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ON IRROTATIONAL D-CONFORMAL CURVATURE TENSOR

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Abstract. The objective of this paper is to study a irrotational D-Conformal curvature tensor on a K-contact, Kenmotsu and trans-Sasakian manifolds.

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1. Introduction

Gatti and Bagewadi [4] have studied irrotational quasi-conformal curvature tensor in K-contact, Kenmotsu and trans-Sasakian manifolds and they have shown that these manifolds are Einsteinian. In this paper we extend the results to irrotational D-Conformal curvature tensor in K-contact, Kenmotsu and trans-Sasakian manifolds.

2. Preliminaries

Definition 2.1. Let M be an n-dimensional almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is (1,1) tensor field, ξ is a vector field, η is a 1-form and g is the associated Riemannian metric such that [2],

$$\eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

$$\phi^2 = -I + \eta \otimes \xi, \quad g(X, \xi) = \eta(X), \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

for any of vector fields X, Y on M. If moreover,

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0 \tag{2.4}$$

where ∇ denotes the Riemannian connection of g, then M is called a K-Contact manifold. In a K-Contact manifold the following relations hold:

$$\nabla_X \xi = -\phi X, \tag{2.5}$$

$$S(X, \xi) = (n-1)\eta(X), \tag{2.6}$$

$$R(X, Y)\xi = \eta(Y)X - -\eta(X)Y. \tag{2.7}$$

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Definition 2.2. An almost contact metric manifold with structure tensors (ϕ, ξ, η, g) is called Kenmotsu manifold if [5]

$$(\nabla_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi X \text{ and} \tag{2.8}$$

$$\nabla_X \xi = X - \eta(X)\xi \tag{2.9}$$

In a Kenmotsu manifold the following relations hold:

$$(\nabla_x \eta)Y = g(X, Y) - \eta(X)\eta(Y) \tag{2.10}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.11}$$

$$R(X, Y)\xi = \eta(X)Y - -\eta(Y)X \tag{2.12}$$

Definition 2.3. An almost contact metric manifold with structure tensors (ϕ, ξ, η, g) is called trans-Sasakian manifold, if it satisfies the condition [3]

$$(\nabla_X \phi)Y = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X] \tag{2.13}$$

From (2.13) it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi) \tag{2.14}$$

$$(\nabla_X \eta)Y = -\alpha\, g(\phi X, Y) + \beta\, g(\phi X, \phi Y). \tag{2.15}$$

In a trans-Sasakian manifold the following relations hold [3]:

$$R(X, Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\phi(X) -\eta(X)\phi(Y)) + (Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y, \tag{2.16}$$

$$2\alpha\beta + \xi\alpha = 0, \tag{2.17}$$

$$S(X, \xi) = ((n-1)(\alpha^2 - \beta^2) - \xi\beta)\,\eta(X) - (n-2)X\beta - (\phi X)\alpha, \tag{2.18}$$

$$Q\xi = ((n-1)(\alpha^2 - \beta^2) - \xi\beta)\xi - (n-2)\, grad\beta\, + \phi(grad\alpha). \tag{2.19}$$

When,

$$\phi(grad\alpha) = (n-2)\, grad\beta \tag{2.20}$$

equations (2.18) and (2.19) reduce to

$$S(X, \xi) = (n-1)(\alpha^2 - \beta^2)\,\eta(X), \tag{2.21}$$

$$Q\xi = (n-1)(\alpha^2 - \beta^2)\xi. \tag{2.22}$$

The D-conformal curvature tensor B on a Riemannian manifold (M^n, g) $(n > 4)$ is defined as [2]:

$$\begin{aligned} B(X,Y)Z &= R(X,Y)Z + \frac{1}{(n-3)}[S(X,Z)Y - S(Y,Z)X + g(X,Z)QY \\ &- g(Y,Z)QX - S(X,Z)\eta(Y)\xi + S(Y,Z)\eta(X)\xi - \eta(X)\eta(Z)QX] \\ &- \frac{(k-2)}{(n-3)}[g(X,Z)Y - g(Y,Z)X] + \frac{k}{(n-3)}[g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X] \end{aligned} \tag{2.23}$$

where $k = \dfrac{r + 2(n-1)}{(n-2)}$, R is the curvature tensor, S is the Ricci tensor and r is the scalar curvature.

