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A GENERAL FIXED POINT THEOREM FOR TWO PAIRS OF MAPPINGS ON TWO METRIC SPACES

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Abstract. A general fixed point theorem for two pairs of mappings on two metric spaces is proved. This result generalizes the main theorem from [2].

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1. Introduction

The following fixed point theorem was proved by Fisher [1].

Theorem 1. [1] Let (X, d) and (Y, ρ) be complete metric spaces,let T be a continuous mappings of X into Y, and let S be a mappings of Y into X satisfying the inequalities

$$d(STx, STx') \leq c \max\{d(x, x'), d(x, STx), d(x', STx'), \rho(Tx, Tx')\}$$

$$\rho(TSy, TSy') \leq c \max\{\rho(y, y'), \rho(y, TSy), \rho(y', TSy'), d(Sy, Sy')\}$$

for all x, x' in X and y, y' in Y,where $0 \leq c < 1$.

Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

A generalization of Theorem 1 is proved by Fisher and Murthy in [2].

Theorem 2. [2] Let (X, d) and (Y, ρ) be complete metric spaces, let A, B be mappings of X into Y, and let S,T be mappings of Y into X satisfying the inequalities

$$d(SAx, TBx') \leq c \max\{d(x, x'), d(x, SAx), d(x', TBx'), \rho(Ax, Bx)\},$$

$$d(BSy, ATy') \leq c \max\{\rho(y, y'), \rho(y, BSy), \rho(y, ATy), d(Sy, Ty'), d(Sy, Ty')\}$$

for all x, x' in X and y, y' in Y, where $0 \leq c < 1$. If one of the mappings A, B, S and T is continuous then SA and TB have a common fixed point z in X and BS and AT have a common fixed point w in Y. Further, $Az = Bz = w$ and $Sw = Tw = z$.

In this paper a generalization of Theorem 2 is proved for pairs of mappings satisfying two implicit relations.

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2. Implicit relations

Let $\mathcal{F}_5$ be the set of all functions $F : R_+^5 \to R$ such that

(Fc): F is continuous in each coordinate variable,

(Fh): there exists $h \in [0,1)$ such that for every $u \geq 0, v \geq 0, w \geq 0$ satisfying $F(u,v,u,v,w) \leq 0$ or $F(u,v,v,u,w) \leq 0$ we have $u \leq h\max\{v,w\}$.

Example 1. $F(t_1,...,t_5) = t_1 - c\max\{t_2,t_3,t_4,t_5\}$, *where* $c \in [0,1)$.

(Fc): Obviously.
(Fh): Let $u > 0, v \geq 0, w \geq 0$ and $F(u,v,u,v,w) = u - c\max\{v,u,w\} \leq 0$.
Then $u \leq c\max\{v,w\}$. If $u > \max\{v,w\}$ then $u(1-c) \leq 0$, a contradiction.
Then $u \leq h\max\{v,w\}$, where $h = c \in [0,1)$. If $u = 0$, then $u \leq h\max\{v,w\}$.
Similarly, $F(u,v,v,u,w) \leq 0$ implies $u \leq h\max\{v,w\}$.

Example 2. $F(t_1,...,t_5) = t_1^2 - c\max\{t_2t_3, t_2t_4, t_3t_4, t_5^2\}$ *where* $c \in [0,1)$.

(Fc): Obviously.
(Fh): Let $u > 0, v \geq 0, w \geq 0$ and $F(u,v,u,v,w) = u^2 - c\max\{uv, v^2, w^2\} \leq 0$.
Then $u^2 \leq c\max\{uv, v^2, w^2\}$.
If $u > \max\{v,w\}$ then $u^2(1-c) \leq 0$, a contradiction.
Thus $u \leq h\max\{v,w\}$, where $h = \sqrt{c} \in [0,1)$. If $u = 0$, then $u \leq h\max\{v,w\}$.
Similarly, if $F(u,v,v,u,w) \leq 0$ then $u \leq \max\{v,w\}$.

