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QUATERNIONIC MAPS BETWEEN A HYPER-KÄHLER MANIFOLD AND A 3-ALMOST CONTACT MANIFOLD

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Abstract. We prove that any quaternionic map between a hyper-Kähler manifold and a 3-almost contact manifold with a certain property is a harmonic map and we give some results about the stability of such a map and about the stability of a quaternionic map between hyper-Kähler manifolds.

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1. Preliminaries

Let us recall that a hyper almost complex manifold is a manifold endowed with three almost complex structures, J_α, $\alpha = \overline{1,3}$, satisfying the quaternionic identities

$$J_\gamma = J_\alpha J_\beta = -J_\beta J_\alpha, \tag{1.1}$$

for any even permutation $\{\alpha, \beta, \gamma\}$ of $\{1, 2, 3\}$. If these three almost complex structures are Kähler then the manifold is called a hyper-Kähler manifold.

For any real numbers a, b, c with $a^2 + b^2 + c^2 = 1$, one obtains a covariant complex structure $aJ_1 + bJ_2 + cJ_3$. As in [3], we shall refer this $\mathbb{S}^2$-family of complex structures as the hyper-Kähler $\mathbb{S}^2$. Therefore, $SO(3)$ acts naturally on the covariant complex structures. Every $SO(3)$ matrix preserves the identities 1.1. A hyper-Kähler manifold is of dimension $4n$.

In order to introduce the 3-almost contact manifolds and the hyperframed manifolds let us recall some basic notions and properties of the framed φ-manifolds.

Let M be an m-dimensional smooth manifold endowed with a tensor field φ of type $(1, 1)$, satisfying the algebraic condition

$$\varphi^3 + \varphi = 0. \tag{1.2}$$

The geometric structure on M defined by φ is called a φ-structure of rank r if the rank r of φ is constant on M and, in this case, M is called a φ-manifold. It follows easily that r is an even number.

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If M is a φ-manifold and if there are $m-r$ vector fields ξ_i and $m-r$ differential 1-forms η_i satisfying

$$\varphi^2 = -I + \sum_{i=1}^{m-r} \eta_i \otimes \xi_i, \quad \eta_i(\xi_j) = \delta_j^i, \tag{1.3}$$

where $i, j = 1, 2, ..., m-r$, M is said to be globally framed or to have a framed φ-structure. In this case M is called a globally framed φ-manifold or, simply, a framed φ-manifold. From (1.3), by some algebraic computations, one obtains,

$$\varphi\xi_i = 0, \quad \eta_i \circ \varphi = 0, \quad \varphi^3 + \varphi = 0. \tag{1.4}$$

If $m = 2n+1$ and $rank\ \varphi = 2n$ one obtains an almost contact structure on M.

Let M be an m-dimensional globally framed φ-manifold with structure tensors (φ, ξ_i, η_i) with $rank\ \varphi = r$, and consider the manifold $M \times \mathbb{R}^{m-r}$. We denote a vector field on $M \times \mathbb{R}^{m-r}$ by $(X, \sum_{i=1}^{m-r} f_i \frac{\partial}{\partial t^i})$ where X is tangent to M, $\{t^1, ..., t^{m-r}\}$ are the usual coordinates on $\mathbb{R}^{m-r}$, and $\{f_1, ..., f_{m-r}\}$ are functions on $M \times \mathbb{R}^{m-r}$. Define an almost complex structure on $M \times \mathbb{R}^{m-r}$ by

$$J(X, \sum_{i=1}^{m-r} f_i \frac{\partial}{\partial t^i}) = (\varphi X - \sum_{i=1}^{m-r} f_i \xi_i, \sum_{i=1}^{m-r} \eta_i(X) \frac{\partial}{\partial t^i}).$$

It is easy to check that $J^2 = -I$. If J is integrable we say that the framed φ-structure is normal. A framed φ-structure is normal if the tensor field S of type (1,2) defined by

$$S = N_\varphi + \sum_{i=1}^{m-r} d\eta_i \otimes \xi_i, \tag{1.5}$$

vanishes, (see [7]), where $N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X,Y]$, for $X, Y \in \chi(M)$, is the Nijenhuis tensor field of φ.

If g is a (semi-)Riemannian metric on M such that

$$g(\varphi X, \varphi Y) = g(X,Y) - \sum_{i=1}^{m-r} \eta_i(X)\eta_i(Y), \tag{1.6}$$

then we say that $(\varphi, \xi_i, \eta_i, g)$ is a metric framed φ-structure and M is called a metric framed φ-manifold. The metric g is called an associated (semi-)Riemannian metric.

The fundamental 2-form Ω of the considered metric framed φ-manifold M, is defined just like in the case of the almost Hermitian and almost contact metric manifold, by $\Omega = g(X, \varphi Y)$, for any $X, Y \in \chi(M)$.

The framed φ-manifold M with structure tensors (φ, ξ_i, η_i) is called a $\mathcal{C}$-manifold if it is normal, $d\Omega = 0$ and $d\eta_i = 0$, $i = 1, ..., m-r$, (see [2]).

If on an almost contact manifold (M, φ, ξ, η) it is defined an associated Riemannian metric g then $(M, \varphi, \xi, \eta, g)$ is called an almost contact metric manifold. If on an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ we have $\Omega = d\eta$, where Ω is the fundamental 2-form on M, then we say that $(M, \varphi, \xi, \eta, g)$ is a contact metric manifold. If for an almost contact metric structure (φ, ξ, η, g) which is normal we have $d\eta = 0$ and $d\Omega = 0$, then $(N, \varphi, \xi, \eta, g)$ is called a cosymplectic manifold.

In [1] the following result is proved.