Definition 2.4. The rotation (curl) of D-Conformal curvature tensor B on a Riemannian manifold is given by

$$\begin{aligned} Rot\, B &= (\nabla_U B)(X,Y,Z) + (\nabla_X B)(U,Y,Z) \\ &+ (\nabla_Y B)(U,X,Z) - (\nabla_Z B)(X,Y,U) \end{aligned} \tag{2.24}$$

By virtue of second Bianchi identity

$$(\nabla_U \mathbf{B})(X,Y)Z + (\nabla_X B)(Y,U)Z) + (\nabla_Y B)(U,X)Z = 0 \tag{2.25}$$

(2.24) reduces to

$$curl\, B = -(\nabla_Z B)(X,Y)U$$

If the D-conformal curvature tensor is irrotational then curl B= 0 and by (2.25) we have

$$(\nabla_Z B)(X,Y)U = 0$$

which implies

$$\nabla_Z \{B(X,Y)U\} = B(\nabla_Z X, Y)U + B(X, \nabla_Z Y)U + B(X,Y)\nabla_Z U \tag{2.26}$$

Put $U = \xi$ in the above equation (2.26), we have

$$\nabla_Z \{B(X,Y)\xi\} = B(\nabla_Z X, Y)\xi + B(X, \nabla_Z Y)\xi + B(X,Y)\nabla_Z \xi \tag{2.27}$$

3. D-Conformal Curvature Tensor in K-Contact Manifold

Lemma 3.1. *Prove that D-Conformal curvature tensor B in K-Contact manifold satisfies*

$$B(X,Y)\xi = k_1(\eta(Y)X - \eta(X)Y) \tag{3.1}$$

where $k_1 = \left(\dfrac{-4}{n-3}\right)$

Proof. Using (2.6) and (2.7) in (2.23) we get (3.1).

Lemma 3.2. *If the D-Conformal curvature tensor in a K-Contact manifold is irrotational then the D-Conformal curvature tensor B is given by*

$$B(X,Y)Z = k_1[g(Y,Z)X - g(X,Z)Y] \tag{3.2}$$

Proof. Using (3.1) and (2.5) in (2.27) and simplifying we have

$$-B(X,Y)\phi Z = k_1[g(X,\phi Z)Y - -g(Y,\phi Z)X] \tag{3.3}$$

Replace Z by ϕZ in (3.3) and by virtue of (2.3) and (3.1) we get (3.2). □

Theorem 3.1. *If the D-Conformal curvature tensor in K-contact manifold is irrotational, then the manifold is η-Einstein and the scalar curvature is given by*

$$r_1 = n[(n-1)(k-1) - -rk]. \tag{3.4}$$

Proof. Using (2.23) and (3.2), the curvature tensor B in K-contact manifold is given by

$$\begin{aligned} R(X,Y)Z &= k_1[g(Y,Z)X - g(X,Z)Y] - \frac{1}{(n-3)}[S(X,Z)Y - S(Y,Z)X \\ &+ g(X,Z)QY - g(Y,Z)QX - S(X,Z)\eta(Y)\xi + S(Y,Z)\eta(X)\xi \\ &- \eta(X)\eta(Z)QY + \eta(Y)\eta(Z)QX\ ] \\ &+ \frac{(k-2)}{(n-3)}[g(X,Z)Y - g(Y,Z)X] \quad - \frac{k}{(n-3)}[g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X] \end{aligned} \tag{3.5}$$

