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CONSTRUCTION OF CODES BY LATTICE VALUED FUZZY SETS*

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Abstract. To every finite lattice L, one can associate a binary block-code, constructed by a particular L-valued fuzzy set. Starting with L, we construct a new lattice such that the corresponding block-code possesses better parameters.

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1. Introduction

Let L be a finite lattice. As usual, we denote the operation infimum by $\wedge$, supremum by $\vee$ and the corresponding ordering relation by $\leqslant$. We recall that an element $a \in L$, $a \neq 1_L$ (1_L is the greatest element of L) is **meet-irreducible** if

$$a = b \wedge c \quad \text{implies} \quad a = b \quad \text{or} \quad a = c.$$

It is well known that every element of a finite lattice can be represented as infimum of meet-irreducible elements.

A partially ordered set $\mathcal{R} = (R, \leqslant)$ with the greatest element 1_R is a **root system** if no two incomparable elements have a lower bound (equivalently, if for all $x \in R$, the set $\{y : x \leqslant y\}$ is totally ordered). From the definition it is evident that all elements of a root system except for the greatest one, are meet-irreducible.

Recall that the **linear sum** of ordered sets $(P, \leqslant)$ and $(Q, \leqslant)$ is the ordered set $(P \cup Q, \leqslant)$, with the ordering relation preserving orders in P and Q, with addition that $p \leqslant q$ for all $p \in P, q \in Q$. The linear sum of ordered sets P and Q is here denoted by $P + Q$.

If S is a nonempty set and L a lattice, then a function $\overline{A} : S \to L$ is L-**fuzzy set** on S. $\overline{A}(x)$ is the membership degree of element $x \in S$ to the fuzzy set $\overline{A}$. For $\overline{A} : S \to L$ and $p \in L$, a p-**level subset** (or p-**cut**) of $\overline{A}$ is defined by

$$A_p = \{x \in S : \overline{A}(x) \geqslant p\}.$$

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By $\overline{A}$ we denote the characteristic function of $\overline{A}_p$: for $x \in S$, $\overline{A}_p(x) = 1$ if and only if $\overline{A}(x) \geqslant p$.

We denote by $\overline{A}(S)$ the set of values of the L-valued fuzzy set $\overline{A}$ on S.

A **binary block-code** V is a nonempty subset of $\{0,1\}^n$, $n \in \mathbb{N}$. The number n is the **length** of V.

The following can be found in [9] and other papers cited in the references.

The set of level functions of a fuzzy set $\overline{A} : S \to L$, for $S = \{1, 2, \ldots, n\}$ is a binary block-code of length n, which we denote by V_L (V for short). To every fuzzy set $\overline{A}$ there corresponds a code V, but a code may correspond to several fuzzy sets.

For every finite lattice L, there is a fuzzy set $\overline{A}$, such that the corresponding code has maximal cardinality $|L|$.

Proposition 1. **([10])** *Let L be a lattice of the finite length, and let $\overline{A} : S \to L$ be an L-valued fuzzy set. Necessary and sufficient condition under which all p-cuts of $\overline{A}$ are different is that the set of all meet-irreducible elements of L is a subset of $\overline{A}(S)$.*

Proposition 2. **([10])** *Necessary and sufficient condition under which for L-valued fuzzy set there is a code V such that $|L| = |V|$ is that the set of all meet-irreducible elements of L is a subset of $\overline{A}(S)$.*

2. Results

Let L be a lattice with $|L| = m$ elements. Let $i \in \mathbb{N}$ ($i \leqslant m$) be the number of meet-irreducible elements of the lattice L. Further, let $S = \{1, 2, \ldots, i\}$. By Proposition 2, one can construct a fuzzy set $\overline{A} : S \to L$, such that the corresponding code V has maximal cardinality $m = |L| = |V|$ and such that the length of the code V is i. Namely, if $a_1, \ldots, a_i$ are meet-irreducible elements of L, then for $k = 1, \ldots, i$, $\overline{A}(k) := a_k$. Then, the number of cut sets of $\overline{A}$ is precisely m, and $|\overline{A}(S)| = i$. Therefore, the corresponding block-code V has the length i and the cardinality m.

Our goal is to improve the above construction in order to get codes with greater cardinality; still, the length of these codes should not increase to much. This goal is based on two algorithms for the construction of new lattices starting with the given lattice L. In the following, we describe the algorithms.

2.1. First Algorithm

Let L be a finite lattice and let $0 \in L$ be the smallest element of L. Let $\mathcal{R} = (R, \leqslant)$ be a finite root system. $\mathcal{R}$ is an upper semilattice.

