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A METHOD OF THE DETERMINATION OF A GEODESIC CURVE ON RULED SURFACE WITH TIME-LIKE RULINGS

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Abstract. A non-linear differential equation is analyzed to determine the geodesic curves on ruled surfaces with time-like rulings in $\mathbb{R}^3_1$. When it is assumed that curvature and torsion of the base curve and components with respect to Frenet's frame of time-like straight-line are constants, for a special integration constant, it appears that the resulting non-linear differential equation can be integrated exactly. Finally, examples are given to show the geodesic curve on ruled surfaces with time-like rulings.

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1. Introduction

Most people have heard the phrase: A straight line is the shortest distance between two points. But in differential geometry, they say the same in a different language. They say instead: Geodesics for the Euclidean metric are straight lines. A geodesic is a curve that represents the extremal value of a distance function in some space. In the Euclidean space, extremal means 'minimal', so geodesics are paths of minimal arc length. In the 3-dimensional Minkowski space, the extremal paths are actually 'maximal' arc length.

Geodesics are important in the relativistic description of gravity. Einstein's Principle of Equivalence, part of the General Theory of Relativity, tells us that geodesics represent the paths of freely-falling particles in a given space. (Freely-falling in this context means moving only under the influence of gravity, with no other forces involved.)

Geodesics are the curves along which geodesic curvature vanishes. This is of course where the geodesic curvature name comes from. Since Lorentzian metric is not positive definite metric, the distance function dS^2 can be positive, negative or zero, whereas the distance function in the Euclidean space can only be positive. Thus, we have to separate our geodesics on the basis of whether the distance function is positive, negative or zero. The geodesics with $\mathrm{dS}^2 < 0$ are called space-like geodesics. The geodesics with $\mathrm{dS}^2 > 0$ are called time-like geodesics, while geodesics with $\mathrm{dS}^2 = 0$ are called null geodesics.

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In this article, the basic concepts of 3-dimensional Minkowski space have been first given. Using geodesic curvature, the differential equation of the geodesics on a time-like ruled surface with time-like rulings has been obtained and solved under some conditions. Finally, examples have been given related to the subject.

2. Preliminaries

Let us consider the Minkowski 3-space $\mathbb{R}^3_1[\mathbb{R}^3, (+,+,-)]$ and let the Lorentzian inner product of $\vec{X} = (x_1, x_2, x_3)$ and $\vec{Y} = (y_1, y_2, y_3) \in \mathbb{R}^3$ be

$$< \vec{X} \, , \vec{Y} >= x_1 y_1 + x_2 y_2 - x_3 y_3 .$$

A vector $\vec{X} \in \mathbb{R}^3_1$ is called a *space-like* vector when $< \vec{X} \, , \vec{X} >> 0$ or $\vec{X} = 0$. It is called *time-like* and *null (lightlike) vector* in case of $< \vec{X} \, , \vec{X} >< 0$ and $< \vec{X} \, , \vec{X} >= 0$ for $\vec{X} \neq 0$, respectively.

The norm of $\vec{X} \in \mathbb{R}^3_1$ is denoted by $\left\| \vec{X} \right\|$ and defined as $\left\| \vec{X} \right\| = \sqrt{\left| < \vec{X} \, , \vec{X} > \right|}$.

For a regular curve in $\mathbb{R}^3_1$, if its tangent vector at every point is space-like, it is called space-like curve. Similarly, if its tangent vector is time-like and null vector, it is called *time-like* and *null curve*, respectively,[3].

Let $\vec{X} = (x_1, x_2, x_3)$ and $\vec{Y} = (y_1, y_2, y_3)$ be any two vectors in $\mathbb{R}^3_1$. *The cross product* of $\vec{X}$ and $\vec{Y}$ is defined by

$$\vec{X} \wedge \vec{Y} = (\, x_2 y_3 - x_3 y_2 \, , \, x_3 y_1 - x_1 y_3 \, , \, x_2 y_1 - x_1 y_2 \,), [1]. \tag{1}$$

A surface in $\mathbb{R}^3_1$ is called a *time-like surface* if the induced metric on the surface is a Lorentz metric, i.e. the normal vector on the time-like surface is a space-like vector, [2].

