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THE HOLOMORPHIC BISECTIONAL CURVATURE OF THE COMPLEX FINSLER SPACES

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Abstract. The notion of holomorphic bisectional curvature for a complex Finsler space (M, F) is defined with respect to the Chern complex linear connection on the pull-back tangent bundle. By means of holomorphic curvature and holomorphic flag curvature of a complex Finsler space, a special approach is emloyed to obtain the characterizations of the holomorphic bisectional curvature. For the class of generalized Einstein complex Finsler spaces some results concerning the holomorphic bisectional curvature are also given.

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1. Introduction

In complex Finsler geometry, it is systematically used the concept of holomorphic curvature in direction η, briefly holomorphic curvature, [1]. In a previous paper, [3], we started to study the holomorphic curvature of complex Finsler spaces with respect to the Chern complex linear connection, briefly Chern $(c.l.c)$, on the holomorphic pull-back tangent bundle $\pi^*(T'M)$. Our goal was to determine the conditions in which a complex Finsler metric has constant holomorphic curvature. We solved this problem for a special class of complex Finsler spaces, called by us generalized Einstein, briefly $(g.E.)$. In another paper [4], we gave a generalization of the holomorphic curvature of the complex Finsler spaces which we called holomorphic flag curvature. But, the holomorphic flag curvature is not the corespondent of the holomorphic bisectional curvature from Hermitian geometry in complex Finsler geometry.

Our objective is to give a characterization of the holomorphic bisectional curvature of a complex Finsler space. The second section of the present paper is devoted to the notion of holomorphic bisectional curvature for such a space. We determine the link between the holomorphic bisectional curvature and two kinds of curvature: holomorphic curvature and holomorphic flag curvature (Proposition 2.3, 2.5). We prove a necessarily and sufficient condition that a complex Finsler space has constant holomorphic bisectional curvature, (Proposition 2.6). In the last section, a special approach is employed to del with the holomorphic bisectional curvature of the $(g.E.)$ complex Finsler spaces.

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We establish some inequalities between the three kinds of curvature (Proposition 3.2, Corollary 3.3). Moreover, we show that the holomorphic bisectional curvature of the Kobayashi metric is ≤ -2.

This section is concerned with recalling the basic notions which are needed; for more information see [1, 8, 3, 4].

Let M be a complex manifold, $\dim_C M = n$, and $T'M$ the holomorphic tangent bundle in which as a complex manifold the local coordinates will be denoted by (z^k, η^k). The complexified tangent bundle of $T'M$ is decomposed in $T_C(T'M) = T'(T'M) \oplus T''(T'M)$.

Considering the restriction of the projection to $\widetilde{T'M} = T'M \setminus \{0\}$, for pulling the holomorphic tangent bundle $T'M$ back, we obtain a holomorphic tangent bundle $\pi' : \pi^*(T'M) \longrightarrow \widetilde{T'M}$, called *the pull-back tangent bundle* over the slit $\widetilde{T'M}$. We denote by $\left\{ \frac{\partial}{\partial z^k}^*, \frac{\partial}{\partial \bar{z}^k}^* \right\}$, and by $\left\{ dz^{*k}, d\bar{z}^{*k} \right\}$, the local frame and its dual.

Let $V(T'M) = \ker \pi_* \subset T'(T'M)$ be the vertical bundle, locally spanned by $\{\frac{\partial}{\partial \eta^k}\}$. A complex nonlinear connection, briefly (*c.n.c.*), determines a supplementary complex subbundle to $V(T'M)$ in $T'(T'M)$, i.e. $T'(T'M) = H(T'M) \oplus V(T'M)$. The adapted frames of the (*c.n.c.*) is $\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}$, where $N_k^j(z, \eta)$ are the coefficients of the (*c.n.c.*). Further on, we shall use the abbreviations $\delta_i = \frac{\delta}{\delta z^i}$, $\dot{\partial}_i = \frac{\partial}{\partial \eta^i}$, $\delta_{\bar{i}} = \frac{\delta}{\delta \bar{z}^i}$, $\dot{\partial}_{\bar{i}} = \frac{\partial}{\partial \bar{\eta}^i}$, and theirs conjugates ([1], [2], [8]).

On $T'M$ let $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$ be the fundamental metric tensor of a complex Finsler space $(M, F^2 = L)$. The isomorphism between $\pi^*(T'M)$ and $T'M$ induces an isomorphism of $\pi^*(T_C M)$ and $T_C M$. Thus, $g_{i\bar{j}}$ defines an Hermitian metric structure $\mathcal{G}(z, \eta) := g_{j\bar{k}} dz^{*j} \otimes d\bar{z}^{*k}$on $\pi^*(T_C M)$, with respect to the natural complex structure. Further, the Hermitian metric structure $\mathcal{G}$ on $\pi^*(T'M)$ induces a Hermitian inner product $h(\chi, \gamma) := Re\mathcal{G}(\chi, \overline{\gamma})$ and the angle $\cos(\chi\gamma) = \frac{Re\mathcal{G}(\chi, \overline{\gamma})}{||\chi|| ||\overline{\gamma}||}$, for any χ, γ the sections on $\pi^*(T'M)$, where $||\chi||^2 = ||\overline{\chi}||^2 = \mathcal{G}(\chi, \overline{\chi})$, see [3].

