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SOME COMBINATORIAL ASPECTS OF THE COMPOSITION OF A SET OF FUNCTIONS[1]

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Abstract. In this paper we determine a number of meaningful compositions of higher order of a set of functions, which is considered in [2], in implicit and explicit forms. The obtained results are applied to vector analysis in order to determine the number of meaningful differential operations of higher order.

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1. The composition of a set of functions

The main topic of considered in this paper is the set of functions $\mathcal{A}_n$, for $n=2,3,\dots$, determined in the following form:

$$(1)\qquad \mathcal{A}_n\ (n=2m):\ \begin{array}{l}\nabla_1 : \mathrm{A}_0 \to \mathrm{A}_1\\ \nabla_2 : \mathrm{A}_1 \to \mathrm{A}_2\\ \vdots\\ \nabla_i : \mathrm{A}_i \to \mathrm{A}_{i+1}\\ \vdots\\ \nabla_m : \mathrm{A}_{m-1} \to \mathrm{A}_m\\ \nabla_{m+1} : \mathrm{A}_m \to \mathrm{A}_{m-1}\\ \vdots\\ \nabla_{n-j} : \mathrm{A}_{j+1} \to \mathrm{A}_j\\ \vdots\\ \nabla_{n-1} : \mathrm{A}_2 \to \mathrm{A}_1\\ \nabla_n : \mathrm{A}_1 \to \mathrm{A}_0,\end{array} \qquad \mathcal{A}_n\ (n=2m+1):\ \begin{array}{l}\nabla_1 : \mathrm{A}_0 \to \mathrm{A}_1\\ \nabla_2 : \mathrm{A}_1 \to \mathrm{A}_2\\ \vdots\\ \nabla_i : \mathrm{A}_i \to \mathrm{A}_{i+1}\\ \vdots\\ \nabla_m : \mathrm{A}_{m-1} \to \mathrm{A}_m\\ \nabla_{m+1} : \mathrm{A}_m \to \mathrm{A}_m\\ \nabla_{m+2} : \mathrm{A}_m \to \mathrm{A}_{m-1}\\ \vdots\\ \nabla_{n-j} : \mathrm{A}_{j+1} \to \mathrm{A}_j\\ \vdots\\ \nabla_{n-1} : \mathrm{A}_2 \to \mathrm{A}_1\\ \nabla_n : \mathrm{A}_1 \to \mathrm{A}_0.\end{array}$$

Additionally, we make an assumption that A_i are non-empty sets, for $i = 0,1,\dots,m$, where $m=[n/2]$. For each set of functions $\mathcal{A}_n$ we determine the number of meaningful compositions of higher order in implicit and explicit forms. Let us define a binary relation ρ "*to be in composition*" over the set of functions

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$\mathcal{A}_n$ with $\nabla_i \rho \nabla_j = \top$ iff the composition $\nabla_j \circ \nabla_i$ is meaningful ($\nabla_i, \nabla_j \in \mathcal{A}_n$). Let us form the matrix $\mathtt{A} = [a_{ij}]$ with

$$a_{ij} = \begin{cases} 1 & : \quad (j = i+1) \vee (i+j = n+1) \\ 0 & : \quad (j \neq i+1) \wedge (i+j \neq n+1) \end{cases} \tag{2}$$

for $i, j \in \{1, 2, \ldots, n\}$. Graph, whose adjacency matrix is $\mathtt{A}$ and $\mathcal{A}_n = \{\nabla_1, \ldots, \nabla_n\}$ is a set of the vertices, is determined. Thus, on the basis of the article [2], some implicit formulas for the number of meaningful compositions of functions from the set $\mathcal{A}_n$ are given by the following statement.

Theorem 1. *Let $P_n(\lambda) = |\mathtt{A} - \lambda\mathtt{I}| = \alpha_0\lambda^n + \alpha_1\lambda^{n-1} + \ldots + \alpha_n$ be the characteristic polynomial of the matrix $\mathtt{A} = [a_{ij}]$, determined by* (2), *and $v_n = [\,1 \cdots 1\,]_{1\times n}$. If we denote by $f(k)$ the number of meaningful composition of k^{th}-order of functions from $\mathcal{A}_n$, then the following formulas are true*:

$$f(k) = v_n \cdot \mathtt{A}^{k-1} \cdot v_n^T \tag{3}$$

and

$$\alpha_0 f(k) + \alpha_1 f(k-1) + \ldots + \alpha_n f(k-n) = 0 \quad (k > n). \tag{4}$$

Remark 1. *Generally, let a graph G, with vertices $\nu_1, \ldots, \nu_n$, be determined by the adjacency matrix $\mathtt{A}$ and let $P_n(\lambda) = |\mathtt{A} - \lambda\mathtt{I}| = \alpha_0\lambda^n + \alpha_1\lambda^{n-1} + \ldots + \alpha_n$ be the characteristic polynomial of the matrix $\mathtt{A}$. If we denote with $a_{ij}^{(k)}$ the number of ν_i, ν_j – walks of length k in the graph G, then for every choice of ν_i and ν_j, the sequence $a_{ij}^{(k)}$ satisfies the same recurrent relation* (4) (*see the first problem in the section* 8.6, *of the supplementary problems page, of the book* [3]).

