Novi Sad J. Math.
Vol. 35, No. 2, 2005, 123-132

NEW FORMULAE FOR $K_i(z)$ FUNCTION

Aleksandar Petojević[1]

Abstract. In this paper we study the function

$$K_i(z) = \frac{1}{(i-1)!}\int_0^{\infty} e^{-x}x^{i-1}\frac{x^z-1}{x-1}dx \qquad (\mathrm{Re}\,(z) > 0,\ i \in \mathbb{N})$$

defined in [13]. We give the generating functions, some representation and the congruences of the function $K_i(z)$. Also, we present some inequalities for the function $K_i(x)$ for positive values of x.

AMS Mathematics Subject Classification (2000): 11B34, 11B50, 11B37.

Key words and phrases: Kurepa's left factorial, $K_i(z)$ function, representation, recurrence relation, generating function, inequalities

1. Introduction

Let the Pochhammer symbol $(z)_n$ be defined by

$$(z)_0 = 1, \quad (z)_n = z(z+1)...(z+n-1) = \frac{\Gamma(z+n)}{\Gamma(z)},$$

where $\Gamma(z)$ is the gamma function

$$\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}\,dt\,, \quad (\mathrm{Re}\,(z) > 0).$$

Recently, [13], we defined the generalization of Kurepa's tree as follows:

Definition 1.1 *Let $n \in \mathbb{N}$ and $i \in \mathbb{N}_0$. Then $TK_i(n)$ denote a finite tree consisting of n levels with the k-th level containing $(i)_k$ nodes, $k = 0, 1, 2 \ldots n-1$.*

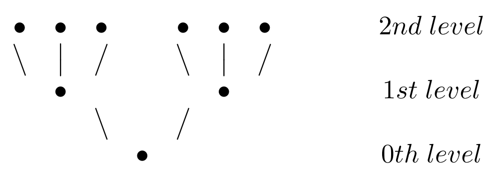

Figure 1: The tree $TK_2(3)$.

[1] University of Novi Sad, Teacher Training Faculty, Podgorička 4, 25000 Sombor, SERBIA and MONTENEGRO, E-mail: apetoje@ptt.yu

Let $K_i(n)$ denote the total number of nodes in the tree $TK_i(n)$. For the numbers $K_i(n)$, the following relations hold:

$$\begin{aligned} K_0(n) &\stackrel{\text{def}}{=} 1\,, \qquad K_1(n) = !n\,, \\ K_i(n) &= \sum_{k=0}^{n-1}(i)_k = \frac{1}{(i-1)!}\sum_{k=i-1}^{n+i-2}k! = i\cdot K_{i+1}(n-1)+1\,, \\ K_i(-n) &= -\frac{(i-n-1)!}{(i-1)!}K_{i-n}(n), \qquad (i>n\in\mathbb{N})\,, \\ K_i(n) &= (-1)^i e^{-1}\left[\Gamma(1-i,-1)-(-1)^n\Gamma(1-i-n,-1)\frac{(i+n-1)!}{(i-1)!}\right] \end{aligned}$$

Here $\Gamma(z,x)$ is the incomplete gamma function defined via

$$\Gamma(z,x) = \int_x^{+\infty} t^{z-1}e^{-t}\,dt\,,$$

and $!n$ is Kurepa's left factorial (see [7])

$$!0 = 0\,, \qquad !n = \sum_{k=0}^{n-1}k! \quad (n\in\mathbb{N})\,.$$

The functions $\{K_i(n)\}_{i=1}^{\infty}$ are periodical functions. In this way we have the following statements:

$$\begin{aligned} K_i(n) &\equiv K_{i+jn}(n)\ \ (mod\ n); \qquad K_{jn-1}(n)\equiv 0\ \ (mod\ n\cdot j); \\ K_i(n) &\equiv 0\ \ (mod\ i+1), \quad (i\in\mathbb{N}_0,\ n\in\mathbb{N}\backslash\{1\})\,. \end{aligned}$$

For every complex number $\mathrm{Re}\,(z)>0$ and $i\in\mathbb{N}$ the function $K_i(z)$ is defined by

$$K_i(z) \stackrel{\text{def}}{=} \frac{1}{(i-1)!}\int_0^{\infty} e^{-x}x^{i-1}\frac{x^z-1}{x-1}\,dx. \tag{1.1}$$