On the other hand, $H(T'M)$ and $\pi^*(T'M)$ are isomorphic. Therefore the structures on $\pi^*(T_C M)$ can be pulled-back to $H(T'M) \oplus \overline{H(T'M)}$. By this isomorphism the natural cobasis dz^{*j} is identified with dz^j. In view of this construction the pull-back tangent bundle $\pi^*(T'M)$ admits a unique complex linear connection ∇, called the Chern (*c.l.c.*), which is metric with respect to $\mathcal{G}$ and of $(1,0)-$ type, [3]:

$$\begin{aligned} \omega_j^i(z, \eta) &= L_{jk}^i(z, \eta) dz^k + C_{jk}^i(z, \eta) \delta \eta^k; \\ L_{jk}^i &= g^{\overline{m}i} \frac{\delta g_{j\overline{m}}}{\delta z^k} \ ; \ C_{jk}^i = g^{\overline{m}i} \frac{\partial g_{j\overline{m}}}{\partial \eta^k}. \end{aligned} \tag{1.1}$$

The Chern (*c.l.c.*) on $\pi^*(T'M)$ determines the Chern-Finsler (*c.n.c.*) on $T'M$, with the coefficients $\overset{CF}{N_k^i} = g^{\overline{m}i} \frac{\partial g_{j\overline{m}}}{\partial z^k} \eta^j$, and its local coefficients of torsion and curvature are

$$(1.2)\qquad \begin{aligned} T^i_{jk} &:= L^i_{jk} - L^i_{kj}\ ; \\ R^i_{j\bar{h}k} &:= -\delta_{\bar{h}} L^i_{jk} - \delta_{\bar{h}}(\overset{CF}{N^l_k}) C^i_{jl}\ ;\ \Xi^i_{j\bar{h}k} := -\delta_{\bar{h}} C^i_{jk} = \Xi^i_{k\bar{h}j}; \\ P^i_{j\bar{h}k} &:= -\dot{\partial}_{\bar{h}} L^i_{jk} - \dot{\partial}_{\bar{h}}(\overset{CF}{N^l_k}) C^i_{jl}\ ;\ S^i_{j\bar{h}k} := -\dot{\partial}_{\bar{h}} C^i_{jk} = S^i_{k\bar{h}j}. \end{aligned}$$

The Riemann type tensor

$$(1.3)\qquad \mathbf{R}(W,\overline{Z},X,\overline{Y}) := \mathcal{G}(R(X,\overline{Y})W,\overline{Z})$$

has the properties:

$$(1.4)\qquad \begin{aligned} \mathbf{R}(W,\overline{Z},X,\overline{Y}) &= W^i\overline{Z}^j X^k \overline{Y}^h R_{i\bar{j}k\bar{h}};\ \ R_{\bar{j}i\bar{h}k} := R^l_{i\bar{h}k} g_{l\bar{j}}; \\ R_{i\bar{j}k\bar{h}} &= -R_{i\bar{j}\bar{h}k} = \overline{R_{j\bar{i}h\bar{k}}} = R_{\bar{j}i\bar{h}k}; \\ \text{If } R^i_{j\bar{h}k} &= R^i_{k\bar{h}j} \text{ then } R_{i\bar{j}k\bar{h}} = R_{k\bar{j}i\bar{h}} = R_{k\bar{h}i\bar{j}}. \end{aligned}$$

According to [1] the complex Finsler space (M,F) is *strongly Kähler* iff $T^i_{jk}=0$, *Kähler* iff $T^i_{jk}\eta^j = 0$ and *weakly Kähler* iff $g_{i\bar{l}}T^i_{jk}\eta^j\overline{\eta}^l = 0$. Note that for a complex Finsler metric which comes from a Hermitian metric on M, so-called *purely Hermitian metric* in [8], i.e. $g_{i\bar{j}} = g_{i\bar{j}}(z)$, the three kinds of Kähler spaces coincide, [11].

We consider $z \in M$ and $\eta \in T'_zM$, $\eta \neq 0$. A flag is given by the tangent vector field η, called flagpole, and another transversal vector field χ. The holomorphic flag curvature of F along of the flag (η,χ), with respect to the Chern (*c.l.c.*), is ([4])

$$(1.5)\qquad \mathcal{K}_F(z,\eta,\chi) := \frac{\mathbf{R}(\eta,\overline{\chi},\eta,\overline{\chi}) + \mathbf{R}(\chi,\overline{\eta},\chi,\overline{\eta})}{\mathcal{G}(\eta,\overline{\eta})\mathcal{G}(\chi,\overline{\chi})},$$

where η and χ are local section of $\pi^*(T'M)$. In particular, if η is colinear with χ then we obtain the holomorphic curvature from [1]

$$(1.6)\qquad \mathcal{K}_F(z,\eta) := \frac{2\mathbf{R}(\eta,\overline{\eta},\eta,\overline{\eta})}{\mathcal{G}^2(\eta,\overline{\eta})} = \frac{2\overline{\eta}^j\eta^k R_{\bar{j}k}}{L^2(z,\eta)}.$$