Theorem 1.1. *An almost contact metric structure (φ, ξ, η, g) is cosymplectic if and only if φ is parallel.*

In the same way one obtains

Theorem 1.2. *If $(M, \varphi, \xi_i, \eta_i, g)$ is a $\mathcal{C}$-manifold then φ is parallel.*

In 1969 in [13] and in 1970 in [11], the authors defined the almost contact 3-structure (or the coquaternionic structure) on an odd dimensional manifold M, as follows.

If the manifold M admits three almost contact structure $(\varphi_\alpha, \xi^\alpha, \eta^\alpha)$, $\alpha = \overline{1,3}$, satisfying

$$\varphi_\gamma = \varphi_\alpha\varphi_\beta - \eta^\beta \otimes \xi^\alpha = -\varphi_\beta\varphi_\alpha + \eta^\alpha \otimes \xi^\beta,$$
$$\xi^\gamma = \varphi_\alpha \xi^\beta = -\varphi_\beta \xi^\alpha, \quad \eta^\gamma = \eta^\alpha \circ \varphi_\beta = -\eta^\beta \circ \varphi_\alpha, \tag{1.7}$$

for any even permutation $\{\alpha, \beta, \gamma\}$ of $\{1, 2, 3\}$, then the manifold is said to have an almost contact 3-structure.

It is proved (see [11]) that there exists an associated metric to each of this three structures. If all structures are cosymplectic, then we call the manifold M a 3-cosymplectic manifold.

As a generalization of the notion of hyper almost complex manifold and the notion of 3-almost contact manifold we defined in [4] and [6] the hyperframed manifolds as follows.

If a differentiable manifold M admits three framed φ_α-structures, $(\varphi_\alpha, \xi_a^\alpha, \eta_a^\alpha)$, such that dim $M - rank\ \varphi_\alpha = n$, for any $\alpha = 1, 2, 3$, satisfying the following, for any even permutation (α, β, γ) of $(1, 2, 3)$,

$$\varphi_\gamma = \varphi_\alpha\varphi_\beta - \sum_{a=1}^n \eta_a^\beta \otimes \xi_a^\alpha = -\varphi_\beta\varphi_\alpha + \sum_{a=1}^n \eta_a^\alpha \otimes \xi_a^\beta,$$
$$\xi_a^\gamma = \varphi_\alpha \xi_a^\beta = -\varphi_\beta \xi_a^\alpha, \quad \eta_a^\gamma = \eta_a^\alpha \circ \varphi_\beta = -\eta_a^\beta \circ \varphi_\alpha, \tag{1.8}$$

then the manifold is said to be a hyperframed manifold. A hyperframed manifold is of dimension $4m + 3n$.

Obviously a 3-almost contact manifold is a hyperframed manifold.

Note that for any real numbers p, q, r with $p^2 + q^2 + r^2 = 1$ we obtain a framed φ-structure $(p\varphi_1 + q\varphi_2 + r\varphi_3, p\xi_a^1 + q\xi_a^2 + r\xi_a^3, p\eta_a^1 + q\eta_a^2 + r\eta_a^3)$, and that every $SO(3)$ matrix preserve 1.7 and 1.8. We shall refer this $\mathbb{S}^2$-family of almost contact structures as the 3-almost contact $\mathbb{S}^2$.

In [6], we prove that there exists a Riemannian metric associated to all three framed φ_α-structures. If the framed φ_α-structures are $\mathcal{C}$-structures we call the manifold M a hyper $\mathcal{C}$-manifold.

2. Quaternionic maps

Definition 2.1. Let (M, J, g) be an almost Kähler manifold and let $(N, \varphi, \xi_a, \eta_a, h)$ be a metric framed φ-manifold. A smooth map $f : M \to N$ is called a $\pm(J, \varphi)$-holomorphic map if $df J = \pm\varphi df$, where $df : TM \to TN$ denotes the induced tangent map of f.

Definition 2.2. Let (M, J_α, g) be a hyper almost Kähler manifold and let $(N, \varphi_\alpha, \xi^\alpha, \eta^\alpha, h)$ be a metric 3-almost contact manifold. We call a smooth map $f : M \to N$ a quaternionic map if

$$A^{\alpha\beta}\varphi_\beta df J_\alpha = df, \tag{2.1}$$

where $A^{\alpha\beta}$ are the entries of a matrix A in $SO(3)$.

It is easy to verify that any $\pm(J, \varphi)$-holomorphic map with respect to an almost complex structure $aJ_1 + bJ_2 + cJ_3$, with $a^2 + b^2 + c^2 = 1$ and an almost contact structure $(p\varphi_1 + q\varphi_2 + r\varphi_3, p\xi^1 + q\xi^2 + r\xi^3, p\eta^1 + q\eta^2 + r\eta^3)$, with $p^2 + q^2 + r^2 = 1$, is a quaternionic map.

Since $SO(3)$ preserves the identities 1.1 and 1.7, we can choose the complex structures J_α for M and the almost contact structures $(\varphi_\beta, \xi^\beta, \eta^\beta)$ for N such that $A^{\alpha\beta} = \delta_{\alpha\beta}$ in 2.1. In the sequel, we shall assume that $A^{\alpha\beta} = \delta_{\alpha\beta}$.

In the following let us consider two manifolds M and N as in Definition 2.2, and suppose that M is compact. As in [3] (for the case of the maps between hyper-Kähler manifolds), for a smooth map $f : M \to N$, consider the energy functional

$$E(f) = \frac{1}{2} \parallel df \parallel^2 = \frac{1}{2}\int_M g^{ij} h_{mn} \partial_i f^m \partial_j f^n * 1,$$

where $*1$ is the volume element of M, the functional

$$E_T(f) = \int_M \sum_\alpha \langle J_\alpha, f^*\varphi_\alpha\rangle * 1 = \frac{1}{2}\int_M \sum_\alpha (J_\alpha)^{pq}(\varphi_\alpha)_{mk}\partial_p f^m \partial_q f^k * 1,$$

and

$$I(f) = \frac{1}{2}\int_M (\mid df - \sum_\alpha \varphi_\alpha df J_\alpha \mid^2 + \langle \sum_\alpha \eta^\alpha \circ df \circ J_\alpha, \sum_\alpha \eta^\alpha \circ df \circ J_\alpha\rangle) * 1.$$

Remark 2.1. Since for a quaternionic map one obtains easily that $\sum_\alpha \eta^\alpha \circ df \circ J_\alpha = 0$, it follows that f is a quaternionic map if and only if $I(f) = 0$.