Example 3. $F(t_1,...,t_5) = t_1^2 - (at_1t_2 + bt_1t_3 + ct_1t_4 + dt_5^2)$ *where* $a,b,c,d > 0$ *and* $0 < a+b+c+d < 1$.

(Fc): Obviously.
(Fh): Let $u > 0, v \geq 0, w \geq 0$ and $F(u,v,u,v,w) = u^2 - (auv + bu^2 + cuv + dw^2) \leq 0$.
Put $m = \max\{v,w\}$, then $(1-b)u^2 - (a+c)um - dm^2 \leq 0$.
If $m = 0$, then $(1-b)u^2 \leq 0$, a contradiction. Thus $m \neq 0$ and $f(t) = (1-b)t^2 - (a+c)t - d \leq 0$, where $t = \frac{u}{m}$.
Since $f(0) < 0$ and $f(1) = 1 - (a+b+c+d) > 0$, let $h_1 \in [0,1)$ be the root of equation $f(t) = 0$, then $f(t) \leq 0$ for $t \leq h_1$ and $u \leq h_1\max\{v,w\}$.
If $u = 0$, then $u \leq h\max\{v,w\}$.
Similarly, if $F(u,v,v,u,w) \leq 0$ then $u \leq h_2\max\{v,w\}$, where $h_2 \in [0,1)$. Thus $u \leq h\max\{v,w\}$, where $h = \max\{h_1,h_2\}$.

In this paper a general fixed point theorem for two pairs of mappings on two metric spaces satisfying implicit relations is proved.

This result generalizes the main result from [2].

3. Main result

Theorem 3. *Let (X, d) and (Y, ρ) be complete metric spaces, let A, B be mappings of X into Y and let S, T be mappings of Y into X satisfying the inequalities*

$$(1) \qquad F(d(SAx, TBx'), d(x, x'), d(x, SAx), d(x', TBx'), \rho(Ax, Bx')) \leq 0$$

$$(2) \qquad G(\rho(BSy, ATy'), \rho(y, y'), \rho(y, BSy), \rho(y', ATy'), d(Sy, Ty')) \leq 0$$

for all x, x' in X and y, y' in Y, where $F, G \in \mathcal{F}_5$. If one of the mappings A, B, S and T is continuous then SA and TB have a common fixed point z in X and BS and AT have a common fixed point w in Y.

Further, $Az = Bz = w$ and $Sw = Tw = z$.

Proof. Let x be an arbitrary point in X and $Ax = y_1$, $Sy_1 = x_1$, $Bx_1 = y_2$, $Ty_2 = x_2$, $Ax_2 = y_3$ and in general let

$$Sy_{2n-1} = x_{2n-1}, Bx_{2n-1} = y_{2n}, Ty_{2n} = x_{2n}, Ax_{2n} = y_{2n+1}$$

for $n = 1, 2, ...$ Let $h = \max\{h_1, h_2\}$, where h_1 and h_2 are real constants satisfying conditions (Fh) and (Gh), respectively.

Using inequalities (1) we have succesively

$$F(d(SAx_{2n}, TBx_{2n-1}), d(x_{2n}, x_{2n-1}), d(x_{2n}, SAx_{2n}), \\ d(x_{2n-1}, TBx_{2n-1}), \rho(Ax_{2n}, Bx_{2n-1})) \leq 0,$$

$$F(d(x_{2n+1}, x_{2n}), d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), \rho(y_{2n}, y_{2n+1})) \leq 0.$$

which implies by (Fh) that

$$(3) \qquad d(x_{2n+1}, x_{2n}) \leq h \max\{d(x_{2n}, x_{2n-1}), \rho(y_{2n}, y_{2n+1})\}.$$

Using inequality (1) again, it follows that

$$(4) \qquad d(x_{2n}, x_{2n-1}) \leq h \max\{d(x_{2n-1}, x_{2n-2}), \rho(y_{2n}, y_{2n-1})\}.$$