From [3], we have

Definition 1.1. *The complex Finsler space (M,F) is called generalized Einstein if $R_{\bar{j}k}$ is proportional to $t_{k\bar{j}}$, i.e. if there exists a real valuated function $K(z,\eta)$, such that*

$$(1.7)\qquad R_{\bar{j}k} = K(z,\eta)t_{k\bar{j}},$$

where $R_{\bar{j}k} := R_{i\bar{j}k\bar{h}}\eta^i\overline{\eta}^h = -g_{l\bar{j}}\delta_{\bar{h}}(\overset{CF}{N^l_k})\overline{\eta}^h$, $t_{k\bar{j}} := L(z,\eta)g_{k\bar{j}} + \eta_k\overline{\eta}_j$, $\eta_k := \frac{\partial L}{\partial \eta^k}$, $\bar{\eta}_j := \frac{\partial L}{\partial \bar{\eta}^j}$.

A $(g.E.)$ complex Finsler space enjoys some of interesting properties which we collect in:

Theorem 1.1. *Let (M, F) be a $(g.E.)$ complex Finsler space. Then*

i) $K(z,\eta) = \frac{1}{4}\mathcal{K}_F(z,\eta)$ and it depends on z alone.

ii) If (M, F) is connected and weakly Kähler, of complex dimension ≥ 2, then it is a space with constant holomorphic curvature.

iii) If the space is of nonzero constant holomorphic curvature, then F is weakly Kähler.

iv) If the space is Kähler of nonzero constant holomorphic curvature, then F is purely Hermitian. Conversely, a purely Hermitian complex Finsler space, which is Kähler of constant holomorphic curvature, is $(g.E.)$.

Note that for the particular case of the complex Finsler spaces which are Kähler of nonzero constant holomorphic curvature, the notions of $(g.E.)$ and purely Hermitian spaces coincide.

Finally, we recall here that in [3] it is proved

Proposition 1.1. *Let (M, F) be a $(g.E.)$ complex Finsler space. Then*

$$\mathcal{K}_F(z,\eta,\chi) = \frac{\mathcal{K}_F(z)}{L(z,\chi)}\left\{Re\left(C_{\bar{j}\bar{h}}\overline{\chi}^j\overline{\chi}^h\right) + \frac{Re\left[\left(\overline{\eta}_j\overline{\chi}^j\right)^2\right]}{L(z,\eta)}\right\}, \tag{1.8}$$

where $\mathcal{K}_F(z)$ is the holomorphic curvature of (M, F) and $C_{\bar{j}\bar{h}} := C_{i\bar{j}\bar{h}}\eta^i$.

Moreover, for a complex Finsler space (M, F), $(g.E.)$ of nonzero holomorphic curvature we have

$$\frac{Re\left(C_{\bar{j}\bar{h}}\overline{\chi}^j\overline{\chi}^h\right)}{L(z,\chi)} + \cos^2\varphi \geq \frac{\mathcal{K}_F(z,\eta,\chi)}{\mathcal{K}_F(z)}, \tag{1.9}$$

where φ is the angle between directions of η and χ.

2. Holomorphic bisectional curvature

We consider $z \in M$, $\eta \in T'_zM$, $\eta \neq 0$ and χ another direction in T'_zM, $\eta \neq \chi$. η and χ are viewed as local sections of $\pi^*(T'M)$, *i.e.* $\eta := \eta^i \frac{\partial}{\partial z^i}^*$ and $\chi := \chi^j \frac{\partial}{\partial z^j}^*$, with $\chi^j = \chi^j(z,\eta)$.

Definition 2.1. *The holomorphic bisectional curvature of the complex Finsler metric F in directions η and χ is given by*

$$\mathcal{B}_F(z,\eta,\chi) := \frac{\mathbf{R}(\eta,\bar{\eta},\chi,\bar{\chi}) + \mathbf{R}(\chi,\bar{\chi},\eta,\bar{\eta})}{\mathcal{G}(\eta,\overline{\eta})\mathcal{G}(\chi,\overline{\chi})}, \tag{2.1}$$

where $\mathcal{G}(\chi,\overline{\chi}) \neq 0$.

Further on, we shall simply call it *holomorphic bisectional curvature*. The holomorphic bisectional curvature $\mathcal{B}_F(z,\eta,\chi)$ depends both on the position $z \in M$ and the two directions η and χ.

Proposition 2.1. *i)* $\mathcal{B}_F(z,\eta,\chi) = \mathcal{B}_F(z,\chi,\eta)$;
ii) $\mathcal{B}_F(z,\eta,\eta) = \mathcal{K}_F(z,\eta)$;
iii) $\mathcal{B}_F(z,\eta,\chi)$ *is real valued;*
iv) $\mathcal{B}_F(z,\frac{\eta}{F},\chi) = \mathcal{B}_F(z,\eta,\chi)$;
v) $\mathcal{B}_F(z,\alpha\eta,\beta\chi) = \mathcal{B}_F(z,\eta,\chi)$, *for any* $\alpha,\beta \in \mathbb{C}$.