Let X_i, $i = 1, 2, \ldots, n$ be an orthonormal basis of the tangent space at any point. Then the sum for $1 \le i \le n$ of the relation (3.5) with $Y = Z = X_i$, yields

$$\begin{aligned} \sum R(X,X_i)X_i &= k_1[g(X_i,X_i)X - g(X,X_i)X_i] \\ &- \frac{1}{(n-3)}[S(X,X_i)X_i - S(X_i,X_i)X \\ &+ g(X,X_i)QX_i - g(X_i,X_i)QX + S(X_i,X_i)\eta(X)\xi] \\ &+ \frac{(k-2)}{(n-3)}[g(X,X_i)X_i - g(X_i,X_i)X] \\ &+ \frac{k}{(n-3)}[g(X_i,X_i)\eta(X)\xi] \end{aligned} \tag{3.6}$$

The Ricci tensor S is given by

$$S(X,Y) = \sum g(R(X,X_i)X_i, Y) + g(X,Y). \tag{3.7}$$

Taking inner product of (3.6) with Y and by virtue of (3.5) and (3.7), we have

$$S(X,Y) = a\,g(X,Y) + b\eta(X)\eta(Y). \tag{3.8}$$

where $a = [(n-1)(k-2) - r + (n-1)]$ and $b = (r - nk)$.

Thus the manifold is η-Einstein. Using (3.7) and making use of the relation

$$S(X,Y) = g(QX,Y), \tag{3.9}$$

we have $QX = [(n-1)(k-1) - nk]X$.

Hence r_1 is given by (3.4). □

4. D-Conformal Curvature Tensor in Kenmotsu Manifold

Lemma 4.1. *Prove that D-Conformal curvature tensor B in Kenmotsu manifold satisfies*

$$B(X,Y)\xi = k_2(\eta(X)Y - \eta(Y)X) \tag{4.1}$$

where $k_2 = \dfrac{-1}{(n-3)}$.

Proof. Using (2.11) and (2.12) in (2.23) we get (4.1).

Lemma 4.2. *If the D-Conformal curvature tensor in a Kenmotsu manifold is irrotational, then the D-Conformal curvature tensor B is given by*

$$B(X,Y)Z = k_2[g(X,Z)Y - g(Y,Z)X] \tag{4.2}$$

Proof. Using (4.1) and (2.9) in (2.27) we have

$$\begin{aligned} \nabla_Z\left[k_2\left\{\eta(X)Y - \eta(Y)X\right\}\right] &= k_2\left[\eta(X)\nabla_Z Y - \eta(\nabla_Z Y)X\right] \\ &+ k_2\left[\eta(\nabla_Z X)Y - \eta(Y)\nabla_Z X\right] \\ &+ B(X,Y)(Z - \eta(Z)\xi) \end{aligned}$$

Simplifying the above equation we get (4.2). □

Theorem 4.1. *If the D-Conformal curvature tensor in Kenmotsu manifold is irrotational, then the manifold is η-Einstein and the scalar curvature is given by*

$$r_1 = n[(n-1)(k-3) - nk]. \tag{4.3}$$

Proof. Using (2.23) and (4.2), the curvature tensor B in Kenmotsu manifold is given by

$$\begin{aligned} R(X,Y)Z &= k_2[g(X,Z)Y - g(Y,Z)X] - \frac{1}{(n-3)}[S(X,Z)Y - S(Y,Z)X \\ &+ g(X,Z)QY - g(Y,Z)QX - S(X,Z)\eta(Y)\xi + S(Y,Z)\eta(X)\xi \\ &- \eta(X)\eta(Z)QY + \eta(Y)\eta(Z)QX\ ] \\ &+ \frac{(k-2)}{(n-3)}[g(X,Z)Y - g(Y,Z)X] - \frac{k}{(n-3)}[g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X] \end{aligned} \tag{4.4}$$

Let X_i, $i = 1, 2, \ldots, n$ be an orthonormal basis of the tangent space at any point. Then the sum for $1 \leq i \leq n$ of the relation (4.4) with $Y = Z = X_i$, yields

$$\begin{aligned}\sum R(X, X_i)X_i &= k_2[g(X, X_i)X_i - g(X_i, X_i)X] \\ &- \frac{1}{(n-3)}[S(X, X_i)X_i - S(X_i, X_i)X \\ &+ g(X, X_i)QX_i - g(X_i, X_i)QX + S(X_i, X_i)\eta(X)\xi\,] \\ &+ \frac{(k-2)}{(n-3)}[g(X, X_i)X_i - g(X_i, X_i)X] \\ &+ \frac{k}{(n-3)}[g(X_i, X_i)\eta(X)\xi]\end{aligned} \tag{4.5}$$

The Ricci tensor S is given by (3.7) and by virtue of (4.5) we have

$$S(X, Y) = a\, g(X, Y) + b\eta(X)\eta(Y). \tag{4.6}$$

where $a = [(n-1)(k-3) - r]$ and $b = (r - nk)$.