Let $\vec{\alpha} = \vec{\alpha}(s)$ be a unit speed space-like curve in $\mathbb{R}^3_1$. Consider the orthonormal Frenet frame $\{\vec{e}_1 \, , \vec{e}_2 \, , \vec{e}_3\}$ associated with the curve $\vec{\alpha} = \vec{\alpha}(s)$, such that $\vec{e}_1 = \vec{e}_1(s)$, $\vec{e}_2 = \vec{e}_2(s)$ and $\vec{e}_3 = \vec{e}_3(s)$ are the tangent vector field, the principal vector field and the binormal vector field, respectively.

In this study, $\vec{e}_3 = \vec{e}_3(s)$ will be taken as time-like. If one takes it space-like, similar procedures will be applied.

The Frenet formulas are given by

$$\vec{e_1}^{\,\prime}(s) = \kappa(s)\vec{e_2}(s), \;\; \vec{e_2}^{\,\prime}(s) = -\kappa(s)\vec{e_1}(s) + \tau(s)\vec{e_3}(s), \;\; \vec{e_3}^{\,\prime}(s) = \tau(s)\vec{e_2}(s) \tag{2}$$

where $\kappa(s)$ and $\tau(s)$ are the curvature and torsion of $\vec{\alpha}(s)$, respectively.

It is easy to see from (1) that

$$\vec{e}_1 \wedge \vec{e}_2 = -\vec{e}_3 \quad , \quad \vec{e}_1 \wedge \vec{e}_3 = -\vec{e}_2 \quad , \quad \vec{e}_2 \wedge \vec{e}_3 = \vec{e}_1 \tag{3}$$

A time-like straight line $\vec{X}$ in $\mathbb{R}^3_1$, such that it is strictly connected to Frenet's frame of the space-like curve $\vec{\alpha} = \vec{\alpha}(s)$, is represented uniquely with respect to this frame, in the form

$$\vec{X}(s) = \sum_{i=1}^{3} x_i(s)\vec{e}_i(s), \quad < \vec{X}(s)\,,\, \vec{X}(s) > \, < \, 0 \tag{4}$$

As $\vec{X}$ moves along $\vec{\alpha} = \vec{\alpha}(s)$ it generates a time-like ruled surface given by the regular parametrization

$$\begin{aligned} &\varphi(\text{s,v}) = \vec{\alpha}(\text{s}) + \text{v}\vec{X}(\text{s}) \\ &x_1^2 + x_2^2 - x_3^2 = -1 \quad , \quad \vec{X}^{'}(s) \neq 0 \end{aligned}$$

where the components $x_i(s)$ $(i = 1, 2, 3)$ are scalar functions of the arc-length parameter of the curve $\vec{\alpha} = \vec{\alpha}(s)$

This ruled surface will be denoted by M. The curve $\vec{\alpha} = \vec{\alpha}(s)$ is called a *base curve* and the various positions of the generating line $\vec{X}$ are called the *rulings* of the surface M.

If consecutive rulings of a ruled surface in $\mathbb{R}_1^3$ intersect, then the surface is said to be *developable*. All other ruled surfaces are called *skew surfaces*. If there exists a common perpendicular to two constructive rulings in the skew surface, then the foot of the common perpendicular on the main ruling is called a *striction point*. The set of striction points on a ruled surface defines *the striction curve*, [5].

The striction curve, $\vec{\beta} = \vec{\beta}(s)$, can be written in terms of the base curve $\vec{\alpha}(s)$ as

$\vec{\beta}(s) = \vec{\alpha}(s) - \phi(s)\vec{X}(s)$ where

$$\phi(s) = \frac{x_1^{'} - x_2\kappa}{< \vec{X}'\,,\, \vec{X}' >} \tag{5}$$

The unit normal vector $\vec{n}$ on the time-like ruled surface M is given by

$$\vec{n} = \frac{\vec{\alpha}^{'}(s) \wedge \vec{X}(s) + v\vec{X}^{'}(s) \wedge \vec{X}(s)}{\left\| \vec{\alpha}'(s) \wedge \vec{X}(s) + v\vec{X}'(s) \wedge \vec{X}(s) \right\|} \tag{6}$$

From (3) and (4) the unit normal vector to the ruled surface M at the point $(s\,,\,o\,)$ is

$$\vec{n}(s\,,\,o\,) = -\frac{x_3\vec{e}_2 + x_2\vec{e}_3}{\sqrt{|x_2^2 - x_3^2|}}$$

Thus, if $x_2 = 0$, $x_3 \neq 0$ then the base curve of M is a geodesic curve.

