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SOME PROPERTIES OF WEAKLY OPEN FUNCTIONS IN BITOPOLOGICAL SPACES

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Abstract. We obtain the further characterizations and properties of weakly open functions in bitopological spaces due to Jelić [5].

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1. Introduction

The notion of weakly continuous functions was introduced in [9]. It is proved in [9] that a function $f : X \to Y$ is weakly continuous if and only if $f^{-1}(V) \subset \mathrm{Int}(f^{-1}(\mathrm{Cl}(V)))$ for every open set V of Y. In [13], Rose introduced the concept of weak openness which is a natural dual to that of weak continuity. In [5], Jelić generalized the notion of weakly open functions in the setting of bitopological spaces.

In this paper, we obtain some new characterizations of weakly open functions between bitopological spaces. Moreover, we investigate some properties of these functions comparing with the related functions.

2. Preliminaries

Throughout the present paper, (X, τ_1, τ_2) (resp. (X, τ)) denote a bitopological (resp. topological) space. Let (X, τ) be a topological space and A be a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. Let (X, τ_1, τ_2) be a bitopological space and A be a subset of X. The closure of A and the interior of A with respect to τ_i are denoted by $i\mathrm{Cl}(A)$ and $i\mathrm{Int}(A)$, respectively, for $i = 1, 2$.

Definition 2.1. A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(1) *(i, j)-regular open* [2] if $A = i\mathrm{Int}(j\mathrm{Cl}(A))$, where $i \neq j, i, j = 1, 2$,

(2) (i, j)-*semi-open* [10] if $A \subset j\mathrm{Cl}(i\mathrm{Int}(A))$, where $i \neq j$, i, $j = 1$, 2,

(3) (i, j)-*preopen* [3] if $A \subset i\mathrm{Int}(j\mathrm{Cl}(A))$, where $i \neq j$, i, $j = 1$, 2,

(4) (i, j)-α-*open* [4] if $A \subset i\mathrm{Int}(j\mathrm{Cl}(i\mathrm{Int}(A)))$, where $i \neq j$, i, $j = 1$, 2.

Definition 2.2. Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. A point $x \in X$ is said to be in the (i, j)-θ-*closure* [6] of A, denoted by (i, j)-$\mathrm{Cl}_\theta(A)$, if $A \cap j\mathrm{Cl}(U) \neq \emptyset$ for every τ_i-open set U containing x, where $i, j = 1, 2$ and $i \neq j$.

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A subset A of X is said to be (i,j)-θ-*closed* if $A = (i,j)\text{-Cl}_\theta(A)$. A subset A of X is said to be (i,j)-θ-*open* if $X - A$ is (i,j)-θ-closed. The (i,j)-θ-*interior* of A, denoted by $(i,j)\text{-Int}_\theta(A)$, is defined as the union of all (i,j)-θ-*open* sets contained in A. Hence $x \in (i,j)\text{-Int}_\theta(A)$ if and only if there exists a τ_i-open set U containing x such that $x \in U \subset j\text{Cl}(U) \subset A$.

Lemma 2.1. *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*
(1) $X - (i,j)\text{-Int}_\theta(A) = (i,j)\text{-Cl}_\theta(X - A)$,
(2) $X - (i,j)\text{-Cl}_\theta(A) = (i,j)\text{-Int}_\theta(X - A)$.

Definition 2.3. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be
(1) *(i, j)-semi-open* [2] if for each τ_i-open set U of X, $f(U)$ is (i,j)-semi-open in Y,
(2) *(i, j)-preopen* [4]) if for each τ_i-open set U of X, $f(U)$ is (i,j)-preopen in Y,
(3) *weakly* (i,j)*-open* [5] if for each τ_i-open set U of X, $f(U) \subset i\text{-Int}(f(j\text{Cl}(U)))$.

A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *pairwise weakly open* if f is weakly $(1,2)$-open and weakly $(2,1)$-open.

3. Characterizations

Lemma 3.1. (Kariofillis [6]). *Let (X, τ_1, τ_2) be a bitopological space. If U is τ_j-open in X, then* $(i,j)\text{-Cl}_\theta(U) = i\text{Cl}(U)$.