Hence the manifold is η-Einstein. Using the relations (3.9) and (4.6) we have $Q = [(n-1)(k-3) - nk]\ X$. Hence r_1 is given by (4.3). □

5. D-Conformal Curvature Tensor in Trans-Sasakian Manifold

Lemma 5.1. *Prove that D-Conformal curvature tensor B in a trans-Sasakian Manifold satisfies*

$$B(X, Y)\xi = k_3(\eta(Y)X - \eta(X)Y) + k_4(\eta(Y)\phi X - \eta(X)\phi Y) \tag{5.1}$$

where $k_3 = \left[\alpha^2 - \beta^2 - 1\right]$ *and* $k_4 = 2\alpha\beta$.

Proof. Using (2.16) and (2.21) in (2.23), we get (5.1). □

Lemma 5.2. *If the D-Conformal curvature tensor B in a trans-Sasakian manifold is irrotational then the D-Conformal curvature tensor B is given by*

$$\begin{aligned}B(X, Y)Z &= k_3\,[g(Y, Z)X - g(X, Z)Y] \\ &+ k_4\,[g(Y, Z)\phi X + \eta(Y)g(\phi Z, X)\xi \\ &- \eta(Y)\ g(X, Z)\phi Y + \eta(X)g(\phi Z, Y)\xi]\end{aligned} \tag{5.2}$$

Proof. Using (5.1) and (2.14) in (2.27) we have

$$\begin{aligned}&\nabla_Z\,[k_3\,[\eta(Y)X - \eta(X)Y] + k_4(\eta(Y)\phi X - \eta(X)\phi Y] \\ &\quad = k_3\,[\eta(Y)\nabla_Z X - \eta(\nabla_Z X)Y] + k_4\,[\eta(Y)\phi(\nabla_Z X) - \eta(\nabla_Z X)\,\phi Y] \\ &\quad +k_3\,[\eta(\nabla_Z Y)X - \eta(X)\nabla_Z Y] + k_4\,[\eta(\nabla_Z Y)\phi X - \eta(X)\,\phi\nabla_Z Y] \\ &\quad +B(X, Y)(-\alpha\phi Z + \beta(Z - \eta(Z)\xi)\end{aligned}$$

Simplifying the above equation by using (2.1), (2.14) and (2.21) and the covariant derivative formula

$$(\nabla_X\phi)Y = \nabla_X\phi(Y) - \phi(\nabla_X Y), \text{ we get}$$

$$\begin{aligned}
&-\alpha B(X,Y)\phi Z + \beta B(X,Y)Z = \\
&\quad k_3[-\alpha g(\phi Z,Y)X + \beta g(Y,Z)X + \alpha g(\phi Z,X)Y - \beta g(X,Z)Y] \\
&+ \ k_4[-\alpha g(\phi Z,Y)\phi X + \beta g(Y,Z)\phi X + \alpha\,\eta(Y)g(Z,X)\xi \\
&+ \ \beta\,\eta(Y)g(\phi Z,X)\xi + \alpha\eta(Y)g(\phi Z,X)\phi Y - \beta\,\eta(Y)g(Z,X)\phi Y \\
&+ \ \eta(X)g(Y,Z)\xi - \beta\,\eta(X)g(\phi Z,Y)\xi
\end{aligned} \tag{5.3}$$