In particular, if $\mathbf{R}$ is symmetric, i.e. $\mathbf{R}(\eta,\overline{\eta},\chi,\overline{\chi}) = \mathbf{R}(\eta,\overline{\chi},\chi,\overline{\eta}) = \mathbf{R}(\chi,\bar{\chi},\eta,\bar{\eta})$ then

$$\mathcal{B}_F(z,\eta,\chi) := \frac{2\mathbf{R}(\eta,\bar{\eta},\chi,\bar{\chi})}{\mathcal{G}(\eta,\overline{\eta})\mathcal{G}(\chi,\overline{\chi})}, \tag{2.2}$$

Moreover, if $\mathbf{R}$ is symmetric, by Proposition 2.5.2 from [1], p. 107, the holomorphic bisectional curvature completely determines the curvature tensor $R^i_{j\bar{h}k}$.

We propose now to determine the relationships between the holomorphic bisectional curvature and the two kinds of holomorphic curvature. For this, we consider the unitary directions l and m, where $l = \frac{\eta}{F(z,\eta)}$ and $m = \frac{\chi}{F(z,\chi)}$; l and m are local sections in $\pi^*(T'M)$. By means of these, we construct the diagonal directions S_{lm} and D_{lm} and their conjugates $S_{\bar{l}\bar{m}} = \bar{l} + \bar{m}$ and $D_{\bar{l}\bar{m}} = \bar{l} - \bar{m}$.

We denote by φ the angle between the directions of the unitary sections l and m. Therefore, we have $\cos\varphi = \frac{Re\mathcal{G}(l,\bar{m})}{||l||||\bar{m}||} = Re\mathcal{G}(l,\bar{m})$.

Proposition 2.2. *i)* $\mathcal{G}(S_{lm},S_{\bar{l}\bar{m}}) = 4\cos^2\frac{\varphi}{2}$;
ii) $\mathcal{G}(D_{lm},D_{\bar{l}\bar{m}}) = 4\sin^2\frac{\varphi}{2}$.

Proof. It follows by direct computation. □

By above considerations, we shall prove the following

Proposition 2.3. *Let (M,F) be a complex Finsler space. Then*

$$\mathcal{B}_F(z,\eta,\chi) = 2\mathcal{B}_F(z,\eta,S_{lm})\cos^2\frac{\varphi}{2} + 2\mathcal{B}_F(z,\eta,D_{lm})\sin^2\frac{\varphi}{2} - \mathcal{K}_F(z,\eta), \tag{2.3}$$

where $\mathcal{B}_F(z,\eta,S_{lm})$ and $\mathcal{B}_F(z,\eta,D_{lm})$ are the holomorphic bisectional curvature of F in the direction η and S_{lm} and η and D_{lm} respectively.

Proof. Taking into account Proposition 2.1, *iv)* and relation (2.1), we obtain

$$\mathcal{B}_F(z,\eta,\chi) = \mathcal{B}_F(z,l,m) = \mathbf{R}(l,\bar{l},m,\bar{m}) + \mathbf{R}(l,\bar{l},m,\bar{m}). \tag{2.4}$$

On the other hand, decomposing $\mathbf{R}(l,\bar{l},S_{lm},S_{\bar{l}\bar{m}})$, $\mathbf{R}(S_{lm},S_{\bar{l}\bar{m}},l,\bar{l})$, $\mathbf{R}(l,\bar{l},D_{lm},D_{\bar{l}\bar{m}})$ and $\mathbf{R}(D_{lm},D_{\bar{l}\bar{m}},l,\bar{l})$, direct computations give:

$$\mathbf{R}(l,\bar{l},S_{lm},S_{\bar{l}\bar{m}}) + \mathbf{R}(S_{lm},S_{\bar{l}\bar{m}},l,\bar{l}) + \mathbf{R}(l,\bar{l},D_{lm},D_{\bar{l}\bar{m}}) + \mathbf{R}(D_{lm},D_{\bar{l}\bar{m}},l,\bar{l})$$
$$= 4\mathbf{R}(l,\bar{l},l,\bar{l}) + 2\left[\mathbf{R}(l,\bar{l},m,\bar{m}) + \mathbf{R}(m,\bar{m},l,\bar{l})\right]$$

$= 2\mathcal{K}_F(z,l) + 2\mathcal{B}_F(z,l,m)$.