In this paper, the striction curve of the ruled surface M will be taken as the base curve. In this case, for the parametric equation of M, we can write

$$\begin{aligned} &\varphi(\text{s,v}) = \vec{\alpha}(\text{s}) + \text{v}\vec{X}(\text{s}) \quad , \quad \vec{X}^{'}(s) \neq 0 \\ &x_1^2 + x_2^2 - x_3^2 = -1 \qquad , \quad x_1^{'} - x_2\kappa = 0 \end{aligned}$$

3. Geodesic Curvature

Let $\vec{\gamma}=\vec{\gamma}(s)$ be a curve on the ruled surface M. Then it can be written as

$$\vec{\gamma}(s)=\vec{\alpha}(s)+v(s)\vec{X}(s) \tag{7}$$

Using (2), for the unit tangent vector along the curve $\vec{\gamma}\vec{\gamma}(s)$, we get

$$T(s)=\frac{\vec{\gamma}^{'}(s)}{\|\vec{\gamma}^{'}(s)\|}=\frac{\eta_1\vec{e}_1+\eta_2\vec{e}_2+\eta_3\vec{e}_3}{\sqrt{R}}$$

where

$$\begin{array}{l} \eta_1=1+v^{'}x_1 \quad , \quad \eta_2=v^{'}x_2+v\varphi_1 \quad , \quad \eta_3=v^{'}x_3+v\varphi_2 \\ \varphi_1=x_2^{'}+x_1\kappa+x_3\tau \quad , \quad \varphi_2=x_3^{'}+x_2\tau \quad \text{and} \\ R=\left|\eta_1^2+\eta_2^2-\eta_3^2\right| . \end{array}$$

Thus, we obtain

$$\begin{aligned} \vec{T}^{'}(s) &= R^{-3/2}\Big(\Big[(\eta_1^{'}-\eta_2\kappa)\vec{e}_1+(\eta_2^{'}+\eta_1\kappa+\eta_3\tau)\vec{e}_2+(\eta_3^{'}+\eta_2\tau)\vec{e}_3\Big]R \\ &- \frac{R^{'}}{2}(\eta_1\vec{e}_1+\eta_2\vec{e}_2+\eta_3\vec{e}_3)\ \Big) \end{aligned}$$

From (6), the unit normal vector field on the ruled surface M along the curve $\vec{\gamma}=\vec{\gamma}(s)$ is

$$\vec{n}(s\,,\,v(s)\,)=\frac{v(x_3\varphi_1-x_2\varphi_2)\vec{e}_1+(-x_3+vx_1\varphi_2)\vec{e}_2+(-x_2+vx_1\varphi_1)\vec{e}_3}{\sqrt{v^2(x_3\varphi_1-x_2\varphi_2)^2+(-x_3+vx_1\varphi_2)^2-(-x_2+vx_1\varphi_1)^2}}$$

Hence, the geodesic curvature of the curve $\vec{\gamma}=\vec{\gamma}(s)$ is obtained as

$$k_g=\frac{1}{R\,\|\vec{n}(s\,,\,v(s)\,)\|}\begin{vmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_1^{'}-\eta_2\kappa & \eta_2^{'}+\eta_1\kappa+\eta_3\tau & \eta_3^{'}+\eta_2\tau \\ v(x_3\varphi_1-x_2\varphi_2) & -x_3+vx_1\varphi_2 & -x_2+vx_1\varphi_1 \end{vmatrix} \tag{8}$$

