac{2}{n-2}\mathcal{A}_{kl}\left\{\tilde{\Omega}^{ir}_{kj}\left(S_{(1)rl} - \frac{S_{(1)}\tilde{g}_{rl}}{2(n-1)}\right)\right\}, \end{cases}$

where:

(4.11) $\mathcal{K}_{(0)jk} = \mathcal{K}_{(0)j}{}^{i}{}_{ki}, \mathcal{K}_{(0)} = g^{jk}\mathcal{K}_{(0)jk}, S_{(1)jk} = S_{(1)j}{}^{i}{}_{ki}, S_{(1)} = \tilde{g}^{jk} S_{(1)jk}$.

If we replace these $\mathcal{K}_{(0)j}{}^{i}{}_{kl}$ and $S_{(1)j}{}^{i}{}_{kl}$ by the tensor fields $\mathcal{K}_{(0)}{}^{*}{}_{j}{}^{i}{}_{kl}$ and $S_{(1)}{}^{*}{}_{j}{}^{i}{}_{kl}$ respectively, defined by:

(4.12) $\begin{cases} \mathcal{K}_{(0)}{}^{*}{}_{j}{}^{i}{}_{kl} = \mathcal{K}_{(0)j}{}^{i}{}_{kl} - \frac{1}{n}\delta^i_j \mathcal{K}_{(0)s}{}^{s}{}_{kl}, \\ S_{(1)}{}^{*}{}_{j}{}^{i}{}_{kl} = S_{(1)j}{}^{i}{}_{kl} - \frac{1}{n}\delta^i_j S_{(1)s}{}^{s}{}_{kl}. \end{cases}$

we can obtain the invariants of the group of transformations of semi-symmetric conformal metrical d-linear connections on $TM$, having the same nonlinear connection N, $\mathcal{C}^s_N$.

**Theorem 4.10.** *For n>2 the following tensor fields: $\mathcal{H}_{(0)}{}^{*}{}_{j}{}^{i}{}_{kl}$ and $M_{(1)}{}^{*}{}_{j}{}^{i}{}_{kl}$ are invariants of the group $\mathcal{C}^s_N$, of transformations, of semi-symmetric conformal metrical d-linear connections on $TM$, having the same nonlinear connection N:*

(4.13) $\begin{cases} \mathcal{H}_{(0)}{}^{*}{}_{j}{}^{i}{}_{kl} = \mathcal{K}_{(0)}{}^{*}{}_{j}{}^{i}{}_{kl} + \frac{2}{n-2}\mathcal{A}_{kl}\left\{\Omega^{ir}_{kj}\left(\mathcal{K}_{(0)}{}^{*}{}_{rl} - \frac{\mathcal{K}^*_{(0)}g_{rl}}{2(n-1)}\right)\right\}, \\ M_{(1)}{}^{*}{}_{j}{}^{i}{}_{kl} = S_{(1)}{}^{*}{}_{j}{}^{i}{}_{kl} + \frac{2}{n-2}\mathcal{A}_{kl}\left\{\tilde{\Omega}^{ir}_{kj}\left(S_{(1)}{}^{*}{}_{rl} - \frac{S^*_{(1)}g_{rl}}{2(n-1)}\right)\right\}, \end{cases}$

*where:*

(4.14) $\mathcal{K}_{(0)}{}^{*}{}_{jk} = \mathcal{K}_{(0)}{}^{*}{}_{j}{}^{i}{}_{ki}$, $\mathcal{K}_{(0)}{}^{*} = g^{jk}\mathcal{K}_{(0)}{}^{*}{}_{jk}$, $S_{(1)}{}^{*}{}_{jk} = S_{(1)}{}^{*}{}_{j}{}^{i}{}_{ki}$, $S_{(1)}{}^{*} = \tilde{g}^{jk} S_{(1)}{}^{*}{}_{jk}$.

Finally we give other invariants of the group $\mathcal{C}^s_N$ :

**Theorem 4.11.** *The following tensor field is an invariant of the group* $\mathcal{C}^s_N$ :

$$C^{*i}{}_{jk} - \tfrac{2}{n-1}\Omega^{ir}_{kj}C^{*m}{}_{rm}, \tag{4.15}$$

*where* $C^{*i}{}_{jk}$ *is given by (4.8).*

**Observation 4.1.** The results obtained in the particular case of the normal d-linear connections, support the findings of R. Miron and M. Haschiguchi [12].

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