In view of Definition 2.1 and Proposition 2.1, the last relation becomes

$$4\mathcal{B}_F(z,l,S_{\bar{l}\bar{m}})\cos^2\frac{\varphi}{2} + 4\mathcal{B}_F(z,l,D_{\bar{l}\bar{m}})\sin^2\frac{\varphi}{2} = 2\mathcal{K}_F(z,\eta) + 2\mathcal{B}_F(z,\eta,\chi),$$

which is (2.3). □

If χ and η are colinearity, i.e. $\chi = \alpha\eta$, $\alpha \in \mathbb{R}^*$, then $\mathcal{B}_F(z,\eta,\chi) = \mathcal{B}_F(z,\eta,\alpha\eta) = \mathcal{B}_F(z,\eta,\eta) = \mathcal{K}_F(z,\eta)$. Conversely, if $\mathcal{B}_F(z,\eta,\chi) \equiv \mathcal{K}_F(z,\eta)$ then, the (2.3) relation, yields

$$\mathcal{K}_F(z,\eta) = \mathcal{B}_F(z,\eta,S_{lm})\cos^2\frac{\varphi}{2} + \mathcal{B}_F(z,\eta,D_{lm})\sin^2\frac{\varphi}{2}. \tag{2.5}$$

Moreover, by relation (2.3), if the holomorphic bisectional curvature is identically vanishing in any direction then the holomorphic curvature is identically vanishing too. Conversely, if $\mathcal{K}_F(z,\eta) = 0$ then

$$\mathcal{B}_F(z,\eta,\chi) = \mathcal{B}_F(z,\eta,S_{lm})\cos^2\frac{\varphi}{2} + \mathcal{B}_F(z,\eta,D_{lm})\sin^2\frac{\varphi}{2}. \tag{2.6}$$

When the holomorphic bisectional curvature is a constant, i.e. it has the same constant value for any choice of z and directions η,χ, but with this assumption, we obtain

Proposition 2.4. *Let (M,F) be a complex Finsler space of constant holomorphic bisectional curvature in any of directions η and χ, i.e. $\mathcal{B}_F(z,\eta,\chi) = c$, $c \in \mathbb{R}$. Then $\mathcal{K}_F(z,\eta) = c$.*

Proof. By (2.3) and by $\mathcal{B}_F(z,\eta,\chi) = c$, for any direction, it results in
$c = 2c\cos^2\frac{\varphi}{2} + 2c\sin^2\frac{\varphi}{2} - \mathcal{K}_F(z,\eta)$. This relation leads to $\mathcal{K}_F(z,\eta) = c$. □

In the remainder of this section, we study the holomorphic bisectional curvature of a complex Finsler space (M,F) with additional symmetry condition of the Riemann type tensor $\mathbf{R}$. A first result is:

Proposition 2.5. *If (M,F) is a complex Finsler space and $\mathbf{R}$ is symmetric then*

$$\begin{aligned}\mathcal{B}_F(z,\eta,\chi) &= 2\mathcal{K}_F(z,S_{lm})\cos^4\frac{\varphi}{2} + 2\mathcal{K}_F(z,D_{lm})\sin^4\frac{\varphi}{2} \\ &\quad -\frac{1}{4}\left[\mathcal{K}_F(z,\eta) + \mathcal{K}_F(z,\chi)\right] - \frac{1}{2}\mathcal{K}_F(z,\eta,\chi),\end{aligned} \tag{2.7}$$

where $\mathcal{K}_F(z,S_{lm})$, $\mathcal{K}_F(z,D_{lm})$ and $\mathcal{K}_F(z,\chi)$ are the holomorphic curvature of F in directions S_{lm}, D_{lm} and χ, respectively.

Proof. Taking into account Proposition 2.1, *iii*) and relation (2.1), we have

$$\mathcal{B}_F(z,\eta,\chi) = \mathcal{B}_F(z,l,m) = 2R(l,\bar{l},m,\bar{m}). \tag{2.8}$$

Decomposing $R(S_{lm}, S_{\bar{l}\bar{m}}, S_{lm}, S_{\bar{l}\bar{m}})$ şi $R(D_{lm}, D_{\bar{l}\bar{m}}, D_{lm}, D_{\bar{l}\bar{m}})$, by direct computations, we obtain:

$R(S_{lm}, S_{\bar{l}\bar{m}}, S_{lm}, S_{\bar{l}\bar{m}}) + R(D_{lm}, D_{\bar{l}\bar{m}}, D_{lm}, D_{\bar{l}\bar{m}}) = 2R(l, \bar{l}, l, \bar{l})$
$+2R(m, \bar{m}, m, \bar{m}) + 2\left[R(l, \bar{m}, l, \bar{m}) + R(m, \bar{l}, m, \bar{l})\right] + 8R(l, \bar{l}, m, \bar{m})$.

By Definition 2.1 and Proposition 2.1, the last relation becomes

$8\mathcal{K}_F(z, S_{lm}) \cos^4 \frac{\varphi}{2} + 8\mathcal{K}_F(z, D_{lm}) \sin^4 \frac{\varphi}{2} = \mathcal{K}_F(z, \eta) + \mathcal{K}_F(z, \chi)$
$+2\mathcal{K}_F(z, \eta, \chi) + 4\mathcal{B}_F(z, \eta, \chi)$,

which leads to (2.7). □

Proposition 2.6. *Let (M, F) be a complex Finsler space of constant holomorphic flag curvature along of any flag (η, χ), i.e. $\mathcal{K}_F(z, \eta, \chi) = c$, $c \in \mathbb{R}$, and* **R** *symmetric. Then*

i)

$$\mathcal{B}_F(z, \eta, \chi) = c \cdot \cos^2 \varphi. \tag{2.9}$$

Moreover,

a) if $c \geq 0$ then $\mathcal{B}_F(z, \eta, \chi) \leq c$;

b) if $c < 0$ then $c < \mathcal{B}_F(z, \eta, \chi)$.

ii) (M, F) is of constant holomorphic bisectional curvature if and only if φ is a constant.