Then, the differential equation of the geodesic curves on the ruled surface M is given by

$$f(s\,,\,v)v^{''}+h(s\,,\,v)(v^{'})^2+g(s\,,\,v)v^{'}+r(s\,,\,v)v-x_2\kappa=0, \tag{9}$$

where

$f(s,\,v)=1+x_1^2+v^2(\varphi_1^2-\varphi_2^2)$,

$h(s,\,v)=v\left[\varphi_2^2-\varphi_1^2-x_2^{'}\varphi_1+x_3^{'}\varphi_2+x_2\tau\varphi_2-(x_1\kappa+x_3\tau)\varphi_1\right]$,

$$\begin{aligned} g(s,\,v) &= vx_1\Big[x_2^{'}\varphi_1-x_2\varphi_1^{'}+x_3\varphi_2^{'}-x_3^{'}\varphi_2 \\ &- 2\tau(x_2\varphi_2-x_3\varphi_1)+2x_1\kappa\varphi_1+\varphi_1^2-\varphi_2^2\Big] \\ &+ v^2(\varphi_2\varphi_2^{'}-\varphi_1\varphi_1^{'})-x_1x_2\kappa \end{aligned}$$

and

$$
\begin{aligned}
r(s,\,v) &= -x_2\varphi_1^{'} + x_3\varphi_2^{'} + x_1\kappa\varphi_1 - x_2\tau\varphi_2 + x_3\tau\varphi_1 \\
&+ v\left[x_1(\varphi_1\varphi_1^{'} - \varphi_2\varphi_2^{'}) + x_2\kappa(\varphi_2^2 - \varphi_1^2)\right] \\
&+ v^2\left[(\varphi_2^{'}\varphi_1 - \varphi_2\varphi_1^{'})(x_3\varphi_1 - x_2\varphi_2)\right. \\
&+ \left.(\varphi_1^3 - \varphi_1\varphi_2^2)(x_1\kappa + x_3\tau) + x_2\tau(\varphi_2^3 - \varphi_1^2\varphi_2)\right]
\end{aligned}
\tag{10}
$$

If $\kappa = 0$, then $\vec{\alpha} = \vec{\alpha}(s)$ is a line. In this case, since the ruled surface M is a part of Lorentzian plane, the geodesics on the surface M are straight lines. In this paper we assume that $\kappa \neq 0$.

In general, the non-linear differential equation (9) can not be solved analytically. Moreover, since we do not have any boundary condition, we can not use any numerical method. Hence, let curvature and torsion of the base curve $\vec{\alpha} = \vec{\alpha}(s)$ be constants and $\vec{X}$ be fixed in $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ such that $x_1 = 0$. Under this assumption, the differential equation (9) takes the form

$$
f(v)v^{''} + h(v)(v^{'})^2 + r(v)v = 0 \tag{11}
$$

where

$$
f(v) = 1 + v^2\varphi_1^2, \quad h(v) = -2v\varphi_1^2, \quad r(v) = \varphi_1^2\left(1 + v^2\varphi_1^2\right), \quad \varphi_1 = x_3\tau \tag{12}
$$

If $\tau = 0$, from (11) we obtain $v = as + b,\ \ a, b \in \mathbb{R}$. Therefore, from (7) the equation of a geodesic curve on the space-like ruled surface M is

$$
\vec{\gamma}(s) = \vec{\alpha}(s) + (as + b)\vec{X}(s)
$$

We assume that $\tau \neq 0$.

By using the substitution $p = p(v) = v^{'} = \frac{dv}{ds}$, we find that equation (11) takes the form

$$
f(v)p\frac{dp}{dv} + h(v)p^2 + r(v)v = 0 \tag{13}
$$

Putting $2\frac{h(v)}{f(v)} = H(v)$ and $-\,2\frac{r(v)}{f(v)}v = R(v)$ in (13), we have the following representation for (13):

$$
2p\frac{dp}{dv} + H(v)p^2 = R(v)
$$

Now, by making the substitution $q = p^2$, we have that the last equation reduces to

$$
\frac{dq}{dv} + H(v)q = R(v) \tag{14}
$$

This is a linear differential equation. It is well known that solution of the equation (14) is $q = e^{-\int H(v)dv}\left[c_1 + \int R(v)e^{\int H(v)dv}dv\right]$,where c_1 is an arbitrary constant.