This function can be extended analytically to the whole complex plane by

$$K_i(z) = K_i(z+1) - \frac{\Gamma(z+i)}{(i-1)!}\,, \tag{1.2}$$

and for $i\in\mathbb{N},\ x\in\mathbb{R}$ satisfy the asymptotic relations

$$\lim_{x\to\infty}\frac{K_i(x)}{\Gamma(x+i-1)} = \frac{1}{(i-1)!}\,, \qquad \lim_{x\to\infty}\frac{K_i(x)}{\Gamma(x+i)} = 0\,.$$

For the function $K_i(z)$ the set of poles is $P_{K_i} = \{ -i,\ -i-2,\ -i-3,\ -i-4,\ \dots\}$. The infinite point is an essential singularity and every pole $z_p \in P_{K_i}$ is simple with the residue

$$res\ K_i(z_p) = \frac{1}{(i-1)!} \sum_{k=i}^{-z_p} \frac{(-1)^{k-i+1}}{(k-i)!}, \qquad (z_p \in P_{K_i}) .$$

Finally, the functional equality

$$K_i(z+1) = (z+i)K_i(z) - (z+i-1)K_i(z-1) , \qquad (i \in \mathbb{N}_0) . \tag{1.3}$$

is valid.

2. The generating functions

For the sequence $\{c_n\}_{n=0}^{\infty}$ the generating function, the exponential generating function and the Direchlet series generating function, denoted respectively by $G(x)$, $g(x)$ and $D(x)$ and are defined as [17, p. 3, p. 21, p. 56]

$$G(x) = \sum_{n=0}^{\infty} c_n x^n , \qquad g(x) = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!} , \qquad D(x) = \sum_{n=1}^{\infty} \frac{c_n}{n^x} .$$

Apart [17], the relevant theory on generating functions can be found in [4] and in [6] Chapter VII.

Remark 2.1 *For a fixed number b, the exponential generating function and the generating function for the Pochhammer symbol $(b)_n$ is given as follows (see [18]):*

$$\sum_{n=0}^{\infty} (b)_n \frac{z^n}{n!} = (1-z)^{-b} , \qquad \sum_{n=0}^{\infty} (b)_n\, x^n \approx - \frac{E_b(-1/x)}{x e^{1/x}} ,$$

where $E_n(x)$ is the exponential integral

$$E_n(x) = \int_1^{\infty} \frac{e^{-xt}}{t^n}\, dt.$$

The second possibility of generating integer sequences by the Pochhammer symbol is that for a fixed $n \in \mathbb{N}$, terms of the sequence are generated by the index $i \in \mathbb{N}_0$, i.e., $\{(i)_n\}_{i=0}^{\infty}$ (see formulae (2.7), (2.8) and (2.9)).

In what follows $\zeta(z)$, $s(n,m)$ and $P_k^n(x)$ are respectively the Riemann zeta function, Stirling number of the first kind and the polynomials defined by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} , \qquad (\mathrm{Re}\,(z) > 1)$$

Table 1: The special cases of $P_k^n(x)$

$P_k^n(x)$	sequences	in [16]
$P_k^1(2)$	0, 1, 4, 12, 32, 80, ...	$A001787$
$P_k^1(3)$	0, 1, 6, 27, 108, 405, ...	$A027471$
$P_k^1(4)$	0, 1, 8, 48, 256, 1280, ...	$A002697$
$P_k^2(1)$	0, 2, 6, 12, 20, 30, ...	$A002378$
$P_k^3(1)$	0, 3, 12, 33, 72, 135, ...	$A054602$
$P_2^n(2)$	0, 4, 10, 18, 28, 40, ...	$A028552$
$P_2^n(3)$	0, 6, 14, 24, 36, 50, ...	$A028557$

$$x(x-1)\cdots(x-n+1)=\sum_{m=0}^{n}s(n,m)x^k\,,$$

$$P_k^n(x)=\sum_{j=0}^{k-1}(n)^{(k-j)}\binom{k}{j}x^j\,,\qquad (n,k\in\mathbb{N}),$$

where $(x)^{(m)}=x(x-1)\cdots(x-m+1)$ is the falling factorial. Several well-known special cases of the polynomials $P_k^n(x)$ are presented in Table 1.