If the smallest element 0 is left out of L, then the direct product $(L \backslash \{0\}) \times \mathcal{R}$ is an upper semilattice. So,

$$L_{\mathcal{R}} \equiv \{0\} + ((L \backslash \{0\}) \times \mathcal{R})$$

is a lattice, where $+$ is a linear sum, and $\times$ direct product of partially ordered sets.

It is obvious that the lattice L is isomorphic to the sublattice L_1 of $L_{\mathcal{R}}$, where

$$L_1 \equiv \{(l, 1_R) \mid l \in (L \backslash \{0\})\} \cup \{0\}.$$

Example 1. *In Figure 1, a lattice L and a root system $\mathcal{R}$ are given.*

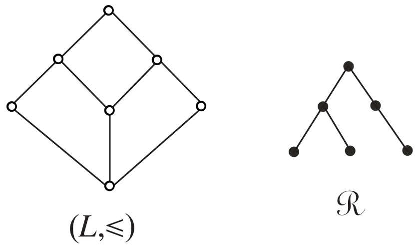

Fig. 1.

In Figure 2, we have the lattice $L_{\mathcal{R}}$.

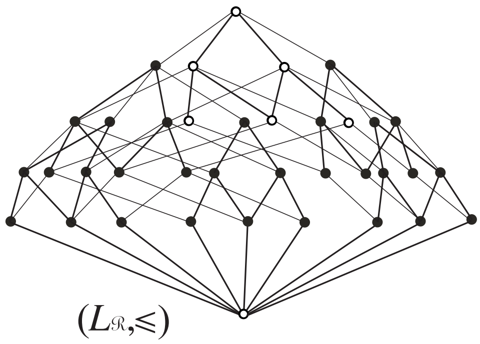

Fig. 2.

From the construction of the lattices $L_{\mathcal{R}}$ the next propositions obviously hold.

Proposition 3. **([4])** *If L is lattice, $|L| = m$, and $\mathcal{R}$ has k elements, then*

$$|L_{\mathcal{R}}| = k \cdot (m - 1) + 1.$$

Proposition 4. *If i and i_k ($k \in \mathbb{N}$) are numbers of meet-irreducible elements of lattices L and $L_{\mathcal{R}}$ respectively, then*

$$i_k = i + k - 1.$$

Proof. From the construction of the lattice $L_{\mathcal{R}}$ it follows that meet-irreducible elements of L_1 are also meet-irreducible elements of $L_{\mathcal{R}}$. This also holds for elements $\{(1_L, r) \mid r \in (R \backslash \{1_R\})\}$. We prove that these are the only meet-irreducible elements of $L_{\mathcal{R}}$.

If l is any meet-irreducible element of L and $r \in R$, we prove that (l, r) is not meet-irreducible element of $L_{\mathcal{R}}$. Indeed, let l_a and r_a be elements that cover l and r respectively in L and R. Then

$$(l, r) = (l_a, r) \wedge (l, r_a)$$

in $L_{\mathcal{R}}$, and (l, r) is meet-irreducible if and only if $l = 1_L$.

It follows that for the number i_k of meet-irreducible elements of $L_{\mathcal{R}}$ we have

$$i_k = i + k - 1.$$

□

Let $\overline{A} : S \to L$ be a fuzzy set such that $\overline{A}(S)$ contains all meet-irreducible elements of L. Then the code V_L corresponding to this fuzzy set has the length $|\overline{A}(S)|$ and the cardinality $|V_L| = |L| = m$.

Let $L_{\mathcal{R}}$ be as above, where $|R| = k$. Fuzzy set corresponding to the code $V_{L_{\mathcal{R}}}$ is of cardinality

$$|V_{L_{\mathcal{R}}}| = |L_{\mathcal{R}}| = k \cdot (m - 1) + 1.$$

2.2. Second Algorithm

Let L be a lattice, and $L_{\mathcal{R}} \equiv \{0\} + ((L \backslash \{0\}) \times \mathcal{R})$ be as defined in the previous part. If the above algorithm is applied to the lattice $L_{\mathcal{R}}$, we get the lattice

$$(L_{\mathcal{R}})_{\mathcal{R}} \equiv \{0\} + (((L \backslash \{0\}) \times \mathcal{R}) \times \mathcal{R}) = L_{\mathcal{R},\mathcal{R}}.$$

Similarly, in the case of n ($n \in \mathbb{N}$) applications of the first algorithm, we have

$$L_{\underbrace{\mathcal{R},\dots,\mathcal{R}}_{n \text{ times}}} \equiv \{0\} + (\dots(((L \backslash \{0\}) \times \mathcal{R}) \times \mathcal{R}) \times \cdots \times \mathcal{R}) =$$
$$= \{0\} + ((L \backslash \{0\}) \times \mathcal{R}^n).$$