Remark 2.2. Note that, if J is an almost Kähler structure on (M, g), with the fundamental 2-form ω, and if (φ, ξ, η, h) is a metric almost contact structure on N such that the fundamental 2-form Ω on N is closed, then any $\pm(J, \varphi)$-holomorphic map between M and N is a minimum of the energy integral in its homotopy class, since

$$E(f) + \int_M \langle J, f^*\varphi\rangle * 1 = \frac{1}{4}\int_M (\mid df - \varphi df J \mid^2 + \langle \eta \circ df, \eta \circ df\rangle) * 1, \tag{2.2}$$

where we use the fact that $\int_M \langle J, f^*\varphi\rangle * 1 = \int_M \langle \omega, f^*\Omega\rangle * 1$ which is a homotopy invariant, (see [5]).

Theorem 2.3. *Let $f : M \to N$ be a smooth map between two manifolds M and N as above. Then*

$$E(f) + E_T(f) = \frac{1}{4} I(f). \tag{2.3}$$

If the fundamental 2-forms on N corresponding to the three almost contact structures are closed and if f is a quaternionic map, then f is a minimum of the energy in its homotopy class.

Proof. After a straightforward computation one obtains

$$I(f) = E(f) + 2E_T(f) + \frac{1}{2}\int_M (\langle \sum_\alpha \varphi_\alpha df J_\alpha, \sum_\alpha \varphi_\alpha df J_\alpha\rangle +$$

$$+ \langle \sum_\alpha \eta^\alpha \circ df \circ J_\alpha, \sum_\alpha \eta^\alpha \circ df \circ J_\alpha\rangle) * 1.$$

Let $\{e_i, J_1 e_i, J_2 e_i, J_3 e_i\}$ be an orthonormal local framed field on M adapted to the hyper almost Kähler structure. One obtains

$$\frac{1}{2}\int_M \langle \sum_\alpha \varphi_\alpha df J_\alpha, \sum_\alpha \varphi_\alpha df J_\alpha\rangle * 1 = \frac{1}{2}\int_M \sum_{i=1}^m [h(\sum_\alpha \varphi_\alpha df J_\alpha(e_i), \sum_\alpha \varphi_\alpha df J_\alpha(e_i)) +$$

$$+ \sum_\beta h(\sum_\alpha \varphi_\alpha df J_\alpha(J_\beta e_i), \sum_\alpha \varphi_\alpha df J_\alpha(J_\beta e_i)) * 1,$$

where $dim\ M = 4m$. Using the definition of the hyper almost Kähler structure and the definition of the almost contact 3-structure, it follows easily

$$\frac{1}{2}\int_M \langle \sum_\alpha \varphi_\alpha df J_\alpha, \sum_\alpha \varphi_\alpha df J_\alpha\rangle * 1 = 3E(f) + 2E_T(f) -$$

$$-\frac{1}{2}\int_M \sum_{i=1}^m \{[\sum_\alpha \eta^\alpha(df J_\alpha(e_i))][\sum_\alpha \eta^\alpha(df J_\alpha(e_i))] +$$

$$+ \sum_\beta [\sum_\alpha \eta^\alpha(df J_\alpha(J_\beta e_i))][\sum_\alpha \eta^\alpha(df J_\alpha(J_\beta e_i))]\} * 1,$$

Hence $E(f)+E_T(f)=\frac{1}{4}I(f)$. For a pair of structures (J^α,φ^β), E_T is a homotopy invariant, (see [5]). If f is a quaternionic map, that means $I(f)=0$, then f is a minimum of the energy in its homotopy class.

Defining the quaternionic maps between a hyper almost Kähler manifold and a hyperframed manifold in the same way as the quaternionic maps with the target manifold a 3-almost contact manifold, one obtains the following

Theorem 2.4. *Let (M,J_α,g) be a compact hyper almost Kähler manifold and let $(N,\varphi_\alpha,\xi_a^\alpha,\eta_a^\alpha,h)$, $r=\overline{1,r}$, be a metric hyperframed manifold. Then, for any smooth map $f:M\to N$, we have $E(f)+E_T(f)=\frac{1}{4}I(f)$, where $E(f)$ and $E_T(f)$ are defined as above and*

$$I(f)=\frac{1}{2}\int_M(\mid df-\sum_\alpha\varphi_\alpha df J_\alpha\mid^2+\sum_{a=1}^r\langle\sum_\alpha\eta_a^\alpha\circ df\circ J_\alpha,\sum_\alpha\eta_a^\alpha\circ df\circ J_\alpha\rangle)*1.$$

If the fundamental 2-forms on N corresponding to the three framed φ_α-structures are closed and if f is a quaternionic map, then f is a minimum of the energy in its homotopy class.

Remark 2.5. Note that a map f defined as in the previous theorem is quaternionic if and only if $I(f)=0$.

Just like in [3], where the target manifold is a hyper-Kähler manifold, a criterion which detects when a quaternionic map is a (J,φ)-holomorphic map with respect to a structure in the hyper-Kähler $\mathbb{S}^2$ and a structure in the 3-almost contact $\mathbb{S}^2$, can be obtained.