Theorem 3.1. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*
(1) f is weakly (i, j)-open;
(2) $f((i,j)\text{-Int}_\theta(A)) \subset i\text{Int}(f(A))$ *for every subset A of X;*
(3) $(i,j)\text{-Int}_\theta(f^{-1}(B)) \subset f^{-1}(i\text{Int}(B))$ *for every subset set B of Y;*
(4) $f^{-1}(i\text{Cl}(B)) \subset (i,j)\text{-Cl}_\theta(f^{-1}(B))$ *for every subset B of Y;*
(5) For each $x \in X$ and each τ_i-open set U of X containing x, there exists a σ_i-open set V of Y containing f(x) such that $V \subset f(j\text{Cl}(U))$.

Proof. (1) $\Rightarrow$ (2): Let A be any subset of X and $x \in (i,j)\text{-Int}_\theta(A)$. Then there exists a τ_i-open set U of X such that $x \in U \subset j\text{Cl}(U) \subset A$. Hence we have $f(x) \in f(U) \subset f(j\text{Cl}(U)) \subset f(A)$. Since f is weakly (i,j)-open, $f(U) \subset i\text{Int}(f(j\text{Cl}(U))) \subset i\text{Int}(f(A))$ and $x \in f^{-1}(i\text{Int}(f(A)))$. Thus $(i,j)\text{-Int}_\theta(A) \subset f^{-1}(i\text{Int}(A))$ and hence $f((i,j)\text{-Int}_\theta(A)) \subset i\text{Int}(f(A))$.

(2) $\Rightarrow$ (3): Let B be any subset of Y. Then by (2), we have $f((i,j)\text{-Int}_\theta(f^{-1}(B))) \subset i\text{Int}(f(f^{-1}(B))) \subset i\text{Int}(B)$. Therefore, $(i,j)\text{-Int}_\theta(f^{-1}(B)) \subset f^{-1}(i\text{Int}(B))$.

(3) $\Rightarrow$ (4): Let B be any subset of Y. Then we have

$$X - (i,j)\text{-Cl}_\theta(f^{-1}(B)) = (i,j)\text{-Int}_\theta(X - f^{-1}(B)) = (i,j)\text{-Int}_\theta(f^{-1}(Y - B))$$
$$\subset f^{-1}(i\text{Int}(Y - B)) = f^{-1}(Y - i\text{Cl}(B)) = X - f^{-1}(i\text{Cl}(B)).$$

Therefore, $f^{-1}(i\mathrm{Cl}(B)) \subset (i,j)\text{-}\mathrm{Cl}_\theta(f^{-1}(B))$.

(4) $\Rightarrow$ (5): Let $x \in X$ and U be any τ_i-open set containing x. Set $B = Y - f(j\mathrm{Cl}(U))$. By (4), we have $f^{-1}(i\mathrm{Cl}(Y - f(j\mathrm{Cl}(U)))) \subset (i,j)\text{-}\mathrm{Cl}_\theta(f^{-1}(Y - f(j\mathrm{Cl}(U))))$. Now, $f^{-1}(i\mathrm{Cl}(Y - f(j\mathrm{Cl}(U)))) = X - f^{-1}(i\mathrm{Int}(f(j\mathrm{Cl}(U))))$. Moreover, by Lemma 3.1 we have

$$\begin{gathered}(i,j)\text{-}\mathrm{Cl}_\theta(f^{-1}(Y - f(j\mathrm{Cl}(U)))) = (i,j)\text{-}\mathrm{Cl}_\theta(X - f^{-1}(f(j\mathrm{Cl}(U)))) \subset \\ (i,j)\text{-}\mathrm{Cl}_\theta(X - j\mathrm{Cl}(U)) = i\mathrm{Cl}(X - j\mathrm{Cl}(U)) = \\ X - i\mathrm{Int}(j\mathrm{Cl}(U)) \subset X - i\mathrm{Int}(U) = X - U.\end{gathered}$$

Therefore, we obtain $U \subset f^{-1}(i\mathrm{Int}(f(j\mathrm{Cl}(U))))$ and $f(U) \subset i\mathrm{Int}(f(j\mathrm{Cl}(U)))$. Since $f(x) \in f(U)$, there exists a σ_i-open set V such that $f(x) \in V \subset f(j\mathrm{Cl}(U))$.