Since $\int H(v)dv = -\ln(1+v^2\varphi_1^2)^2$ and $\int \mathrm{R(v)e}^{\int H(v)dv}dv = \frac{1}{1+v^2\varphi_1^2}$, we obtain the following solution for the differential equation (11):

$$s = \pm \int \frac{dv}{\sqrt{c_1(1+v^2\varphi_1^2)^2 + (1+v^2\varphi_1^2)}} \tag{15}$$

It is well known that the left-hand side of (15) is the elliptic integral (see for example [4]) which can not be integrated exactly except for the following case:

If case $c_1 = 0$, we have

$$v = \pm \frac{1}{\varphi_1}\sinh(\varphi_1 s) \tag{16}$$

From (7) and (15), the equation of a geodesic curve on the surface M is

$$\vec{\gamma}(s) = \overrightarrow{\alpha}(s) + \frac{1}{\varphi_1}\sinh(\varphi_1 s)\overrightarrow{X}(s) \tag{17}$$

Since the tangent vector of the geodesic curve $\overrightarrow{\gamma} = \vec{\gamma}(s)$ is $\overrightarrow{\gamma_s} = \vec{e}_1 + v\varphi_1\vec{e}_2 + v'x_3\vec{e}_3$, we have the following result:

Corollary 1.
i) If $1 + v^2\varphi_1^2 - x_3^2(v')^2 > 0$, then $\overrightarrow{\gamma} = \vec{\gamma}(s)$ is a space-like curve,
ii) If $1 + v^2\varphi_1^2 - x_3^2(v')^2 < 0$, then $\overrightarrow{\gamma} = \vec{\gamma}(s)$ is a time-like curve,
iii) If $1 + v^2\varphi_1^2 - x_3^2(v')^2 = 0$, then $\overrightarrow{\gamma} = \vec{\gamma}(s)$ is a null curve.

Example 1. Let $\overrightarrow{\alpha}(s) = \left(\frac{2\sqrt{3}}{3}s,\ \cosh(\frac{\sqrt{3}}{3}s),\ \sinh(\frac{\sqrt{3}}{3}s)\right)$be a space-like curve such that $\kappa = 1/3$ and $\tau = 2\ /\ 3$. The short calculations give $\vec{e}_1 = \left(\frac{2\sqrt{3}}{3},\ \frac{\sqrt{3}}{3}\sinh(\frac{\sqrt{3}}{3}s), \frac{\sqrt{3}}{3}\cosh(\frac{\sqrt{3}}{3}s)\ \right),\ \vec{e}_2 = \left(0,\ \cosh(\frac{\sqrt{3}}{3}s),\ \sinh(\frac{\sqrt{3}}{3}s)\right)$ and $\vec{e}_3 = \left(-\frac{\sqrt{3}}{3},\ -\frac{2\sqrt{3}}{3}\sinh(\frac{\sqrt{3}}{3}s), -\frac{2\sqrt{3}}{3}\cosh(\frac{\sqrt{3}}{3}s)\ \right)$.

Let (0, 0, 1) be expression of unit time-like vector $\overrightarrow{X} = \overrightarrow{X}(s)$ with respect to the orthonormal frame $\{\vec{e}_1\,,\,\vec{e}_2\,,\,\vec{e}_3\}$. Therefore, we have a ruled surface given by the parametric equation

$$\begin{aligned}\varphi_1(s,\,v) \quad &= \quad (\frac{2\sqrt{3}}{3}s - v\frac{\sqrt{3}}{3},\ \ \cosh(\frac{\sqrt{3}}{3}s) - v\frac{2\sqrt{3}}{3}\sinh(\frac{\sqrt{3}}{3}s),\\ &\qquad \sinh(\frac{\sqrt{3}}{3}s) - v\frac{2\sqrt{3}}{3}\cosh(\frac{\sqrt{3}}{3}s))\end{aligned}$$

It is clear that φ is a time-like ruled surface with time-like rulings. From (16), it follows that the equation of a geodesic curve on the surface φ is

$$\begin{aligned}\gamma(s) &= \Big(\frac{2\sqrt{3}}{3}s - \frac{\sqrt{3}}{2}\sinh(\frac{2}{3}s),\ \ \cosh(\frac{\sqrt{3}}{3}s) - \sqrt{3}\sinh(\frac{2}{3}s)\sinh(\frac{\sqrt{3}}{3}s),\\ &\quad \sinh(\frac{\sqrt{3}}{3}s) - \sqrt{3}\sinh(\frac{2}{3}s)\cosh(\frac{\sqrt{3}}{3}s)\Big)\end{aligned}$$

Because of Corollary 1, $\overrightarrow{\gamma} = \overrightarrow{\gamma}(s)$ is a null curve (Fig. 1).