Example 2. *Let L be the lattice from Example 1 (Fig. 2) and $\mathcal{R}$ root system with 3 elements (Fig. 3). Then lattice $L_{\mathcal{R},\mathcal{R}}$ is given in Fig. 3.*

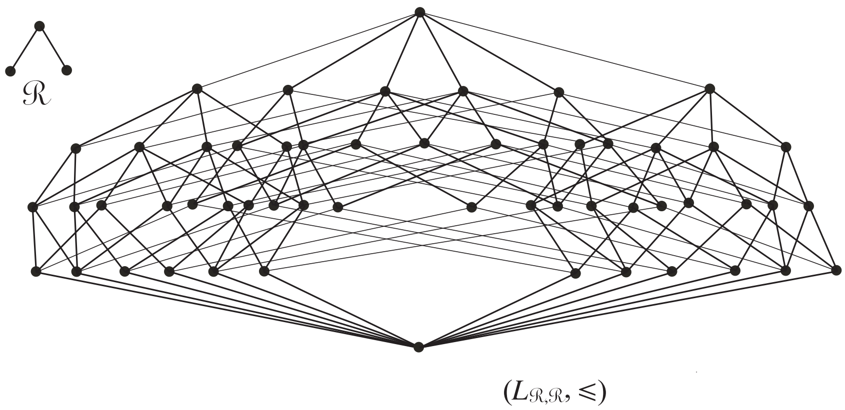

Fig. 3.

Proposition 5. *Let L be a lattice with $|L| = m$ and let $L_{\mathcal{R}}$ be as above, then*

$$|L_{\underbrace{\mathcal{R},\ldots,\mathcal{R}}_{n \text{ times}}}| = k^n \cdot (m-1) + 1.$$

Proof. For $|L| = m$ and $n = 1$, the lattice $L_{\mathcal{R}}$ has the cardinality $k \cdot (m-1) + 1$ (Proposition 3), and our formula holds.

Let $|L_{\underbrace{\mathcal{R},\ldots,\mathcal{R}}_{(n-1) \text{ times}}}| = k^{n-1} \cdot (m-1) + 1$.

From the construction we have

$$\begin{aligned} |L_{\underbrace{\mathcal{R},\ldots,\mathcal{R}}_{n \text{ times}}}| &= |\{0\} + (((L\backslash\{0\}) \times \mathcal{R}^{n-1}) \times \mathcal{R})| = \\ &= k \cdot (|L_{\underbrace{\mathcal{R},\ldots,\mathcal{R}}_{(n-1) \text{ times}}}| - 1) + 1 = \\ &= k \cdot (k^{n-1} \cdot (m-1) + 1 - 1) + 1 = \\ &= k^n \cdot (m-1) + 1. \end{aligned}$$

□

From the above propositions it follows

$$\begin{aligned} |L_{\mathcal{R}}| &= k \cdot (|L| - 1) + 1 \quad \text{and} \\ |L_{\underbrace{\mathcal{R},\ldots,\mathcal{R}}_{n \text{ times}}}| &= k^n \cdot (|L| - 1) + 1. \end{aligned}$$

So, by the second algorithm we have that $L_{\underbrace{\mathcal{R},\ldots,\mathcal{R}}_{n \text{ times}}} = L_{\mathcal{R}^n}$.

Proposition 6. *Let L be a lattice with i meet-irreducible elements and let $L_{\mathcal{R}^n}$ be as above, then the number of meet-irreducible elements of lattice $L_{\mathcal{R}^n}$, denoted by i_{k^n}, is given by formula*

$$i_{k^n} = i + n \cdot (k-1).$$

Proof. From $L_{\mathcal{R}^n} = \{0\} + ((L\backslash\{0\}) \times \mathcal{R}^n)$ and Proposition 4 it follows that i_{k^n} is equal to the sum of numbers of meet-irreducible elements of L (i of them) and meet-irreducible elements of $\mathcal{R}^n$ ($n \cdot (k-1)$ of them). □

Corollary 1. *Let V_L be a block code of length i and of cardinality $|V_L| = |L| = m$, then block code $V_{L_{\mathcal{R}^n}}$ is of length $i + n \cdot (k-1)$ and of cardinality $|V_{L_{\mathcal{R}^n}}| = |L_{\mathcal{R}^n}| = k^n \cdot (m-1) + 1 \quad (|R| = k)$.*

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