Theorem 2.6. *Let $f:M\to N$ be a quaternionic map between a hyper-Kähler manifold and a 3-almost contact manifold. Let A be a 3×3-matrix whose (α,β)-entries are $-\int_M\langle J_\alpha,f^*\varphi_\beta\rangle*1$ for $\alpha,\beta=1,2,3$. Then*

$$(tr\ A)^2\geqslant max\{eigenvalues\quad of\quad AA^t\}$$

and the equality holds if and only if f is a (J,φ)-holomorphic map with respect to a structure in the hyper-Kähler $\mathbb{S}^2$ and a structure in the 3-almost contact $\mathbb{S}^2$.

Proof. Set $J=X^\alpha J_\alpha$ with $\mid X\mid=1$ and $\varphi=Y^\beta\varphi_\beta$, $\eta=\sum_\beta Y^\beta\eta^\beta$, $\xi=\sum_\beta Y^\beta\xi^\beta$, with $\mid Y\mid=1$. Then, from 2.2 one obtains

$$E(f)=XAY^t+\frac{1}{4}\int_M(\mid df-\varphi df J\mid^2+\langle\eta\circ df\circ J,\eta\circ df\circ J\rangle)*1.$$

Since f is quaternionic, from 2.3, we have $E(f)=tr\ A$. It follows that $tr\ A\geqslant XA^tY$, for any unit vectors X,Y. The equality holds if and only if f is holomorphic with respect J and φ.

All eigenvalues of AA^t are nonnegative. Let $4\lambda^2$ be an eigenvalue of AA^t with $\lambda \geqslant 0$. Then there is a unit vector Y_λ in $\mathbb{R}^3$ such that $AA^tY_\lambda^t = 4\lambda^2Y_\lambda^t$. Hence

$$Y_\lambda(A^tAY_\lambda^t) = Y_\lambda(AA^tY_\lambda^t) = 4\lambda^2Y_\lambda Y_\lambda^t.$$

We have $\mid AY_\lambda^t \mid = 2\lambda$. Suppose $\lambda \neq 0$ and we choose $X_\lambda^t = \frac{1}{2\lambda}AY_\lambda^t$. It follows $X_\lambda AY_\lambda^t = 2\lambda$. Then $tr\ A \geqslant 2\lambda$. Hence $(tr\ A)^2 \geqslant max\{eigenvalues\ \ of\ \ AA^t\}$. If all the eigenvalues of AA^t are 0 that is trivially true.

For the second part of the theorem let us consider the Lagrange multiplier

$$F(X,Y) = XAY^t - \lambda(\mid X \mid^2 - 1) - \mu(\mid Y \mid^2 - 1).$$

If XAY^t attains its maximum at two unit vector fields $V, W \in \mathbb{R}^3$, then $F_X = 0$, $F_Y = 0$ in $X = V$, $Y = W$. One obtains $AW^t = 2\lambda V^t$, $VA = 2\mu W$. Then $2\lambda = 2\lambda \mid V \mid^2 = VAW^t = 2\mu \mid W \mid^2 = 2\mu$. This implies $A^tAW^t = 2\lambda A^tV^t = 4\lambda\mu W^t = 4\lambda^2W^t$. That is $4\lambda^2$ is an eigenvalue of A^tA. If f is (J,φ)-holomorphic with respect to a complex structure in the hyper-Kähler $\mathbb{S}^2$, on M and an almost contact structure in the 3-almost contact $\mathbb{S}^2$, on N, then XAY^t attains its maximum $tr\ A$. On the other hand, $tr\ A = XAY^t = 2\lambda$ and then $4\lambda^2$ is an eigenvalue of A^tA.

Conversely, if $tr\ A = max\{eigenvalues\ \ of\ \ AA^t\}$, we take $2\lambda = tr\ A$ and it follows that $4\lambda^2$ is an eigenvalue of A^tA. Suppose $\mid Y \mid = 1$ and $A^tAY^t = 4\lambda^2Y^t$. It follows that $YA^tAY^t = 4\lambda^2$ and then $\mid AY^t \mid^2 = 4\lambda^2$. Taking $X^t = \frac{1}{2\lambda}AY^t$ one obtains $XAY^t = 2\lambda = tr\ A$. Hence f is a (J,φ)-holomorphic map with respect to a complex structure in the hyper-Kähler $\mathbb{S}^2$, on M and an almost contact structure in the 3-almost contact $\mathbb{S}^2$, on N. Note that if A is the zero matrix, then the quaternionic map is a constant.

3. The stability of the quaternionic maps

Let $f : M \to N$ be a smooth map between two Riemannian manifolds (M,g) and (N,h). We should recall some notions and results related to the induced bundle over M of TN, as they are presented in [14].

Let $f^{-1}(TN)$ be the induced bundle over M of TN defined as follows, denote by $\pi : TN \to N$ the projection. Then

$$f^{-1}TN = \{(x,u) \in M \times TN, \pi(u) = f(x), x \in M\} = \bigcup_{x \in M} T_{f(x)}N.$$

The set of all C^∞-sections of $f^{-1}TN$, denoted by $\Gamma(f^{-1}TN)$ is

$$\Gamma(f^{-1}TN) = \{V : M \to TN, C^\infty - map,\ V(x) \in T_{f(x)}N, x \in M\}.$$

Denote by ∇^M, ∇^N, the Levi-Civita connections on (M,g) and (N,h) respectively. Then, for a smooth map f between (M,g) and (N,h), we define the induced connection $\widetilde{\nabla}$ on the induced bundle $f^{-1}TN$ as follows, for $X \in \chi(M), V \in \Gamma(f^{-1}TN)$, define $\widetilde{\nabla}_XV \in \Gamma(f^{-1}TN)$ by $\widetilde{\nabla}_XV = \nabla^N_{dfX}V$.

Then the connection $\widetilde{\nabla}$ and the metric h are compatible, that is, for $V_1, V_2 \in \Gamma(f^{-1}TN), X \in \chi(M)$ we have $X(h(V_1,V_2)) = h(\widetilde{\nabla}_XV_1,V_2) + h(V_1,\widetilde{\nabla}_XV_2)$.