2. Some explicit formulas for the number of composition

In this part we give some explicit formulas for the number of meaningful compositions of functions from the set $\mathcal{A}_n$. The following statements are true.

Lemma 1. *The characteristic polynomial $P_n(\lambda)$ of the matrix $\mathtt{A} = [a_{ij}]$, determined by* (2), *fulfills the following recurrent relation*

$$P_n(\lambda) = \lambda^2\big(P_{n-2}(\lambda) - P_{n-4}(\lambda)\big). \tag{5}$$

Proof. Expanding the determinant $P_n(\lambda) = |\mathtt{A} - \lambda\mathtt{I}|$ by the first column we have

$$P_n(\lambda) = -\lambda C_{n-1}(\lambda) + (-1)^{n+1} D_{n-1}(\lambda), \tag{6}$$

where $C_{n-1}(\lambda)$ and $D_{n-1}(\lambda)$ are suitable minors of the elements a_{11} i a_{n1} of the determinant $P_n(\lambda)$. Continuing the expansion of the determinant $C_{n-1}(\lambda)$ by the ending row we can conclude that

$$C_{n-1}(\lambda) = -\lambda P_{n-2}(\lambda). \tag{7}$$

Further, notice that the determinant $D_n(\lambda)$ has minor $P_{n-3}(\lambda)$ as follows

$$D_n(\lambda) = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ -\lambda & 1 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & & & & & & 0 & 0 \\ 0 & & & & & & 0 & 0 \\ \vdots & & & P_{n-3}(\lambda) & & & \vdots & \vdots \\ 0 & & & & & & 0 & 0 \\ 0 & & & & & & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & -\lambda & 1 \end{vmatrix}. \tag{8}$$

If in the previous determinant we multiply the first row by -1 and add it to the n^{th}-row and then, if in the next step, we expand determinant by ending column, we can conclude

$$D_n(\lambda) = (-1)^{n-1}\lambda^2 P_{n-3}(\lambda). \tag{9}$$

On the basis of expansion (6) and formulas (7), (9) it is true that

$$P_n(\lambda) = \lambda^2\big(P_{n-2}(\lambda) - P_{n-4}(\lambda)\big). \tag{10}$$

□

Lemma 2. *Characteristic polynomial $P_n(\lambda)$ of the matrix $\mathtt{A} = [a_{ij}]$, determined by* (2)*, has the following explicit representation*

$$P_n(\lambda) = \begin{cases} \displaystyle\sum_{k=1}^{[\frac{n+2}{4}]+1} (-1)^{k-1} \binom{\frac{n}{2}-k+2}{k-1} \lambda^{n-2k+2} & : n=2m, \\ \displaystyle\sum_{k=1}^{[\frac{n+2}{4}]+2} (-1)^{k-1} \left(\binom{\frac{n+3}{2}-k}{k-1} + \binom{\frac{n+3}{2}-k}{k-2} \lambda \right) \lambda^{n-2k+2} & : n=2m+1. \end{cases} \tag{11}$$

Proof. Let us determine a few initial characteristic polynomials in the following forms:

$$\begin{aligned} P_2(\lambda) &= \lambda^2 - 1 = \sum_{k=1}^{2} (-1)^{k-1} \binom{3-k}{k-1} \lambda^{4-2k}, \\ P_4(\lambda) &= \lambda^4 - 2\,\lambda^2 = \sum_{k=1}^{2} (-1)^{k-1} \binom{4-k}{k-1} \lambda^{6-2k}; \end{aligned} \tag{12}$$

and

$$\begin{aligned} P_3(\lambda) &= \lambda^3 - \lambda^2 - \lambda = \sum_{k=1}^{3} (-1)^{k-1} \left(\binom{3-k}{k-1} \lambda^{5-2k} + \binom{3-k}{k-2} \lambda^{6-2k} \right), \\ P_5(\lambda) &= \lambda^5 - \lambda^4 - 2\,\lambda^3 + \lambda^2 = \sum_{k=1}^{3} (-1)^{k-1} \left(\binom{4-k}{k-1} \lambda^{7-2k} + \binom{4-k}{k-2} \lambda^{8-2k} \right). \end{aligned} \tag{13}$$

Then the statement of this lemma follows by mathematical induction on the basis of the recurrent relation (5). □

From Theorem 1 and Lemma 2, the following statement follows.