(5) $\Rightarrow$ (1): Let U be any τ_i-open set of X and $x \in U$. By (5), there exists a σ_i-open set V of Y containing $f(x)$ such that $V \subset f(j\mathrm{Cl}(U))$. Hence we have $f(x) \in V \subset i\mathrm{Int}(f(j\mathrm{Cl}(U)))$ for each $x \in U$. Therefore, we obtain $f(U) \subset i\mathrm{Int}(f(j\mathrm{Cl}(U)))$. This shows that f is weakly (i,j)-open.

Theorem 3.2. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is weakly (i, j)-open;

(2) $f(i\mathrm{Int}(F)) \subset i\mathrm{Int}(f(F))$ for each τ_j-closed set F of X;

(3) $f(U) \subset i\mathrm{Int}(f(j\mathrm{Cl}(U)))$ for every (i,j)-preopen set U of X;

(4) $f(U) \subset i\mathrm{Int}(f(j\mathrm{Cl}(U)))$ for every (i,j)-α-open set U of X.

Proof. (1) $\Rightarrow$ (2): Assume that f is weakly (i,j)-open. Let F be a τ_j-closed set of X. Then $i\mathrm{Int}(F)$ is τ_i-open and by (1) we have $f(i\mathrm{Int}(F)) \subset i\mathrm{Int}(f(j\mathrm{Cl}(i\mathrm{Int}(F)))) \subset i\mathrm{Int}(f(F))$.

(2) $\Rightarrow$ (3): Let U be any (i,j)-preopen set of X. Then by (2) we obtain $f(U) \subset f(i\mathrm{Int}(j\mathrm{Cl}(U))) \subset i\mathrm{Int}(f(j\mathrm{Cl}(U)))$.

(3) $\Rightarrow$ (4): This is obvious since every (i,j)-α-open set is (i,j)-preopen.

(4) $\Rightarrow$ (1): Let U be any τ_i-open set of X. Then U is (i,j)-α-open in X and hence $f(U) \subset i\mathrm{Int}(f(j\mathrm{Cl}(U)))$. Therefore, f is weakly (i,j)-open.

Theorem 3.3. *For a bijective function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is weakly (i, j)-open;

(2) $i\mathrm{Cl}(f(j\mathrm{Int}(F))) \subset f(F)$ for every τ_i-closed set F of X;

(3) $i\mathrm{Cl}(f(U)) \subset f(i\mathrm{Cl}(U))$ for every τ_j-open set U of X.

Proof. (1) $\Rightarrow$ (2): Let F be any τ_i-closed set of X. Then $X - F$ is τ_i-open and

$$\begin{gathered}Y - f(F) = f(X - F) \subset i\mathrm{Int}(f(j\mathrm{Cl}(X - F))) = \\ i\mathrm{Int}(f(X - j\mathrm{Int}(F))) = i\mathrm{Int}(Y - f(j\mathrm{Int}(F))) = Y - i\mathrm{Cl}(f(j\mathrm{Int}(F))).\end{gathered}$$

This implies that $i\mathrm{Cl}(f(j\mathrm{Int}(F))) \subset f(F)$.

(2) $\Rightarrow$ (3): Let U be any τ_j-open set of X. By (2) we have

$$i\mathrm{Cl}(f(U)) = i\mathrm{Cl}(f(j\mathrm{Int}(U))) \subset i\mathrm{Cl}(f(j\mathrm{Int}(i\mathrm{Cl}(U)))) \subset f(i\mathrm{Cl}(U)).$$

Therefore, $i\mathrm{Cl}(f(U)) \subset f(i\mathrm{Cl}(U))$.