Theorem 3.1. *Let $f : M \to N$ be a quaternionic map between two hyper-Kähler manifolds, (M, J_α, g) and $(N, \mathcal{J}_\alpha, h)$. If M is compact, then*

$$\int_M h(J_f V, V) * 1 = \frac{1}{4}\int_M h(DV, DV) * 1 \geqslant 0,$$

where $V \in \Gamma(f^{-1}TN)$, and J_f is the Jacobi operator of f defined by

$$J_f V = -\sum_{i=1}^{m}(\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i} - \widetilde{\nabla}_{\nabla_{e_i}e_i})V - \sum_{i=1}^{m} R^N(V, dfe_i)dfe_i, V \in \Gamma(f^{-1}TN),$$

where R^N denote the curvature tensor on N. For each $V \in \Gamma(f^{-1}TN)$, DV is an element of $\Gamma(f^{-1}TN \otimes T^*M)$ defined by

$$DV(X) = \sum_\alpha \mathcal{J}_\alpha \widetilde{\nabla}_{J_\alpha X} V - \widetilde{\nabla}_X V, X \in \chi(M),$$

Then

1) f is weakly stable, that is, each eigenvalue of J_f is nonnegative.

2) $\ker J_f = \{V \in \Gamma(f^{-1}TN), DV = 0\}$.

Proof. Let $\{e_i, J_1e_i, J_2e_i, J_3e_i\}_{i=1}^m$ be an orthonormal local frame field on M adapted to the hyper-Kähler structure on M.

According to the definition of DV, one obtains, for any $V \in \Gamma(f^{-1}TN)$,

$$h(DV, DV) = \sum_{i=1}^{m}\{h(DV(e_i), DV(e_i)) + \sum_\alpha h(DV(J_\alpha e_i), DV(J_\alpha e_i))\} = \tag{3.1}$$

$$= \sum_{i=1}^{m}\{h(\sum_\alpha \mathcal{J}_\alpha \widetilde{\nabla}_{J_\alpha e_i}V - \widetilde{\nabla}_{e_i}V, \sum_\alpha \mathcal{J}_\alpha \widetilde{\nabla}_{J_\alpha e_i}V - \widetilde{\nabla}_{e_i}V) +$$

$$+ \sum_\alpha h(\sum_\beta \mathcal{J}_\beta \widetilde{\nabla}_{J_\beta J_\alpha e_i}V - \widetilde{\nabla}_{J_\alpha e_i}V, \sum_\beta \mathcal{J}_\beta \widetilde{\nabla}_{J_\beta J_\alpha e_i}V - \widetilde{\nabla}_{J_\alpha e_i}V)\} =$$

$$= \sum_{i=1}^{m}\{4h(\widetilde{\nabla}_{e_i}V, \widetilde{\nabla}_{e_i}V) + 4\sum_\alpha h(\widetilde{\nabla}_{J_\alpha e_i}V, \widetilde{\nabla}_{J_\alpha e_i}V) - 8h(\widetilde{\nabla}_{e_i}V, \sum_\alpha \mathcal{J}_\alpha \widetilde{\nabla}_{J_\alpha e_i}V) +$$

$$+8\sum_{\alpha\neq\beta} h(\mathcal{J}_\alpha \widetilde{\nabla}_{J_\alpha e_i}V, \mathcal{J}_\beta \widetilde{\nabla}_{J_\beta e_i}V)\},$$

since the two manifolds are hyper-Kähler.

Next we shall prove that

$$R^N(V, dfe_i)dfe_i + \sum_\alpha R^N(V, dfJ_\alpha e_i)dfJ_\alpha e_i = \tag{3.2}$$

$$= -\sum_\alpha \mathcal{J}_\alpha R^N(dfe_i, dfJ_\alpha e_i)V + \mathcal{J}_3 R^N(dfJ_2e_i, dfJ_1e_i)V +$$

$$+\mathcal{J}_2 R^N(dfJ_1e_i, dfJ_3e_i)V + \mathcal{J}_1 R^N(dfJ_3e_i, dfJ_2e_i)V.$$

Since f is a quaternionic map one obtains

$$R^N(V, dfe_i)dfe_i + \sum_\alpha R^N(V, dfJ_\alpha e_i)dfJ_\alpha e_i =$$

$$= \sum_\alpha \mathcal{J}_\alpha R^N(V, dfe_i)dfJ_\alpha e_i + \sum_\alpha \sum_\beta \mathcal{J}_\beta R^N(V, dfJ_\alpha e_i)dfJ_\beta J_\alpha e_i =$$

$$= -\sum_\alpha \mathcal{J}_\alpha R^N(dfe_i, V)dfJ_\alpha e_i + \sum_\alpha \mathcal{J}_\alpha R^N(dfJ_\alpha e_i, V)dfe_i + A =$$

$$= -\sum_\alpha \mathcal{J}_\alpha R^N(dfe_i, dfJ_\alpha e_i)V + A,$$

where A is the sum of the last three terms in the right side of 3.2 and where we used the formulas, for $X, Y, Z \in \chi(N)$

$$R^N(X,Y)Z + R^N(Y,X)Z = 0, \quad R^N(X,Y)Z + R^N(Y,Z)X + R^N(Z,X)Y = 0.$$

Since

$$\int_M h(J_f V, V) * 1 = \int_M \sum_{i=1}^m \{h(\widetilde{\nabla}_{e_i} V, \widetilde{\nabla}_{e_i} V) + \sum_\alpha h(\widetilde{\nabla}_{J_\alpha e_i} V, \widetilde{\nabla}_{J_\alpha e_i} V) -$$

$$-h(R^N(V, dfe_i)dfe_i, V) - \sum_\alpha h(R^N(V, dfJ_\alpha e_i)dfJ_\alpha e_i, V)\} * 1, (\text{see}[14]),$$

we have

$$\int_M [h(J_f V, V) - \frac{1}{4} h(DV, DV)] * 1 = \tag{3.3}$$

$$= \int_M \{\sum_{i=1}^m \{h(\sum_\alpha \mathcal{J}_\alpha R^N(dfe_i, dfJ_\alpha e_i)V, V) -$$

$$-h(A, V) + 2h(\widetilde{\nabla}_{e_i} V, \sum_\alpha \mathcal{J}_\alpha \widetilde{\nabla}_{J_\alpha e_i} V) - 2\sum_{\alpha \neq \beta} h(\mathcal{J}_\alpha \widetilde{\nabla}_{J_\alpha e_i} V, \mathcal{J}_\beta \widetilde{\nabla}_{J_\beta e_i} V)\} * 1.$$