Theorem 2. *Let* $\mathtt{A} = [a_{ij}]$ *be the matrix determined by* (2). *Then the number of meaningful compositions of* k^{th}*-order of functions from* $\mathcal{A}_n$ *fulfills the recurrent relation* (4), *whereas* α_i*'s* $(i = 0, 1, \ldots, n)$ *are coefficients of the characteristic polynomial* $P_n(\lambda)$ *determined by* (11).

Further, the following general statement is true.

Lemma 3. *Let* $\mathtt{A}^k = [a_{ij}^{(k)}]$ *be the* k^{th}*-power of the matrix* $\mathtt{A} = [a_{i,j}] \in \mathbf{C}^{n\times n}$ $(k \in N)$ *and let*

$$P_n(\lambda) = |\mathtt{A} - \lambda\mathtt{I}| = \alpha_0\lambda^n + \alpha_1\lambda^{n-1} + \ldots + \alpha_n, \tag{14}$$

be the characteristic polynomial $P_n(\lambda)$ *of the matrix* $\mathtt{A}$. *If for each pair of indexes* $(i, j) \in \{1, 2, \ldots, n\}^2$ *the sequence* $g_{ij}(m)$, *for* $m > n$, *is determined as a solution of the recurrent relation*

$$\alpha_0 g_{ij}(m) + \alpha_1 g_{ij}(m-1) + \ldots + \alpha_n g_{ij}(m-n) = 0, \tag{15}$$

with the initial values $g_{ij}(1) = a_{ij}^{(1)}$, $g_{ij}(2) = a_{ij}^{(2)}$, $\ldots$, $g_{ij}(n) = a_{ij}^{(n)}$, *then the matrix* $\mathtt{G}_m = [g_{ij}(m)] \in \mathbf{C}^{n\times n}$ *is the* m^{th}*-power of the matrix* $\mathtt{A}$ $(m \in N)$.

Proof. We prove the equality $G_m = A^m$ by induction over $m \in N$. Indeed, for $m = 1, \ldots, n$ the statement is true. Let $m > n$. Let us assume that $G_k = A^k$ is true for each $k < m$. Then for $k = m$, let us note that $g_{ij}(k)$ fulfils

$$g_{ij}(m) = -\frac{1}{\alpha_0}\Big(\alpha_1 g_{ij}(m-1) + \ldots + \alpha_n g_{ij}(m-n)\Big), \tag{16}$$

where $\alpha_0 = (-1)^n$. From the previous equality, by the Cayle-Hamilton theorem, it follows that

$$\begin{aligned} G_m &= -\frac{1}{\alpha_0}\Big(\alpha_1 G_{m-1} + \ldots + \alpha_n G_{m-n}\Big) \\ &= -\frac{1}{\alpha_0}\Big(\alpha_1 A^{m-1} + \ldots + \alpha_n A^{m-n}\Big) = A^m. \end{aligned} \tag{17}$$

□

By Theorem 1 and Lemmas 3, 2, the following statement follows.

Theorem 3. *Let* $\mathtt{A} = [a_{ij}]$ *be the matrix determined by* (2) *and let* $\mathtt{A}^m = [a_{ij}^{(m)}]$ *be the* m^{th}*-power of the matrix* $\mathtt{A}$ *determined for each pair of the indexes* $(i, j) \in \{1, 2, \ldots, n\}^2$, *for* $m > n$, *by an explicit form of the elements* $a_{ij}^{(m)}$ *based on a recurrent relation*

$$\alpha_0 a_{ij}^{(m)} + \alpha_1 a_{ij}^{(m-1)} + \ldots + \alpha_n a_{ij}^{(m-n)} = 0. \tag{18}$$

Initial values $a_{ij}^{(k)}$ *are determined as* (i, j)*-elements of the matrix* $\mathtt{A}^k$ $(k = 1, 2, \ldots, n)$ *and* α_i*'s* $(i = 0, 1, \ldots, n)$ *are coefficients of the characteristic polynomial* $P_n(\lambda)$ *determined by* (11). *Then, by the formula* (3), *the number* $f(k)$ *of meaningful composition of* k^{th}*-order of functions from* $\mathcal{A}_n$, *is explicitly determined.*