Proof. *i)* Because, $\mathcal{B}_F(z, \eta, \chi) = c$, $c \in \mathbb{R}$, for any directions η and χ, then $\mathcal{K}_F(z, \eta) = c$. Therefore, relation (2.7) becomes

$\mathcal{B}_F(z, \eta, \chi) = 2c\left(\cos^4 \frac{\varphi}{2} + \sin^4 \frac{\varphi}{2}\right) - c = 2c\left(1 - 2\sin^2 \frac{\varphi}{2} \cos^2 \frac{\varphi}{2}\right) - c$
$= c - c\sin^2 \varphi = c\cos^2 \varphi$.

ii) Immediately result by (2.9) relation. □

Colorallary 2.1. *Let (M, F) be a complex Finsler space with* **R** *symmetric. If along any flag and in any direction we have*

$$|\mathcal{K}_F(z, \eta, \chi)| \leq c \text{ and } |\mathcal{K}_F(z, \eta)| \leq c, \; c > 0,$$

then $|\mathcal{B}_F(z, \eta, \chi)| \leq 3c$.

Proof. Indeed,

$|\mathcal{B}_F(z, \eta, \chi)| \leq 2c\left(\cos^4 \frac{\varphi}{2} + \sin^4 \frac{\varphi}{2}\right) + c = c\left(2 + \cos^2 \varphi\right) \leq 3c.$ □

Some special results for the holomorphic bisectional curvature will be obtained subsequently, when we study a particular fruity case.

3. The holomorphic bisectional curvature of a $(g.E.)$ complex Finsler space

We establish some inequalities between the holomorphic bisectional curvature and holomorphic curvature of a $(g.E.)$ complex Finsler space. For the beginning, let us express the holomorphic bisectional curvature of a $(g.E.)$ complex Finsler space by means of the holomorphic curvature of the same space.

In a local coordinate, the holomorphic bisectional curvature of complex Finsler metric F in directions η and χ is given by

$$\mathcal{B}_F(z,\eta,\chi)=\frac{\left(\eta^i\overline{\eta}^j\chi^k\overline{\chi}^h+\chi^i\overline{\chi}^j\eta^k\overline{\eta}^h\right)R_{i\bar{j}k\bar{h}}}{L(z,\eta)L(z,\chi)}, \tag{3.1}$$

with $L(z,\chi)=g_{i\bar{j}}\chi^i\overline{\chi}^j\neq 0$, and the angle φ between directions of η and χ is

$$\cos\varphi=\frac{\eta_i\chi^i+\bar{\eta}_j\overline{\chi}^j}{2\sqrt{L(z,\eta)L(z,\chi)}}. \tag{3.2}$$

Proposition 3.1. *Let (M,F) be a $(g.E.)$ complex Finsler space. Then*

$$\mathcal{B}_F(z,\eta,\chi)=\frac{\mathcal{K}_F(z)}{2}\left\{1+\frac{\left|\eta_j\chi^j\right|^2}{L(z,\eta)L(z,\chi)}\right\}, \tag{3.3}$$

where $\mathcal{K}_F(z)$ is the holomorphic curvature of (M,F).