(3) $\Rightarrow$ (1): Let U be any τ_i-open set of X. Then, we have

$$Y - i\mathrm{Int}(f(j\mathrm{Cl}(U))) = i\mathrm{Cl}(Y - f(j\mathrm{Cl}(U))) = i\mathrm{Cl}(f(X - j\mathrm{Cl}(U))) \subset f(i\mathrm{Cl}(X - j\mathrm{Cl}(U))) = f(X - i\mathrm{Int}(j\mathrm{Cl}(U))) \subset f(X - i\mathrm{Int}(U)) = f(X - U) = Y - f(U).$$

This implies $f(U) \subset i\mathrm{Int}(f(j\mathrm{Cl}(U))$. Therefore, f is weakly (i,j)-open.

4. Relations with other forms of open functions

Definition 4.1. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *pairwise open* [1] if the induced functions $f_1 : (X, \tau_1) \to (Y, \sigma_1)$ and $f_2 : (X, \tau_2) \to (Y, \sigma_2)$ are open.

Definition 4.2. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *almost (i, j)-open* [2] if $f(U)$ is σ_i-open in Y for every (i,j)-regular open set U of X.

A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *pairwise almost open* if it is almost $(1,2)$-open and almost $(2,1)$-open.

Remark 4.1. It is known that [5]

pairwise openness $\Rightarrow$ almost pairwise openness $\Rightarrow$ weakly pairwise openness.

Definition 4.3. A bitopological space (X, τ_1, τ_2) is said to be (i,j)-*semi-regular* [14] if for each $x \in X$ and each τ_i-open set U containing x, there exists an (i,j)-regular open set V of X such that $x \in V \subset U$. (X, τ_1, τ_2) is said to be *pairwise semi-regular* if it is $(1,2)$-semi-regular and $(2,1)$-semi-regular.

Theorem 4.1. *Let a bitopological space (X, τ_1, τ_2) be pairwise semi-regular. Then a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is pairwise open if and only if it is almost pairwise open.*

Proof. Necessity. This is shown in [5].

Sufficiency. Suppose that f is almost (i,j)-open. Let U be any τ_i-open set of X. Since X is (i,j)-semi-regular, for each $x \in U$ there exists an (i,j)-regular open set U_x such that $x \in U_x \subset U$. Since f is almost (i,j)-open, $f(U_x)$ is σ_i-open in Y. Since $f(U) = \cup\{f(U_x) : x \in U\}$, it follows that $f(U)$ is σ_i-open. Therefore, $f_i : (X, \tau_i) \to (Y.\sigma_i)$ is open for $i = 1$, 2 and hence $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is pairwise open.

Definition 4.4. A bitopological space (X, τ_1, τ_2) is said to be (i,j)-*almost regular* [14] if for each $x \in X$ and each (i,j)-regular open set U containing x, there exists an (i,j)-regular open set V of X such that $x \in V \subset j\mathrm{Cl}(V) \subset U$. (X, τ_1, τ_2) is said to be *pairwise almost regular* if it is $(1,2)$-almost regular and $(2,1)$-almost regular

Theorem 4.2. *Let a bitopological space* (X, τ_1, τ_2) *be pairwise almost regular. Then a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is almost pairwise open if and only if it is weakly pairwise open.*

Proof. Necessity. This is shown in Lemma 2.1 of [5].

Sufficiency. Suppose that f is weakly pairwise open. Let U be any (i, j)-regular open set of X. Since X is (i, j)-almost regular, for each $x \in U$ there exists an (i, j)-regular open set U_x such that $x \in U_x \subset j\mathrm{Cl}(U_x) \subset U$. Since every (i, j)-regular open set is τ_i-open and f is weakly (i, j)-open, we obtain

$$f(U) = \cup\{f(U_x) : x \in U\} \subset \cup\{i\mathrm{Int}(f(j\mathrm{Cl}(U_x))) : x \in U\}$$
$$\subset \{i\mathrm{Int}(\cup f(j\mathrm{Cl}(U_x))) : x \in U\} = \{i\mathrm{Int}(f(\cup j\mathrm{Cl}(U_x))) : x \in U\} \subset i\mathrm{Int}(f(U)).$$

Therefore, $f(U) \subset i\mathrm{Int}(f(U))$ and hence $f(U)$ is σ_i-open. Thus, f is almost (i, j)-open for $i \neq j; i, j = 1, 2$.