Theorem 2.2 *For a fixed number $n\in\mathbb{N}$ we have*

$$\sum_{i=0}^{+\infty}K_i(n)x^i = 1+\sum_{k=0}^{n-1}k!\,\frac{x}{(1-x)^{k+1}}\qquad(|x|<1) \tag{2.4}$$

$$\sum_{i=0}^{+\infty}K_i(n)\frac{x^i}{i!} = e^x+\sum_{k=1}^{n-1}[e^xx^k]^{(k-1)}\qquad(x\in\mathbb{R}) \tag{2.5}$$

$$\sum_{i=1}^{+\infty}\frac{K_i(n)}{i^x} = \zeta(x)+\sum_{k=1}^{n-1}\sum_{j=1}^{k}(-1)^{j+k}s(k,j)\cdot\zeta(x-j)\,. \tag{2.6}$$

Proof. Firstly, for a fixed number $n\in\mathbb{N}$ the equation

$$\sum_{i=0}^{\infty}(i)_nx^i=n!\,\frac{x}{(1-x)^{n+1}}\qquad(|x|<1) \tag{2.7}$$

is well known.

Secondly, let $[f(x)]^{(k)}$ be the k^{th} derivative of a function $f(x)$ and $g_n(x)=xe^x\left[x^{n-1}+P_{n-1}^n(x)\right]$. By induction on $i\in\mathbb{N}$ we have

$$[g_n(x)]^{(i)} = e^xx^n+e^x\sum_{j=1}^{i}\binom{i}{i-j}x^{n-i}\prod_{m=0}^{j-1}(n-m)+$$

$$+ \quad e^x \sum_{j=0}^{n-2} \binom{n-1}{j} x^{j+1} \prod_{m=0}^{n-2-j} (n-m) +$$

$$+ \quad e^x \sum_{s=0}^{i-1} \binom{i}{s+1} \sum_{j=s}^{n-2} \frac{(j+1)!}{(j-s)!} \binom{n-1}{j} x^{j-s} \prod_{m=0}^{n-2-j} (n-m).$$

Hence

$$\begin{aligned} [g_n(0)]^{(i)} &= \sum_{s=0}^{i-1} \binom{i}{s+1} (s+1)! \binom{n-1}{s} \prod_{m=0}^{n-2-s} (n-m) \\ &= i!\,(n-1)!\,n! \sum_{s=0}^{i-1} \frac{1}{(i-s-1)!\,(s+1)!\,(n-s-1)!\,s!} \\ &= i!\,(n-1)!\,n! \cdot \frac{(n+i-1)!}{i!\,(i-1)!\,n!\,(n-1)!} = \frac{(n+i-1)!}{(i-1)!} = (i)_n\,. \end{aligned}$$

Applying the standard formula for the Taylor series expansion about the point $x = 0$ we arrive at the formula

$$\sum_{i=0}^{\infty} (i)_n \frac{x^i}{i!} = xe^x \left[x^{n-1} + P_{n-1}^n(x) \right] = [e^x x^n]^{(n-1)} \qquad (x \in \mathbb{R})\,. \tag{2.8}$$

Thirdly, using the equation

$$\sum_{i=1}^{\infty} \frac{(i)_{n+1}}{i^x} = n \sum_{i=1}^{\infty} \frac{(i)_n}{i^x} + \sum_{i=1}^{\infty} \frac{(i)_n}{i^{x-1}}$$

and the recurrence relation for Stirling numbers of the first kind (see [18])

$$s(n+1, j) = s(n, j-1) - n \cdot s(n, j)$$

we have

$$\sum_{i=1}^{\infty} \frac{(i)_n}{i^x} = \sum_{j=1}^{n} (-1)^{j+n} s(n, j) \zeta(x - j)\,. \tag{2.9}$$