We have

$$-h(\mathcal{J}_\alpha R^N(dfe_i, dfJ_\alpha e_i)V, V) = h(R^N(dfe_i, dfJ_\alpha e_i)V, \mathcal{J}_\alpha V) = \tag{3.4}$$

$$= h(\widetilde{\nabla}_{e_i} \widetilde{\nabla}_{J_\alpha e_i} V - \widetilde{\nabla}_{J_\alpha e_i} \widetilde{\nabla}_{e_i} V - \widetilde{\nabla}_{[e_i, J_\alpha e_i]} V, \mathcal{J}_\alpha V) =$$

$$= e_i(h(\widetilde{\nabla}_{J_\alpha e_i} V, \mathcal{J}_\alpha V)) - h(\widetilde{\nabla}_{J_\alpha e_i} V, \widetilde{\nabla}_{e_i} \mathcal{J}_\alpha V) -$$

$$-J_\alpha e_i(h(\widetilde{\nabla}_{e_i} V, \mathcal{J}_\alpha V)) + h(\widetilde{\nabla}_{e_i} V, \widetilde{\nabla}_{J_\alpha e_i} \mathcal{J}_\alpha V) - h(\widetilde{\nabla}_{\nabla_{e_i} J_\alpha e_i} V, \mathcal{J}_\alpha V) +$$

$$+h(\widetilde{\nabla}_{\nabla_{J_\alpha e_i} e_i} V, \mathcal{J}_\alpha V),$$

for any $\alpha = 1, 2, 3$, since $\widetilde{\nabla}$ and h are compatible, where ∇ denote the Levi-Civita connection on M. In the same way we can compute $h(A, V)$. If we denote

$$\sum_{i=1}^{m}\{h(\sum_{\alpha}\mathcal{J}_\alpha R^N(dfe_i, dfJ_\alpha e_i)V, V) - h(A, V)\} = B,$$

we obtain $B = B_1 + B_2$, where

$$B_1 = \sum_{i=1}^{m}\{\sum_{\alpha}[-e_i h(\widetilde{\nabla}_{J_\alpha e_i}V, \mathcal{J}_\alpha V) + J_\alpha e_i h(\widetilde{\nabla}_{e_i}V, \mathcal{J}_\alpha V) + h(\widetilde{\nabla}_{\nabla_{e_i}J_\alpha e_i}V, \mathcal{J}_\alpha V) -$$
$$-h(\widetilde{\nabla}_{\nabla_{J_\alpha e_i}e_i}V, \mathcal{J}_\alpha V)] + J_2 e_i h(\widetilde{\nabla}_{J_1 e_i}V, \mathcal{J}_3 V) - J_1 e_i h(\widetilde{\nabla}_{J_2 e_i}V, \mathcal{J}_3 V) -$$
$$-h(\widetilde{\nabla}_{\nabla_{J_2 e_i}J_1 e_i}V, \mathcal{J}_3 V) + h(\widetilde{\nabla}_{\nabla_{J_1 e_i}J_2 e_i}V, \mathcal{J}_3 V) + J_1 e_i h(\widetilde{\nabla}_{J_3 e_i}V, \mathcal{J}_2 V) -$$
$$-J_3 e_i h(\widetilde{\nabla}_{J_1 e_i}V, \mathcal{J}_2 V) - h(\widetilde{\nabla}_{\nabla_{J_1 e_i}J_3 e_i}V, \mathcal{J}_2 V) + h(\widetilde{\nabla}_{\nabla_{J_3 e_i}J_1 e_i}V, \mathcal{J}_2 V) +$$
$$+J_3 e_i h(\widetilde{\nabla}_{J_2 e_i}V, \mathcal{J}_1 V) - J_2 e_i h(\widetilde{\nabla}_{J_3 e_i}V, \mathcal{J}_1 V) - h(\widetilde{\nabla}_{\nabla_{J_3 e_i}J_2 e_i}V, \mathcal{J}_1 V) +$$
$$+h(\widetilde{\nabla}_{\nabla_{J_2 e_i}J_3 e_i}V, \mathcal{J}_1 V)\},$$