Definition 4.5. A bitopological space (X, τ_1, τ_2) is said to be (i, j)-*regular* [7] if for each $x \in X$ and each τ_i-open set U containing x, there exists a τ_i-open set V such that $x \in V \subset j\mathrm{Cl}(V) \subset U$. (X, τ_1, τ_2) is said to be *pairwise regular* if it (1, 2)-regular and (2, 1)-regular.

Corollary 4.1. *Let* (X, τ_1, τ_2) *be a pairwise regular space. For a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) f is pairwise open;
(2) f is almost pairwise open;
(3) f is weakly pairwise open.

Proof. This is an immediate consequence of Theorems 4.1 and 4.2 since every pairwise regular space is pairwise semi-regular and pairwise almost regular.

Definition 4.6. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *strongly continuous* [8] if $f(\mathrm{Cl}(A)) \subset f(A)$ for every subset A of X.

Theorem 4.3. *If a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is weakly (i, j)-open and strongly j-continuous, then f is i-open.*

Proof. Let U be any τ_i-open set of X. Since f is weakly (i, j)-open and strongly j-continuous, we have $f(U) \subset i\mathrm{Int}(f(j\mathrm{Cl}(U))) \subset i\mathrm{Int}((f(U))$. Therefore, $f(U) = i\mathrm{Int}(f(U))$ and $f(U)$ is σ_i-open. Hence f is i-open.

Definition 4.7. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to have *the weak (i, j)-interiority condition* if $i\mathrm{Int}(f(j\mathrm{Cl}(U))) \subset f(U)$ for every τ_i-open set U of X.

Theorem 4.4. *If a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is weakly (i, j)-open and satisfies the weak (i, j)-interiority condition, then f is i-open.*

Proof. Let U be any τ_i-open set of X. Since f is weakly (i,j)-open and satisfies the weak (i,j)-interiority condition, we have $f(U) \subset i\mathrm{Int}(f(j\mathrm{Cl}(U))) = i\mathrm{Int}(i\mathrm{Int}(f(j\mathrm{Cl}(U))))) \subset i\mathrm{Int}((f(U))$. Therefore, $f(U) = i\mathrm{Int}(f(U))$ and $f(U)$ is σ_i-open. Hence f is i-open.

5. Some properties of weakly (i,j)-open functions

Definition 5.1. A bitopological space (X,τ_1,τ_2) is said to be (i,j)-*hyperconnected* if $j\mathrm{Cl}(U) = X$ for every τ_i-open set U of X.

Theorem 5.1. *Let (X,τ_1,τ_2) be an (i, j)-hyperconnected space. Then a function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is weakly (i, j)-open if and only if f(X) is σ_i-open in Y.*

Proof. Necessity. Let f be weakly (i,j)-open. Since X is τ_i-open, $f(X) \subset i\mathrm{Int}(f(j\mathrm{Cl}(X))) = i\mathrm{Int}(f(X))$. Therefore, $f(X)$ is σ_i-open in Y.

Sufficiency. Suppose that $f(X)$ is σ_i-open in Y. Let U be τ_i-open in X. Then $f(U) \subset f(X) = i\mathrm{Int}(f(X)) = i\mathrm{Int}(f(j\mathrm{Cl}(U)))$. Therefore, $f(U) \subset i\mathrm{Int}(f(j\mathrm{Cl}(U)))$. This shows that f is weakly (i,j)-open.

Definition 5.2. A function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is said to be *(i, j)-contra-closed* if $f(F)$ is σ_i-open in Y for every τ_j-closed set F of X.

Theorem 5.2. *If a function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is (i, j)-contra-closed, then f is weakly (i, j)-open.*

Proof. Let U be any τ_i-open set of X. Then $j\mathrm{Cl}(U)$ is τ_j-closed in X. Hence, we have $f(U) \subset f(j\mathrm{Cl}(U)) \subset i\mathrm{Int}(f(j\mathrm{Cl}(U)))$. Therefore, f is weakly (i,j)-open.

Definition 5.3. A function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is said to be *(i, j)-contra-open* if $f(U)$ is σ_j-closed in Y for every τ_i-open set U of X.