Finally, the theorem now follows from (2.7), (2.8) and (2.9). □

Remark 2.3 *Equation (2.5) is given in [13]. On the basis of equation (2.4) and the well-known relation [1, p. 88, entry 6.5.19.]*

$$\Gamma(-n, x) = \frac{(-1)^n}{n!} \left[\Gamma(0, x) - e^{-x} \sum_{m=0}^{n-1} (-1)^m \frac{m!}{x^{m+1}} \right] \quad (n \in \mathbb{N})$$

we get representation of generating function of the sequences $\{K_i(n)\}_{i=0}^{+\infty}$ *via an incomplete gamma function:*

$$\sum_{i=0}^{+\infty} K_i(n)x^i = 1 + x \cdot e^{x-1}\Big[(-1)^n n! \cdot \Gamma(-n, x-1) - \Gamma(0, x-1)\Big].$$

3. The representation and some congruences

Let γ be Euler's constant. Then the following statement is true.

Theorem 3.1 *For* $z \in \mathbb{C}$ *we have*

$$K_i(z) = \frac{1}{(i-1)!} \cdot \left[-!(i-1) - \frac{\pi}{e}\cot \pi z + \frac{1}{e}\left(\sum_{n=1}^{+\infty}\frac{1}{n!n} + \gamma\right) + \sum_{n=0}^{+\infty}\Gamma(z+i-n-1)\right].$$

Proof. For $\mathrm{Re}\,(z) > 1$ and $i \in \mathbb{N}$, according to definition (1.1) we have

$$iK_{i+1}(z-1) + 1 = 1 + \frac{1}{(i-1)!}\int_0^{\infty} e^{-x}x^i \frac{x^{z-1}-1}{x-1} = K_i(z).$$

Consequently, using the relation (1.2) we have

$$K_i(z) = i \cdot K_{i+1}(z-1) + 1 \qquad (z \in \mathbb{C},\, i \in \mathbb{N}). \tag{3.10}$$

For $i = 1$ theorem is true (see [15, p. 472]). These formulas were mentioned also in the book [10]. By means of the relation (3.10) and induction on $i \in \mathbb{N}$ the result of the theorem is obtained. □

Lemma 3.2 *For* $n \in \mathbb{N}$ *we have*

$$\sum_{i=1}^{n-1} K_i(n) \equiv \sum_{i=1}^{n-1} \frac{!n - !(i-1)}{(i-1)!} \pmod{n}.$$

Proof. The relations (1.2) and

$$\int_0^{\infty} e^{-x}x^{i-2} \cdot (x^z - 1)\, dx = \Gamma(z+i-1) - \Gamma(i-1), \qquad (\mathrm{Re}\,(i) > 1,\ \mathrm{Re}\,(z) > 0),$$

yields

$$K_i(z) = \frac{1}{i-1}K_{i-1}(z) + \frac{\Gamma(z+i-1)}{(i-1)!} - \frac{1}{i-1} \qquad (1 < i \in \mathbb{N},\ z \in \mathbb{C}). \tag{3.11}$$

Hence

$$\begin{aligned} K_{n-m}(n) \;&\equiv\; \frac{1}{(n-m-1)!}K_1(n) \\ &- \sum_{k=m+1}^{n-1}\prod_{j=m+1}^{k}\frac{1}{n-j} \pmod{n}, \quad (0 \le m \le n-2) \end{aligned}$$

i.e.,

$$K_i(n) \equiv \frac{!n - !(i-1)}{(i-1)!} \pmod{n}, \qquad (1 \le i \le n).$$

□

Remark 3.3 *Applying relations (1.3) and (3.10), for* $2 < i \in \mathbb{N}$ *and* $z \in \mathbb{C}$, *we have*

$$K_i(z) = \frac{z+i-1}{i-1} K_{i-1}(z) - \frac{z+i-2}{(i-2)(i-1)} K_{i-2}(z) + \frac{z}{(i-2)(i-1)}. \tag{3.12}$$

Six integer sequences in [16] are special cases of the function $K_i(n) : K_0(n) = K_i(1)$, $K_1(n)$, $K_2(n)$, $K_i(2)$, $K_i(3)$ *and* $K_i(4)$. *The sequence* $\{K_i(5)\}_{i=0}^{+\infty}$

$$1,\ 34,\ 153,\ 436,\ 985,\ 1926,\ \ldots$$

cannot currently be found in [16]. Using relation (3.12) the formula for $K_i(5)$ *numbers is given as follows* $K_i(5) = i^4 + 7i^3 + 15i^2 + 10i + 1$.