3. Examples from vector analysis

We present some examples of counting the numbers of meaningful differential operations of higher order in vector analysis according to [2]. Let us start with the sets of functions

$$\mathrm{A}_i = \{\mathrm{f} : \mathbf{R}^n \longrightarrow \mathbf{R}^{\binom{n}{i}} \,|\, f_1, \ldots, f_{\binom{n}{i}} \in C^\infty(\mathbf{R}^n)\}, \tag{19}$$

for $i = 0, 1, \ldots, m$, with $m = [n/2]$. Let $\Omega^r(\mathbf{R}^n)$ be the space (module) of differential forms of degree $r = 0, 1, \ldots, n$ on the space $\mathbf{R}^n$ over the ring A_0. For each r let us choose the order of basis elements $dx_{i_1} \wedge \ldots \wedge dx_{i_r}$. For each i we determine

$$\varphi_i : \Omega^i(\mathbf{R}^n) \rightarrow \mathrm{A}_i \ (0 \leq i \leq m) \text{ and } \varphi_{n-i} : \Omega^{n-i}(\mathbf{R}^n) \rightarrow \mathrm{A}_i \ (0 \leq i < n-m), \tag{20}$$

by forming $\binom{n}{i}$-tuple of coefficients with respect to the basis elements in the given order. Notice that, in comparison to [2], we additionally consider the order of basis elements. Consequently, previously introduced functions are isomorphisms from the space of differential forms into the space of vector functions. Next, we use the well-known fact that $\Omega^i(\mathbf{R}^n)$ and $\Omega^{n-i}(\mathbf{R}^n)$ are spaces of the same dimension $\binom{n}{i}$, for $i = 0, 1, \ldots, m$. They can be identified with A_i, using the corresponding isomorphism (20). Let us define *differential operations of the first order* via exterior differentiation operator d as follows

$$\nabla_r = \varphi_r \circ d \circ \varphi_{r-1}^{-1} \quad (1 \leq r \leq n). \tag{21}$$

Thus, the following diagrams commute:

$$\begin{array}{ccc} \Omega^{i-1} & \xrightarrow{\ d\ } & \Omega^{i} \\ \uparrow{\scriptstyle \varphi_{i-1}^{-1}} & & \downarrow{\scriptstyle \varphi_i} \\ A_{i-1} & \xrightarrow{\ \nabla_i\ } & A_i \end{array} \quad (1 \leq i \leq m) \qquad \begin{array}{ccc} \Omega^{n-(i+1)} & \xrightarrow{\ d\ } & \Omega^{n-i} \\ \uparrow{\scriptstyle \varphi_{n-(i+1)}^{-1}} & & \downarrow{\scriptstyle \varphi_{n-i}} \\ A_{i+1} & \xrightarrow{\ \nabla_{n-i}\ } & A_i \end{array} \quad (0 \leq i < n-m) \tag{22}$$

Hence, the differential operations ∇_r determine functions so that (1) is fulfilled. Let us define *differential operations of the higher order* as meaningful compositions of higher order of functions from the set $\mathcal{A}_n = \{\nabla_1, \ldots, \nabla_n\}$. Let us consider in the next sections the concrete dimensions $n = 3, 4, 5, \ldots, 10$.

Three-dimensional vector analysis. In the real three-dimensional space $\mathbf{R}^3$ we consider the following sets

$$\mathrm{A}_0 = \{f : \mathbf{R}^3 \longrightarrow \mathbf{R} \,|\, f \in C^\infty(\mathbf{R}^3)\} \text{ and } \mathrm{A}_1 = \{\vec{f} : \mathbf{R}^3 \longrightarrow \mathbf{R}^3 \,|\, \vec{f} \in \vec{C}^\infty(\mathbf{R}^3)\}. \tag{23}$$

Let dx, dy, dz, respectively, be the basis vectors of the space of 1-forms and let $dy \wedge dz$, $dz \wedge dx$, $dx \wedge dy$ respectively be the basis vectors of the space of 2-forms. Thus, over the sets A_0 and A_1 there exist $m = 3$ differential operations of the first order

$$\begin{aligned} \nabla_1 f &= \operatorname{grad} f : \mathrm{A}_0 \longrightarrow \mathrm{A}_1, \\ \nabla_2 \vec{f} &= \operatorname{curl} \vec{f} : \mathrm{A}_1 \longrightarrow \mathrm{A}_1, \\ \nabla_3 \vec{f} &= \operatorname{div} \vec{f} : \mathrm{A}_1 \longrightarrow \mathrm{A}_0. \end{aligned} \tag{24}$$