Similarly, using inequality (2) we get

$$(5) \qquad \rho(y_{2n}, y_{2n+1}) \leq h \max\{d(x_{2n-1}, x_{2n}), \rho(y_{2n-1}, y_{2n})\}$$

and

$$(6) \qquad \rho(y_{2n-1}, y_{2n}) \leq h \max\{d(x_{2n-2}, x_{2n-1}), \rho(y_{2n-2}, y_{2n-1})\}.$$

Using inequalities (3) and (5) we have

$$(7) \qquad d(x_{2n+1}, x_{2n}) \leq h \max\{d(x_{2n}, x_{2n-1}), \rho(y_{2n-1}, y_{2n})\}$$

and similarly, from inequalities (4) and (6), we have

$$(8) \qquad d(x_{2n}, x_{2n-1}) \leq h \max\{d(x_{2n-2}, x_{2n-1}), \rho(y_{2n-2}, y_{2n-1})\}.$$

It now follows from inequalities (5),(6),(7) and (8) that

$$d(x_{n+1}, x_n) \leq h \max\{d(x_n, x_{n-1}), \rho(y_n, y_{n-1})\},$$

$$\rho(y_{n+1}, y_n) \leq h \max\{d(x_n, x_{n-1}), \rho(y_n, y_{n-1})\}$$

and easy induction argument shows that

$$d(x_{2n+1}, x_{2n}) \leq h^{n-1} \max\{d(x_1, x_2), \rho(y_1, y_2)\},$$

$$\rho(y_{2n+1}, y_{2n}) \leq h^{n-1} \max\{d(x_1, x_2), \rho(y_1, y_2)\}$$

for $n = 1, 2, \ldots$. Since $0 \leq h < 1$, it follows that $\{x_n\}$ and $\{y_n\}$ are the Cauchy sequences with the limits z in X and w in Y, respectively.

Now suppose that A is continuous. Then

$$\lim Ax_{2n} = Az = \lim y_{2n+1} = w$$

and so $Az = w$.

Using inequality (1) we have successively

$$F(d(SAz, TBx_{2n-1}), d(z, x_{2n-1}), d(z, SAz), d(x_{2n-1}, TBx_{2n-1}), \rho(Az, Bx_{2n-1})) \leq 0,$$

$$F(d(Sw, x_{2n}), d(z, x_{2n-1}), d(z, Sw), d(x_{2n-1}, x_{2n}), \rho(w, y_{2n})) \leq 0.$$

Letting n tend to infinity, we have

$$F(d(Sw, z), 0, d(z, Sw), 0, 0) \leq 0$$

which implies by (Fh) that $z = Sw = SAz$.

Now using inequality (2) we have successively

$$G(\rho(BSw, ATy_{2n}), \rho(w, y_{2n}), \rho(w, BSw), \rho(y_{2n}, ATy_{2n}), d(Sw, Ty_{2n})) \leq 0,$$

$$G(\rho(Bz, y_{2n+1}), \rho(w, y_{2n}), \rho(w, Bz), \rho(y_{2n}, y_{2n+1}), d(z, x_{2n})) \leq 0.$$

Letting n tend to infinity, we have

$$G(\rho(Bz, w), 0, \rho(w, Bz), 0, 0) \leq 0$$

which implies by (Gh) that $w = Bz = BSw$.

Using inequality (1) we have successively

$$F(d(SAz, TBz), d(z, z), d(z, SAz), d(z, TBz), \rho(Az, Bz)) \leq 0,$$

$$F(d(z, Tw), 0, 0, d(z, Tw), 0) \leq 0$$

which implies by (Fh) that $z = Tw$ and $z = Tw = TBz$.

Using inequality (2) we have succesively

$$G(\rho(BSw, ATw), \rho(w, w), \rho(w, BSw), \rho(w, ATw), d(Sw, Tw)) \leq 0,$$

$$G(\rho(w, ATw), 0, 0, \rho(w, ATw), 0) \leq 0$$

which implies by (Fh) that $w = ATw$.

The same results hold also if one of the mappings B,S,T is continuous. □

Corollary 1. *Theorem 2[2].*

Proof. The proof follows from Theorem 3 and Example 1. □

References

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