Proof. Because (M,F) is a $(g.E.)$ complex Finsler space, by Propositions 3.3, $iii)$ and 3.4 from [3], we obtain:
$R_{\bar{j}l\bar{h}k}\eta^l\overline{\eta}^j=K(z)\left(L(z,\eta)g_{k\bar{h}}+\eta_k\bar{\eta}_h\right)$,
$R_{\bar{j}l\bar{h}k}\eta^k\overline{\eta}^h=K(z)\left(L(z,\eta)g_{l\bar{j}}+\eta_l\bar{\eta}_j\right)-C_{\bar{j}r|k}C^{\bar{r}}_{l|\bar{h}}\eta^k\overline{\eta}^h$
$=K(z)\left(L(z,\eta)g_{l\bar{j}}+\eta_l\bar{\eta}_j\right)+\dot{T}_{\bar{j}l}$where $T_{\bar{j}k}:=g_{i\bar{j}}T^i_{lk}\eta^l$ and $\dot{T}_{\bar{j}k}:=T_{\bar{j}k|\overline{m}}\overline{\eta}^m$.
By Jacobi identity
$\left[\dot{\partial}_i,[\delta_j,\delta_{\bar{k}}]\right]+\left[\delta_j,[\delta_{\bar{k}},\dot{\partial}_i]\right]+\left[\delta_{\bar{k}},[\dot{\partial}_i,\delta_j]\right]=0$, we have
$-\dot{\partial}_iR^l_{\bar{k}j}-P^r_{\bar{k}i}L^l_{rj}-\delta_jP^l_{\bar{k}i}-\delta_{\bar{k}}L^l_{ij}=0$. We interchange i with j
$-\dot{\partial}_iR^l_{\bar{k}j}+\dot{\partial}_jR^l_{\bar{k}i}-P^r_{\bar{k}i}L^l_{rj}+P^r_{\bar{k}j}L^l_{ri}-\delta_jP^l_{\bar{k}i}+\delta_iP^l_{\bar{k}j}-\delta_{\bar{k}}T^l_{ij}=0$.
Multiplying the last relation by $\bar{\eta}^k$, we obtain
$-\dot{\partial}_i\left(R^l_{\bar{k}j}\bar{\eta}^k\right)+\dot{\partial}_j\left(R^l_{\bar{k}i}\bar{\eta}^k\right)-T^l_{ij|\bar{k}}\bar{\eta}^k=0$.
But $R^l_{\bar{k}j}\bar{\eta}^k=R_{\overline{m}r\bar{k}j}g^{\overline{m}l}\eta^r\overline{\eta}^k=K(z)\left(L(z,\eta)\delta^l_j+\eta_j\eta^l\right)$ and
$\dot{\partial}_i\left(R^l_{\bar{k}j}\overline{\eta}^k\right)=K(z)\left(\eta_i\delta^l_j+C_{i\bar{k}j}\overline{\eta}^k\eta^l+\eta_j\delta^l_i\right)$.
Taking into account above relations it results in $T^l_{ij|\bar{k}}\overline{\eta}^k=0$ and from here $\dot{T}_{\bar{j}k}=0$.
Plugging into (3.1) it results:
$\mathcal{B}_F(z,\eta,\chi)=\frac{2K(z)\left(L(z,\eta)g_{l\bar{j}}+\eta_l\bar{\eta}_j\right)\chi^i\overline{\chi}^j}{L(z,\eta)L(z,\chi)}$
$=\frac{2K(z)}{L(z,\eta)L(z,\chi)}\left[L(z,\eta)L(z,\chi)+\left|\eta_j\chi^j\right|^2\right]$.
But, $K(z)=\frac{1}{4}\mathcal{K}_F(z)$, so that the last relation is (3.3). □

We note that, if $\mathcal{K}_F(z)=0$ then, by relation (3.3), we have $\mathcal{B}_F(z,\eta,\chi)=0$.

Example 1. In [3] we considered the complex version of *Antonelli-Shimada metric* on a domain from $\widetilde{T'M}$, $\dim_C M = 2$, such that its metric tensor be nondegenerated

$$L_{AS}(z,w;\eta,\theta) := e^{2\sigma(z,w)}\left(|\eta|^4 + |\theta|^4\right)^{\frac{1}{2}}, \text{ with } \eta,\theta \neq 0, \tag{3.4}$$

where $z := z^1$, $w := z^2, \eta := \eta^1$, $\theta := \eta^2$, $\sigma(z,w)$ is a real valued function and $|\eta^i|^2 := \eta^i\bar{\eta}^i$, $\eta^i \in \{\eta,\theta\}$. We showed that the (3.4) metric is not $(g.E.)$ and its holomorphic curvature is $\mathcal{K}_F = -\frac{4}{L_{AS}}\frac{\partial^2\sigma}{\partial z^k\partial\bar{z}^h}\eta^k\bar{\eta}^h$, where $z^i \in \{z,w\}, \eta^i \in \{\eta,\theta\}$. If $\frac{\partial^2\sigma}{\partial z^k\partial\bar{z}^h} = 0$ then the (3.4) metric is not purely Hermitian or weakly Kähler, but it is $(g.E.)$ with $\mathcal{K}_F = \mathcal{B}_F = 0$. Moreover, it is locally Minkowski if and only if σ is a constant. □

Taking into account (3.2) we have
$\left[2Re\left(\bar{\eta}_j\overline{\chi}^j\right)\right]^2 = \left(\eta_j\chi^j + \bar{\eta}_j\overline{\chi}^j\right)^2 = 4L(z,\eta)L(z,\chi)\cos^2\varphi$
and from here we obtain
$Re\left[\left(\bar{\eta}_j\overline{\chi}^j\right)^2\right] + \left|\eta_j\chi^j\right|^2 = 2L(z,\eta)L(z,\chi)\cos^2\varphi.$
The complex number theory permit us to write

$$\left|\eta_j\chi^j\right|^2 = L(z,\eta)L(z,\chi)\cos^2\varphi + \left[Im\left(\bar{\eta}_j\overline{\chi}^j\right)\right]^2, \tag{3.5}$$

which leads to

Colorallary 3.1. *Let (M,F) be a complex Finsler space $(g.E.)$, of nonzero constant holomorphic curvature, $\mathcal{K}_F = c$, $c \in \mathbb{R}^*$. Then*

$$\mathcal{B}_F(z,\eta,\chi) = \frac{c}{2}\left\{1 + \cos^2\varphi + \frac{\left[Im\left(\bar{\eta}_j\overline{\chi}^j\right)\right]^2}{L(z,\eta)L(z,\chi)}\right\}. \tag{3.6}$$

In particular, if in the relation (3.6), $Im\left(\bar{\eta}_j\overline{\chi}^j\right) = 0$ then it results in $\mathcal{B}_F(z,\eta,\chi) = \frac{c}{2}(1+\cos^2\varphi)$. Moreover, if $Re\left(\bar{\eta}_j\overline{\chi}^j\right) = 0$ then $\mathcal{B}_F(z,\eta,\chi) = \frac{c}{2}$.