Under the previous choice of order of basis vectors of spaces of 1-forms and 2-forms, the previously defined operations of the first order coincide with differential operations of first order in the classical vector analysis. It is a known fact that there are $m = 5$ differential operations of the second order. In the article [1] it is proved that there exists $m = 8$ differential operations of the third order. Further, in the article [2], it is proved that there exist F_{k+3} differential operations of the k^{th}-order, where F_k is k^{th}-Fibonacci's number. Here we give the proof of the previous statement, based on results of the second part of this paper. Namely, using Theorem 3, we obtain

$$\mathtt{A}^k = \begin{bmatrix} F_{k-1} & F_k & F_k \\ F_{k-1} & F_k & F_k \\ F_{k-2} & F_{k-1} & F_{k-1} \end{bmatrix} \tag{25}$$

and

$$f(k) = v_3 \cdot \mathtt{A}^{k-1} \cdot v_3^T = F_{k-3} + 4F_{k-2} + 4F_{k-1} = F_{k+3}. \tag{26}$$

Multidimensional vector analysis. In the real n-dimensional space $\mathbf{R}^n$ the number of differential operations is determined by the corresponding recurrent formulas, which for the dimension $n = 3, 4, 5, \ldots, 10$, we cite according to [2]:

dimension:	recurrent relations for the number of meaningful operations:
$n = 3$	$f(k+2) = f(k+1) + f(k)$
$n = 4$	$f(k+2) = 2f(k)$
$n = 5$	$f(k+3) = f(k+2) + 2f(k+1) - f(k)$
$n = 6$	$f(k+4) = 3f(k+2) - f(k)$
$n = 7$	$f(k+5) = f(k+3) + 3f(k+2) - 2f(k+1) - f(k)$
$n = 8$	$f(k+4) = 4f(k+2) - 3f(k)$
$n = 9$	$f(k+5) = f(k+4) + 4f(k+3) - 3f(k+2) - 3f(k+1) + f(k)$
$n = 10$	$f(k+6) = 5f(k+4) - 6f(k+2) + f(k)$

For the dimensions $n = 3$, as we have shown above, the number of differential operations of higher order is determined via Fibonacci numbers. Also, this is true for the dimension $n = 6$. Namely, using Theorem 3, we obtain

$$(27)\quad \mathtt{A}^k = \begin{cases} \begin{bmatrix} F_{2p-1} & 0 & F_{2p} & 0 & F_{2p} & 0 \\ 0 & F_{2p} & 0 & F_{2p-1} & 0 & F_{2p} \\ F_{2p-2} & 0 & F_{2p-1} & 0 & F_{2p-1} & 0 \\ 0 & F_{2p} & 0 & F_{2p-1} & 0 & F_{2p} \\ F_{2p-1} & 0 & F_{2p} & 0 & F_{2p} & 0 \\ 0 & F_{2p-1} & 0 & F_{2p-2} & 0 & F_{2p-1} \end{bmatrix} & : \; k = 2p, \\ \begin{bmatrix} 0 & F_{2p+1} & 0 & F_{2p} & 0 & F_{2p+1} \\ F_{2p} & 0 & F_{2p+1} & 0 & F_{2p+1} & 0 \\ 0 & F_{2p} & 0 & F_{2p-1} & 0 & F_{2p} \\ F_{2p} & 0 & F_{2p+1} & 0 & F_{2p+1} & 0 \\ 0 & F_{2p+1} & 0 & F_{2p} & 0 & F_{2p+1} \\ F_{2p-1} & 0 & F_{2p} & 0 & F_{2p} & 0 \end{bmatrix} & : \; k = 2p+1; \end{cases}$$

and

$$(28)\qquad f(k) = v_6 \cdot \mathtt{A}^{k-1} \cdot v_6^T = 2F_{k-3} + 8F_{k-2} + 8F_{k-1} = 2 \cdot F_{k+3}.$$

For higher dimensions, that is for $n = 4, 5, 7, 8, 9, 10$, the roots of corresponding characteristic polynomials are not related to Fibonacci numbers.

Finally, let us outline that for all dimensions $n = 3, 4, 5, 6, 7, 8, 9, 10$, the values of the function $f(k)$, for initial values of the argument k, are given in database of integer sequences [4] as sequences $A020701$ ($n = 3$), $A090989$ ($n = 4$), $A090990$ ($n = 5$), $A090991$ ($n = 6$), $A090992$ ($n = 7$), $A090993$ ($n = 8$), $A090994$ ($n = 9$), $A090995$ ($n = 10$), respectively.

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