and

$$B_2 = \sum_{i=1}^{m}\{\sum_{\alpha}[h(\widetilde{\nabla}_{J_\alpha e_i}V, \widetilde{\nabla}_{e_i}\mathcal{J}_\alpha V) - h(\widetilde{\nabla}_{e_i}V, \widetilde{\nabla}_{J_\alpha e_i}\mathcal{J}_\alpha V)] -$$
$$-h(\widetilde{\nabla}_{J_1 e_i}V, \widetilde{\nabla}_{J_2 e_i}\mathcal{J}_3 V) + h(\widetilde{\nabla}_{J_2 e_i}V, \widetilde{\nabla}_{J_1 e_i}\mathcal{J}_3 V) - h(\widetilde{\nabla}_{J_3 e_i}V, \widetilde{\nabla}_{J_1 e_i}\mathcal{J}_2 V) +$$
$$+h(\widetilde{\nabla}_{J_1 e_i}V, \widetilde{\nabla}_{J_3 e_i}\mathcal{J}_2 V) - h(\widetilde{\nabla}_{J_2 e_i}V, \widetilde{\nabla}_{J_3 e_i}\mathcal{J}_1 V) + h(\widetilde{\nabla}_{J_3 e_i}V, \widetilde{\nabla}_{J_2 e_i}\mathcal{J}_1 V)\} =$$
$$= 2\{\sum_{i=1}^{m}[\sum_{\alpha}h(\mathcal{J}_\alpha\widetilde{\nabla}_{e_i}V, \widetilde{\nabla}_{J_\alpha e_i}V) + h(\widetilde{\nabla}_{J_1 e_i}\mathcal{J}_3 V, \widetilde{\nabla}_{J_2 e_i}V) +$$
$$+h(\widetilde{\nabla}_{J_3 e_i}\mathcal{J}_2 V, \widetilde{\nabla}_{J_1 e_i}V) + h(\widetilde{\nabla}_{J_2 e_i}\mathcal{J}_1 V, \widetilde{\nabla}_{J_3 e_i}V)]\} =$$
$$= -2\sum_{i=1}^{m}[h(\widetilde{\nabla}_{e_i}V, \sum_{\alpha}\mathcal{J}_\alpha\widetilde{\nabla}_{J_\alpha e_i}V) - 2\sum_{\alpha\neq\beta}h(\mathcal{J}_\alpha\widetilde{\nabla}_{J_\alpha e_i}V, \mathcal{J}_\beta\widetilde{\nabla}_{J_\beta e_i}V)],$$

since the two manifolds are hyper-Kähler.

Consider the vector field $X \in \chi(M)$ defined by 1

$$g(X, Y) = -\sum_{\alpha}h(\widetilde{\nabla}_{J_\alpha Y}V, \mathcal{J}_\alpha V),$$

for any $Y \in \chi(M)$. By a straightforward computation one obtains $div\ X = B_1$. From the expressions of B_1 and B_2 by using the Green's formula, it follows that

$$\int_M h(J_f V, V) * 1 = \frac{1}{4}\int_M h(DV, DV) * 1.$$

Theorem 3.2. *Let $f : M \to N$ be a quaternionic map between a hyper-Kähler manifold, (M, J_α, g) and a 3-cosymplectic manifold $(N, \varphi_\alpha, \xi^\alpha, \eta^\alpha, h)$. If M is compact, then*

$$\int_M h(J_f V, V) * 1 = \frac{1}{4}\int_M h(DV, DV) * 1 + \frac{1}{12}\int_M \sum_\alpha tr\ (\eta^\alpha \otimes \eta^\alpha)(DV, DV) * 1,$$

where, for any $V \in \Gamma(f^{-1}TN)$, $DV(X) = \sum_\alpha \varphi_\alpha \widetilde{\nabla}_{J_\alpha X} V - \widetilde{\nabla}_X V$, $X \in \chi(M)$, and

$$(\eta^\alpha \otimes \eta^\alpha)(DV, DV)(X, Y) = \eta^\alpha(DV(X))\eta^\alpha(DV(Y)).$$

Then

1) f is weakly stable, that is, each eigenvalue of J_f is nonnegative.

2) $\ker J_f = \{V \in \Gamma(f^{-1}TN), DV = 0\}$.

Proof. After a similar computation with that in Theorem 3.1, based on the definitions of the hyper-Kähler structure and of the 3-cosymplectic structure, on the definitions of the Levi-Civita connections on the domain and the target manifolds and on Theorem 1.1, one obtains that in the expression of $\int_M h(J_f V, V) * 1 - \frac{1}{4}\int_M h(DV, DV) * 1$ the following new terms arise

$$\sum_{i=1}^{m}\{\int_M \frac{1}{4}[\sum_\alpha\sum_\beta \eta_\alpha(\widetilde{\nabla}_{J_\beta e_i}V)\eta_\alpha(\widetilde{\nabla}_{J_\beta e_i}V) + \sum_\alpha \eta_\alpha(\widetilde{\nabla}_{e_i}V)\eta_\alpha(\widetilde{\nabla}_{e_i}V)] * 1+$$

$$+\int_M \frac{1}{2}[\sum_{\alpha\neq\beta} \eta_\alpha(\widetilde{\nabla}_{J_\alpha e_i}V)\eta_\beta(\widetilde{\nabla}_{J_\beta e_i}V) - \sum_{\alpha\neq\beta} \eta_\alpha(\widetilde{\nabla}_{J_\beta e_i}V)\eta_\beta(\widetilde{\nabla}_{J_\alpha e_i}V)] * 1-$$

$$-\int_M \frac{1}{2}[\eta_1(\widetilde{\nabla}_{e_i}V)(\eta_3(\widetilde{\nabla}_{J_2 e_i}V) - \eta_2(\widetilde{\nabla}_{J_3 e_i}V))+$$

$$+\eta_2(\widetilde{\nabla}_{e_i}V)(\eta_1(\widetilde{\nabla}_{J_3 e_i}V) - \eta_3(\widetilde{\nabla}_{J_1 e_i}V)) + \eta_3(\widetilde{\nabla}_{e_i}V)(\eta_2(\widetilde{\nabla}_{J_1 e_i}V) - \eta_1(\widetilde{\nabla}_{J_2 e_i}V))] * 1.$$

After a straightforward computation, one obtains that this is

$$\frac{1}{12}\int_M \sum_\alpha tr\ (\eta^\alpha \otimes \eta^\alpha)(DV, DV) * 1.$$

In the same way one obtains

Theorem 3.3. *Let $f : M \to N$ be a quaternionic map between a hyper-Kähler manifold, (M, J_α, g) and a hyper $\mathcal{C}$-manifold $(N, \varphi_\alpha, \xi^\alpha_a, \eta^\alpha_a, h)$, with $a = \overline{1, r}$. If M is compact, then*

$$\int_M h(J_f V, V) * 1 = \frac{1}{4}\int_M h(DV, DV) * 1 + \frac{1}{12}\int_M \sum_{a=1}^{r}\sum_\alpha tr\ (\eta^\alpha_a \otimes \eta^\alpha_a)(DV, DV) * 1.$$

Then

1) f is weakly stable, that is, each eigenvalue of J_f is nonnegative.