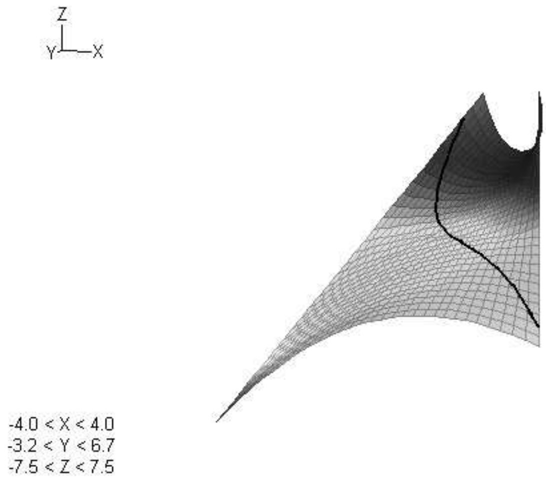

Fig. 1. Ruled surface φ_1 and its geodesic curve

Example 2.

$$\begin{aligned}\varphi_2(s,\,v) &= \Big(-\cosh(\frac{2\sqrt{5}}{5}s) + v\frac{3\sqrt{5}}{5}\sinh(\frac{2\sqrt{5}}{5}s),\ \ \frac{3\sqrt{5}}{5}s - v\frac{2\sqrt{5}}{5},\\ &- \ \sinh(\frac{2\sqrt{5}}{5}s) + v\frac{3\sqrt{5}}{5}\cosh(\frac{2\sqrt{5}}{5}s)\Big)\end{aligned}$$

is a time-like ruled surface with time-like rulings. A geodesic curve on the surface φ_2 is

$$\begin{aligned}\gamma(s) &= \Big(-\cosh(\frac{2\sqrt{5}}{5}s) + \frac{\sqrt{5}}{2}\sinh(\frac{6}{5}s)\sinh(\frac{2\sqrt{5}}{5}s),\ \ \frac{3\sqrt{5}}{5}s - \frac{\sqrt{5}}{3}\sinh(\frac{6}{5}s),\\ &- \ \sinh(\frac{2\sqrt{5}}{5}s) + \frac{\sqrt{5}}{2}\sinh(\frac{6}{5}s)\cosh(\frac{2\sqrt{5}}{5}s)\Big)\end{aligned}$$

Because of Corollary 1, $\overrightarrow{\gamma} = \overrightarrow{\gamma}(s)$ is a null curve (Fig. 2).

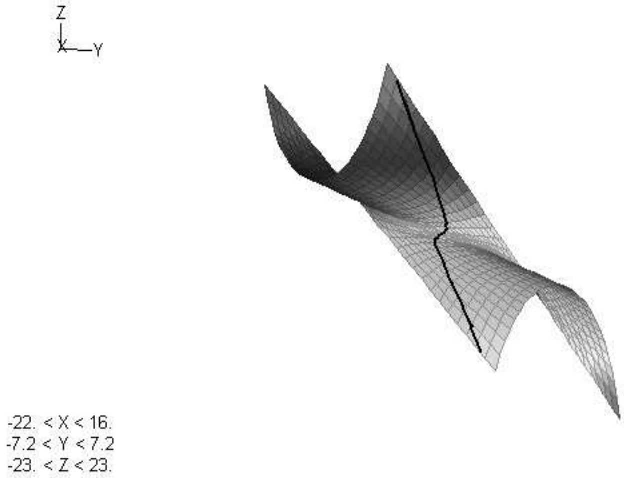

Fig. 2. Ruled surface φ_2 and its geodesic curve

References

[1] Akutagawa, K., Nishikawa, S., The Gauss Map and Space-like Surfaces with Prescribed Mean Curvature in Minkowski 3-Space. Tohoku Math. J. 42 (1990), 67-82.

[2] Beem, J. K., Ehrlich, P. E.,Global Lorentzian Geometry. New York: Marcel Dekker. Inc. 1981.

[3] O'Neill, B., Semi Riemannian Geometry. New York, London: Accedemic Press 1983.

[4] Gradshteyn, I. S., Ryzhık, I. M., Table of Integrals, Series and Products. New York: Academic Press 1980

[5] Turgut, A., Hacisalihoğlu, H. H., Time-like Ruled Surfaces in the Minkowski 3-Space. Far East J. Math. Sci. Vol. 5 No. 1 (1997), 83-90.

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