Replacing Z by ϕZ in (5.3) and simplifying we get

$$\begin{aligned}
&\alpha B(X,Y)Z - \alpha\eta(Z)k_3[\eta(Y)X - \eta(X)Y] \\
&-\alpha\,\eta(Z)k_4[\eta(Y)\phi X - \eta(X)\phi Y] + \beta B(X,Y)\phi Z = \\
&\quad k_3[\alpha g(Z,Y)X - \alpha\eta(Z)\eta(Y)X) + \beta g(\phi Z,Y)X \\
&- \ \alpha g(Z,X)Y + \alpha\eta(Z)\eta(X)Y - \beta g(\phi Z,X)Y] + k_4[\alpha g(Z,Y)\phi X \\
&- \ \alpha\,\eta(Z)\eta(Y)\phi X + \beta g(\phi Z,Y)\phi X + \alpha\,\eta(Y)g(\phi Z,X)\xi \\
&- \ \beta\,\eta(Y)g(Z,X)\xi - \alpha\,\eta(Y)g(Z,X)\phi Y + \alpha\,\eta(Z)\eta(X)\eta(Y)\phi Y \\
&- \ \beta\,\eta(Y)g(\phi Z,X)\phi Y + \alpha\,\eta(X)g(\phi Z,Y)\xi + \beta\,\eta(X)g(Z,Y)\xi
\end{aligned} \tag{5.4}$$

Multiplying equation (5.3) by α and (5.4) by β and adding we get (5.2). □

Theorem 5.1. *If the D-Conformal curvature tensor in a trans-Sasakian manifold is irrotational, then the manifold is η-Einstein and scalar curvature is given by*

$$r_1 = n[(n-1)(k-2) - (n-1)(n-3)(\alpha^2 - \beta^2 - 1) - nk]. \tag{5.5}$$

Proof. Using (2.23) and (5.2) the curvature tensor B in trans-Sasakian manifold is given by

$$\begin{aligned}
&R(X,Y)Z = \\
&\quad k_3[g(Y,Z)X - g(X,Z)Y] + k_4\,[g(Y,Z)\phi X + \eta(Y)g(\phi Z,X)\xi \\
&- \ \eta(Y)\,g(X,Z)\phi Y + \eta(X)g(\phi Z,Y)\xi] \\
&- \ \frac{1}{(n-3)}\,[S(X,Z)Y - S(Y,Z)X + g(X,Z)QY - g(Y,Z)QX \\
&- \ S(X,Z)\eta(Y)\xi + S(Y,Z)\eta(X)\xi - \eta(X)\eta(Z)QY + \eta(Y)\eta(Z)QX\,] \\
&+ \ \frac{(k-2)}{(n-3)}\,[g(X,Z)Y - g(Y,Z)X] - \frac{k}{(n-3)}\,[g(X,Z)\eta(Y)\xi \\
&- \ g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X]
\end{aligned} \tag{5.6}$$

Let X_i, $i = 1, 2, \ldots, n$ be an orthonormal basis of the tangent space at any point. Then the sum for $1 \leq i \leq n$ of the relation (5.6) with $Y = Z = X_i$, yields

$$
\begin{aligned}
\sum R(X, X_i)X_i = & \\
& k_3[g(X_i, X_i)X - -g(X, X_i)X_i] \\
- & \frac{1}{(n-3)}[S(X, X_i)X_i - S(X_i, X_i)X + g(X, X_i)QX_i \\
- & g(X_i, X_i)QX + S(X_i, X_i)\eta(X)\xi\ ] \\
+ & \frac{(k-2)}{(n-3)}[g(X, X_i)X_i - g(X_i, X_i)X] \\
+ & \frac{k}{(n-3)}[g(X_i, X_i)\eta(X)\xi]
\end{aligned} \tag{5.7}
$$

The Ricci tensor S is given by (3.7) and by virtue of (5.7) we have

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) \tag{5.8}$$

where $a = [(n-1)(k-2) - r - (n-1)(n-3)(\alpha^2 - \beta^2 - 1)]$ and $b = (r - nk)$.

Hence the manifold is Einsteinian. Using (3.9) and (5.8) we have

$$QX = [(n-1)(k-2) - (n-1)(n-3)(\alpha^2 - \beta^2 - 1) - nk]X.$$

Hence r_1 is given by (5.5). □

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