Theorem 5.3. *If a function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is (i, j)-preopen and (i, j)-contra-open, then f is i-open.*

Proof. Let U be any τ_i-open set of X. Since f is (i,j)-preopen, $f(U) \subset i\mathrm{Int}(j\mathrm{Cl}(f(U)))$. Since f is (i,j)-contra-open, $f(U)$ is σ_j-closed. Therefore, $f(U) \subset i\mathrm{Int}(j\mathrm{Cl}(f(U))) = i\mathrm{Int}(f(U))$. Hence $f(U)$ is σ_i-open in Y. This shows that f is i-open.

Lemma 5.1. *If $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is a bijective and (i, j)-semi-open function, then $j\mathrm{Int}(i\mathrm{Cl}(f(F))) \subset f(F)$ for every τ_i-closed set F of X.*

Proof. Let F be any τ_i-closed set of X. Then $X - F$ is τ_i-open in X. Since f is (i,j)-semi-open, $f(X-F) \subset j\mathrm{Cl}(i\mathrm{Int}(f(X-F)))$. Therefore, $Y - f(F) = f(X-F) \subset j\mathrm{Cl}(i\mathrm{Int}(f(X-F))) = j\mathrm{Cl}(i\mathrm{Int}(Y-f(F))) = Y - j\mathrm{Int}(i\mathrm{Cl}(f(F)))$. Therefore, we obtain $j\mathrm{Int}(i\mathrm{Cl}(f(F))) \subset f(F)$.

Theorem 5.4. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an (i, j)-preopen and (j, i)-semi-open bijection, then f is weakly (i, j)-open.*

Proof. Let U be any τ_i-open set of X. Then $j\mathrm{Cl}(U)$ is τ_j-closed in X. Since f is (j, i)-semi-open, by Lemma 5.1 $i\mathrm{Int}(j\mathrm{Cl}(f(j\mathrm{Cl}(U)))) \subset f(j\mathrm{Cl}(U))$. Since f is (i, j)-preopen, $f(U) \subset i\mathrm{Int}(j\mathrm{Cl}(f(U)))$. Therefore, $f(U) \subset i\mathrm{Int}(f(j\mathrm{Cl}(U)))$. Hence f is weakly (i, j)-open.

Corollary 5.1. *If $f : (X, \tau) \to (Y, \sigma)$ is a preopen and semi-open bijection, then f is weakly open.*

Remark 5.1. Corollary 5.1 is a dual form of Theorem 1 of [12].

Definition 5.4. A bitopological space (X, τ_1, τ_2) is said to be *pairwise connected* [11] if it cannot be expressed as the union of two nonempty disjoint sets U and V such that U is τ_i-open and V is τ_j-open.

Theorem 5.5. *If (Y, σ_1, σ_2) is pairwise connected and $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a pairwise weakly open bijection, then (X, τ_1, τ_2) is pairwise connected.*

Proof. Suppose that (X, τ_1, τ_2) is not pairwise connected. Then there exist a τ_i-open set U_1 and a τ_j-open set U_2 such that $U_1 \neq \emptyset, U_2 \neq \emptyset, U_1 \cap U_2 = \emptyset$ and $U_1 \cup U_2 = X$. Since f is bijective, we have $f(U_1) \neq \emptyset$, $f(U_2) \neq \emptyset$, $f(U_1) \cap f(U_2) = \emptyset$ and $f(U_1) \cup f(U_2) = Y$. Since f is pairwise weakly open, $f(U_1) \subset i\mathrm{Int}(f(j\mathrm{Cl}(U_1)))$ and $f(U_2) \subset j\mathrm{Int}(f(i\mathrm{Cl}(U_2)))$. Since U_1 and U_2 are τ_j-closed and τ_i-closed, respectively, we have $f(U_1) \subset i\mathrm{Int}(f(U_1))$ and $f(U_2) \subset j\mathrm{Int}(f(U_2))$ and hence $f(U_1) = i\mathrm{Int}(f(U_1))$ and $f(U_2) = j\mathrm{Int}(f(U_2))$. Therefore, $f(U_1)$ is σ_i-open and $f(U_2)$ is σ_j-open. This is contrary to the hypothesis that (Y, σ_1, σ_2) is pairwise connected. Therefore, (X, τ_1, τ_2) is pairwise connected.

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