Remark 3.4 *The total number of arrangements of a set with n elements (see [2], [5], [14] and [12]), the derangement numbers (sequence A000166 in [16]) and the harmonic number, denoted respectively by* a_n, S_n *and* H_n, *and are defined as*

$$a_n = n! \sum_{k=0}^{n} \frac{1}{k!}; \qquad S_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \quad (n \ge 0); \qquad H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

The following congruences are easy to find after some simple calculation:

$$\sum_{i=1}^{n-1} K_i(n) \equiv \sum_{i=0}^{n-3} a_i \pmod{n} \qquad (2 < n \in \mathbb{N}) \tag{3.13}$$

$$\equiv \sum_{k=1}^{n-1} (-1)^k S_k \pmod{n} \tag{3.14}$$

$$\equiv\ !n - H_{n-2} + \sum_{i=1}^{n-2} \frac{K_i(n)}{i} \pmod{n}. \tag{3.15}$$

Question 3.5 *For* $k \in \mathbb{N}_0 \backslash \{1\}$ *is it correct that*

$$\sum_{i=0}^{n-1} K_i(n) \equiv 0 \pmod{n} \Leftrightarrow n = 2^k\ ?$$

Question 3.6 *For all a prime number p is it correct that*

$$\sum_{i=0}^{p} K_i(p) \equiv 0 \pmod{p}\ ?$$

4. Some inequalities for $K_i(x)$ function

For positive values of x, based on the functional equation (1.2) and the inequality [8, p. 299, (4.4)]

$$K_1(x) \leq 1 + 2\Gamma(x)$$

the following inequality is true:

$$K_1(x-1) \leq 1 + \Gamma(x)\,.$$

Analogously, on the basis of the functional equation (1.2) and inequalities [9, p. 3, (4.3)]

$$K_1(x) \leq \frac{9}{5}x\,, \qquad (x \in [0,1]) \tag{4.16}$$

and [9, p. 4, (4.8)]

$$K_1(x) \leq 2\Gamma(x)$$

the main result in [9, p. 4, Theorem 4.4)] is true:

$$K_1(x-1) \leq \Gamma(x)\,, \qquad (x \geq 3) \tag{4.17}$$

while the equality is true for $x = 3$.

Here we give elementary proof of the inequality which is an improvement of the inequality (4.17) as follows.

Lemma 4.1 *For $x \geq 1$ we have*

$$K_1(x-1) \leq \frac{9}{5} + \frac{!([x]-1)}{([x]-1)!} \cdot \Gamma(x)\,, \tag{4.18}$$

where $[x]$ denotes the integer part of x.

Proof. Let $n \in \mathbb{N}$. For $x = n$ the lemma is true. Let $x \in (n, n+1)$. Then the functional equation (1.2) yields

$$\begin{aligned} K_1(x-1) &= K_1(x-2) + \Gamma(x-1) = K_1(x-3) + \Gamma(x-2) + \Gamma(x-1) \\ &\;\vdots \\ &= K_1(x-s) + \Gamma(x-s+1) + \Gamma(x-s+2) + \cdots + \Gamma(x-1) \end{aligned}$$

where $x \geq s \in \mathbb{N}$. Hence, for $s = n$ we have

$$K_1(x-1) = K_1(x-n) + \sum_{k=1}^{n-1} \Gamma(x-k)\,. \tag{4.19}$$

Also, the functional equation $\Gamma(x+1) = x\Gamma(x)$ yields

$$\Gamma(x-k) = \Gamma(x) \cdot \prod_{t=1}^{k} \frac{1}{x-t}\,, \qquad (k = 1, 2, \ldots, n-1)\,.$$

Hence, using relation (4.19) we have

$$\begin{aligned} K_1(x-1) &= K_1(x-n)+\Gamma(x)\cdot\sum_{k=1}^{n-1}\prod_{t=1}^{k}\frac{1}{x-t} \\ &\le K_1(x-n)+\Gamma(x)\cdot\sum_{k=1}^{n-1}\prod_{t=1}^{k}\frac{1}{n-t} \\ &= K_1(x-n)+\Gamma(x)\cdot\frac{!(n-1)}{(n-1)!}\,. \end{aligned}$$

Since $x-n\in(0,1)$ the result now follows from (4.16). □

Corrollary 4.2 *For $5\le x\in\mathbb{R}$ we have*

$$K_1(x-1)\le\frac{9}{5}+\frac{\Gamma(x)}{2}\,. \tag{4.20}$$

Proof. For $4\le n\in\mathbb{N}$ induction on n we have $\dfrac{!n}{n!}\le\dfrac{1}{2}$.