By Proposition 1.1 we obtain

Colorallary 3.2. *Let (M,F) be a complex Finsler space $(g.E.)$, of nonzero constant holomorphic curvature, $\mathcal{K}_F = c$, $c \in \mathbb{R}^*$. Then*

$$\frac{\mathcal{B}_F(z,\eta,\chi)}{c} + \frac{\mathcal{K}_F(z,\eta,\chi)}{2c} = \frac{1}{2}\left\{1 + 2\cos^2\varphi + \frac{Re\left(C_{\bar{j}\bar{h}}\overline{\chi}^j\overline{\chi}^h\right)}{L(z,\chi)}\right\}, \tag{3.7}$$

where $\mathcal{K}_F(z,\eta,\chi)$ is the holomorphic flag curvature along the flag (η,χ).

Proposition 3.2. *Let (M,F) be a complex Finsler space $(g.E.)$ of nonzero constant holomorphic curvature, $\mathcal{K}_F = c$, $c \in \mathbb{R}^*$. Then*

$$\frac{\mathcal{B}_F(z,\eta,\chi)}{c} \geq \frac{1}{2}. \tag{3.8}$$

Moreover,
if $c > 0$ *then* $\mathcal{B}_F(z,\eta,\chi) \geq \frac{c}{2} > 0$;
if $c < 0$ *then* $0 > \frac{c}{2} \geq \mathcal{B}_F(z,\eta,\chi)$.

Proof. Taking into account relation (1.9), we obtain
$\frac{\mathcal{B}_F(z,\eta,\chi)}{c} \geq \frac{1}{2}\left(1+\cos^2\varphi\right) \geq \frac{1}{2}$. □

Example 2. In Proposition 3.5 from [3] we proved that if (M,F) is $(g.E.)$ complex Finsler space with $\mathcal{K}_F = -4$ then F is Kobayashi metric. Therefore the holomorphic bisectional curvature of Kobayashi metric is
$\mathcal{B}_{F_K}(z,\eta,\chi) = -2\left\{1+\cos^2\varphi + \frac{\left[Im(\bar{\eta}_j\overline{\chi}^j)\right]^2}{L(z,\eta)L(z,\chi)}\right\}$ and
$\mathcal{B}_{F_K}(z,\eta,\chi) \leq -2$. □

In the remainder of this section, we consider the particular class of the $(g.E.)$ complex Finsler space which is Kähler with nonzero constant holomorphic curvature. Therefore, relation (3.7) is reduced to

Colorallary 3.3. *Let* (M,F) *be a complex Kähler-Finsler space* $(g.E.)$ *of nonzero constant holomorphic curvature,* $\mathcal{K}_F = c$, $c \in \mathbb{R}^*$. *Then*

$$\frac{1}{2} \leq \frac{\mathcal{B}_F(z,\eta,\chi)}{c} + \frac{\mathcal{K}_F(z,\eta,\chi)}{2c} = \frac{1}{2} + \cos^2\varphi \leq \frac{3}{2}; \tag{3.9}$$

Example 3. We consider the complex Finsler metrics

$$L := \frac{|\eta|^2 + \varepsilon(|z|^2|\eta|^2 - <z,\eta>\overline{<z,\eta>})}{(1+\varepsilon|z|^2)^2}, \tag{3.10}$$

defined over the disk $\Delta_r^n = \left\{z \in \mathbf{C}^n,\ |z| < r,\ \ r := \sqrt{\frac{1}{|\varepsilon|}}\right\}$ if $\varepsilon < 0$; on $\mathbf{C}^n$ if $\varepsilon = 0$; and on the complex projective space $P^n(\mathbf{C})$ if $\varepsilon > 0$, where $|z|^2 := \sum_{k=1}^n z^k\overline{z}^k$, $<z,\eta> := \sum_{k=1}^n z^k\overline{\eta}^k$. Particularly, for $\varepsilon = -1$ we obtain the *Bergman metric* on the unit disk $\Delta^n := \Delta_1^n$, for $\varepsilon = 0$ the *Euclidian metric* on $\mathbf{C}^n$, and for $\varepsilon = 1$ the *Fubini-Study metric* on $P^n(\mathbf{C})$.

The (3.10) metrics are $(g.E.)$, Kähler with $\mathcal{K}_F = 4\varepsilon$. From Proposition 3.2 we obtain: if $\varepsilon < 0$ then $\mathcal{B}_F(z,\eta,\chi) \leq 2\varepsilon$ and if $\varepsilon > 0$ then $2\varepsilon \leq \mathcal{B}_F(z,\eta,\chi)$. □

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