2) $\ker J_f = \{V \in \Gamma(f^{-1}TN), DV = 0\}$.

4. Quaternionic maps between a hyper-Kähler manifold with dimension 4 and a 3-cosymplectic manifold with dimension 3

Let M be a differential manifold which is of real dimension 4. Then the only hyper-Kähler structure on M is defined by the tensor fields of type $(1,1)$ whose matrix expressions take the following form, (see [9])

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ J_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{4.1}$$

$$J_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Concerning the 3-cosymplectic structures which can be defined on a 3-dimensional manifold we can state the following.

Let N be a 3-dimensional manifold. Let us consider on N the almost contact structures given in local coordinates by

$$\varphi_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \ \eta^1 = dx^1, \ \xi^1 = \frac{\partial}{\partial x^1},$$

$$\varphi_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \eta^2 = dx^2, \ \xi^2 = \frac{\partial}{\partial x^2}, \tag{4.2}$$

$$\varphi_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \eta^3 = dx^3, \ \xi^3 = \frac{\partial}{\partial x^3}.$$

It can be easily verified that this structures define an almost contact 3-structure on N. Moreover, let us consider the induced almost complex structures on $N \times \mathbb{R}$, defined by $J_i = (X, f\frac{d}{dt}) = (\varphi_i X - f\xi^i, \eta^i(X)\frac{d}{dt})$, $i = \overline{1,3}$, where $X \in \chi(N)$ and $f \in C^\infty(N \times \mathbb{R})$. It is proved in [1] that these tensor fields define three almost contact structures on $N \times \mathbb{R}$. After a straightforward computation one obtains that J_i actually are the tensor fields which define the unique hyper-Kähler structure on $N \times \mathbb{R}$. It follows that the almost contact structures defined on N are normal structures and then on N we obtained a 3-cosymplectic structure, (see [1]).

In the following let us consider a new 3-cosymplectic structure on N, $(\psi_i, \zeta^i, \theta^i)$, $i = \overline{1,3}$, such that the fundamental 2-forms, ω_i, $i = \overline{1,3}$, corresponding to each almost contact structure, are closed. Then, taking the induced almost complex structures on $N \times \mathbb{R}$ defined as above, and the metric G on $N \times \mathbb{R}$, defined by $G = g + dtdt$, where g is the Riemannian metric associated to all almost contact structures on N, which gives the fundamental 2-forms ω_i on N, one obtains for the corresponding fundamental 2-forms, Ω_i, $i = \overline{1,3}$, on $N \times \mathbb{R}$, that $\Omega_i = \omega_i - \theta^i \wedge dt$. Hence, if the 1-forms θ^i are closed and the 2-forms ω_i are also closed, then $d\Omega_i = 0$. But, using a Hitchin's lemma one obtains that the induced almost complex structures on $N \times \mathbb{R}$ are normal and then they define a hyper-Kähler structure on $N \times \mathbb{R}$, (see [8], [10]). Since this structure is unique and since the induced Kähler structures on $N \times \mathbb{R}$ are uniquely determined by the cosymplectic structures on N, it follows that the structure defined by 4.2 is the only 3-cosymplectic structure on N.

Let $f : M \to N$ be a smooth map between a 4-dimensional hyper-Kahler manifold and a 3-dimensional 3-cosymplectic manifold. Then, using 2.1, with $A^{\alpha\beta} = \delta_{\alpha\beta}$, 4.1 and 4.2, one obtains that f is quaternionic map if and only if

$$\left\{ \begin{array}{l} f_1^1 + f_2^2 + f_3^3 = 0 \\ f_1^2 - f_2^1 - f_4^3 = 0 \\ f_1^3 + f_4^2 - f_3^1 = 0 \\ -f_4^1 + f_2^3 - f_3^2 = 0 \end{array} \right. , \tag{4.3}$$

where $f_i^a = \frac{\partial f^a}{\partial x^i}$, $a = \overline{1,3}, i = \overline{1,4}$.

Example. Let M and N be $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively, endowed with the hyper-Kähler and the 3-cosymplectic structures and let $f : \mathbb{R}^4 \to \mathbb{R}^3$ be a smooth map, given by

$$f(x^1, x^2, x^3, x^4) = (-2x^1, x^2, x^3).$$

It is easy to see from 4.3 that f is a quaternionic map. The matrix of df is

$$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since the rank of this matrix is 3 and the rank of a $\pm(J,\varphi)$-holomorphic map must be even, it follows that f is a quaternionic map which is non-$\pm(J,\varphi)$-holomorphic with respect to any almost complex structure on $\mathbb{R}^4$ and to any almost contact structure on $\mathbb{R}^3$.

Using Theorem 3.2, after a straightforward computation, one obtains, for a vector field $V \in \Gamma(f^{-1}T\mathbb{R}^3)$, that $V \in Ker\ J_f$ if and only if

$$\left\{ \begin{array}{l} V_3^2 - V_2^3 = 0 \\ V_3^1 + 2V_1^3 = 0 \\ V_2^1 + 2V_1^2 = 0 \\ 2V_1^1 + V_2^2 + V_3^3 = 0 \end{array} \right. ,$$

where $V = V^i \frac{\partial}{\partial x^i}$ and $V_i^j = \frac{\partial V^j}{\partial x^i}$, $i, j = \overline{1,4}$, V^i being smooth functions on $\mathbb{R}^4$.

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