Question 4.3 *For $0\le y\le x$ is it correct that*

$$K_1(y)\le K_1(x)\ ?$$

Finally, according to relation (3.11) we give a generalization of Lemma 4.1 as follows.

Theorem 4.4 *For $x\ge 1$ and $i\in\mathbb{N}$ we have*

$$(i-1)!\cdot K_i(x-1)\le\frac{9}{5}+\frac{!([x]-1)}{([x]-1)!}\cdot\Gamma(x)-\,!(i-1)+\sum_{k=0}^{i-2}\Gamma(x+k)\,. \tag{4.21}$$

Corrollary 4.5 *For $5\le x\in\mathbb{R}$ and $i\in\mathbb{N}$ we have*

$$(i-1)!\cdot K_i(x-1)\le\frac{9}{5}+\frac{\Gamma(x)}{2}-\,!(i-1)+\sum_{k=0}^{i-2}\Gamma(x+k)\,. \tag{4.22}$$

Acknowledgement

This work was supported in part by the Ministry of Science and Environmental Protection of the Republic of Serbia under Grant # 2002: *Applied Orthogonal Systems, Constructive Approximation and Numerical Methods.*

References

[1] Abramowitz, M., Stegun, I. A., Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. 9th printing., New York: Dover 1972.

[2] Cameron, P. J., Sequences realized by oligomorphic permutation groups. Journal of Integer Sequences, Vol. 3 (2000), Article 00.1.5.

[3] Carlitz, L., A note on the left factorial function. Math. Balkanica 5 (1975), 37-42.

[4] Comtet, L., Advanced Combinatorics; The art of finite and infinite expansions. Boston: D. Reidel Publ. Co. 1974.

[5] Gandhi, J. M., On logarithmic numbers. Math. Student 31 (1963), 73–83.

[6] Graham, R., Knuth D., Patashnik, I. O., Concrete mathematics. Reading, MA: Addison-Wesley 1989.

[7] Kurepa, Đ., On the left factorial function $!n$. Math. Balkanica 1 (1971), 147-153.

[8] Kurepa, Đ., Left factorial function in complex domain. Math. Balkanica 3 (1973), 297-307.

[9] Malešević, B. J., Some inequalities of the Kurepa's function. J. Inequal. Pure and Appl. Math. 5(4) Art. 84 (2004), p.5.

[10] Marichev, O. I., Handbook of Integral Transformation of Higher Transcendental Functions: Theory and Algorithmic Tables. Chichester: Ellis Horwood Ltd. 1983.

[11] Milovanović, G. V., Petojević, A., Generalized factorial function, numbers and polynomials and related problems. Math. Balkanica, New Series Vol. 16, Fasc 1-4 (2002), 113-130.

[12] Petojević, A., The function ${}_vM_m(s; a, z)$ and some well-known sequences. Journal of Integer Sequences Vol. 5 (2002), Article 02.1.7, p.16.

[13] Petojević, A., The $\{K_i(z)\}_{i=1}^{\infty}$ functions. Rocky Mountain J. Math. (to appear, RMJM#2671)

[14] Singh, D., The numbers *L(m,n)* and their relations with prepared Bernoulli and Eulerian numbers. Math. Student 20 (1952), 66-70.

[15] Slavić, D. V., On the left factorial function of the complex argument. Math. Balkanica 3 (1973), 472-477.

[16] Sloane, N. J. A., On-Line Encyclopedia of Integer Sequences. published elec. at `http://www.research.att.com/~njas/sequences/`

[17] Wilf, H. S., Generating functionology. New York: Academic Press 1990.

[18] Wolfram Research, Mathematica. published elec. at `http://functions.wolfram.com`

Received by the editors June 13, 2005