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ON AN EINSTEIN PROJECTIVE SASAKIAN MANIFOLD

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Abstract. In this paper, we have proved that a projectively flat Sasakian manifold is an Einstein manifold. Also, if an Einstein-Sasakian manifold is projectively flat, then it is locally isometric with a unit sphere $S^n(1)$. It has also been proved that if in an Einsten-Sasakian manifold the relation $K(X,Y).P = 0$ holds, then it is locally isometric with a unit sphere $S^n(1)$.

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1. Introduction

Let (M^n, g) be a contact Riemannian manifold with a contact form η, the associated vector field ξ, a (1-1) tensor field ϕ and the associated Riemannian metric g. If ξ is a killing vector field, then M^n is called a K-contact Riemannian manifold ([1], [2]). A K-contact Riemannian manifold is called a Sasakian manifold [2] if

$$(D_X\phi)(Y) = g(X,Y)\xi - \eta(Y)X \tag{1}$$

holds, where D denotes the operator of covariant differentiation with respect to g. We deal with a type of Sasakian manifold in which

$$K(X,Y).P = 0, \tag{2}$$

where P is the projective curvature tensor (see [5]) defined by

$$P(X,Y)Z = K(X,Y)Z - \frac{1}{n-1}[Ric(Y,Z)X - Ric(X,Z)Y], \tag{3}$$

K is the Riemannian curvature tensor, Ric is the Ricci tensor of type (0,2) and $K(X,Y)$ is considered as derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y. In this connection we mention the works of K. Sekigawa [3] and Z.L. Szabo [4] who studied Riemannian manifolds satisfying the conditions similar to it. It is easy to see that $K(X,Y).K = 0$ implies $K(X,Y).P = 0$. So it is meaningful to undertake the study of manifolds satisfying the condition (2).

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Let R and r denote the Ricci tensor of type (1,1) and the scalar curvature of M^n respectively. It is known that in a Sasakian manifold M^n, besides the relation (1), the following relations also hold (see [1], [2]):

$$\phi(\xi) = 0 \tag{4}$$

$$\eta(\xi) = 1 \tag{5}$$

$$g(\xi, X) = \eta(X) \tag{6}$$

$$Ric(X, \xi) = (n-1)\eta(X) \tag{7}$$

$$g(K(\xi, X)Y, \xi) = g(X, Y) - \eta(X)\eta(Y) \tag{8}$$

$$K(\xi, X)\xi = -X + \eta(X)\xi \tag{9}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{10}$$

$$K(\xi, X)Y = g(X, Y) - \eta(Y)X \tag{11}$$

and

$$\eta(\phi X) = 0 \tag{12}$$

for any vector fields X, Y.

The above results will be used in the next section.

2. Sasakian manifold satisfying $P(X, Y)Z = 0$

Let us suppose that in a Sasakian manifold

$$P(X, Y)Z = 0. \tag{13}$$

Then it follows from (3) that

$$K(X, Y)Z = \frac{1}{n-1}[Ric(Y, Z)X - Ric(X, Z)Y] \tag{14}$$

or,

$$g(K(X, Y)Z, U) = \frac{1}{n-1}[Ric(Y, Z)g(X, U) - Ric(X, Z)g(Y, U)]. \tag{15}$$

Taking $X = U = \xi$ in (15) and then using (5), (6), (7) and (8), we get

$$g(Y, Z) - \eta(Y)\eta(Z) = \frac{1}{n-1}[Ric(Y, Z) - (n-1)\eta(Y)\eta(Z)] \tag{16}$$

Consequently,

$$Ric(Y, Z) = kg(Y, Z), \tag{17}$$

where $k = (n-1)$. Thus we have the following result:

Theorem 1. *A projectively flat Sasakian manifold is an Einstein manifold.*

Next, we prove the following:

Theorem 2. *The scalar curvature r of a projectively flat Sasakian manifold M^n is constant.*

Proof. From (17), we have

$$R(Y) = (n-1)Y, \tag{18}$$

where $Ric(Y, Z) = g(R(Y), Z)$. Contracting (18) with respect to 'Y', we have $r = n(n-1)$, which proves the result. □

Theorem 3. *A projectively flat Einstein-Sasakian manifold M^n $(n \geq 2)$ is locally isometric with a unit sphere $S^n(1)$.*

Proof. Let the Riemannian manifold be Einstein, i.e.

$$Ric(X, Y) = kg(X, Y),$$

where k is a constant. Then (14) reduces to

$$K(X, Y)Z = \frac{k}{n-1}[g(Y, Z)X - g(X, Z)Y] \tag{19}$$

or,

$$g(K(X, Y)Z, V) = \frac{k}{n-1}[g(Y, Z)g(X, V) - g(X, Z)g(Y, V)]. \tag{20}$$

Taking $X = V = \xi$ in (20) and then using (5), (6) and (8), we get

$$g(Y, Z) - \eta(Y)\eta(Z) = \frac{k}{n-1}[g(Y, Z) - \eta(Z)\eta(Y)]$$

or,

$$[\frac{k}{n-1} - 1][g(Y, Z) - \eta(Y)\eta(Z)] = 0.$$

This shows that either $k = n-1$ or $g(Y, Z) = \eta\ (Y)\ \eta(Z)$. Now, if $g(Y, Z) = \eta\ (Y)\ \eta\ (Z)$, then from (10), we get $g(\phi Y, \phi Z) = 0$, which is not possible. Therefore, $k = n-1$ and, putting this value of k in (19), we get the result. □

3. An Einstein Sasakian manifold satisfying $K(X,Y).P = 0$

Let the Riemannian manifold M be an Einstein manifold, then (3) gives

$$P(X,Y)Z = K(X,Y)Z - \frac{k}{n-1}[g(Y,Z)X - g(X,Z)Y]. \tag{21}$$

We have,

$$\begin{aligned} \eta(P(X,Y)Z) &= g(P(X,Y)Z,\xi) \\ &= g(K(X,Y)Z - \frac{k}{n-1}[g(Y,Z)X - g(X,Z)Y],\xi) \\ &= \eta(X)g(Z,Y) - \eta(Y)g(Z,X) - \frac{k}{n-1}[\eta(X)g(Z,Y) - \eta(Y)g(Z,X)] \end{aligned}$$

or,

$$\eta(P(X,Y)Z) = [\frac{k}{n-1} - 1][\eta(Y)g(Z,X) - \eta(X)g(Z,Y)]. \tag{22}$$

Taking $X = \xi$ in (22) and then using (5) and (6), we get

$$\eta(P(\xi,Y)Z) = [\frac{k}{n-1} - 1][\eta(Y)\eta(Z) - g(Z,Y)]. \tag{23}$$

Again, taking $Z = \xi$ in (22) and then using (5) and (6), we get

$$\eta(P(X,Y)\xi) = 0. \tag{24}$$

Now,

$$\begin{aligned} (K(X,Y)P)(U,V)W = K(X,Y)P(U,V)W - P(K(X,Y)U,V)W- \\ -P(U,K(X,Y)V)W - P(U,V)K(X,Y)W. \end{aligned}$$

In view of (2), we get

$$\begin{aligned} K(X,Y)P(U,V)W - P(K(X,Y)U,V)W &- \\ -P(U,K(X,Y)V)W - P(U,V)K(X,Y)W &= 0. \end{aligned} \tag{25}$$

Therefore,

$$\begin{aligned} g(K(\xi,Y)P(U,V)W,\xi) - g(P(K(\xi,Y)U,V)W,\xi) &- \\ -g(P(U,K(\xi,Y)V)W,\xi) - g(P(U,V)K(\xi,Y)W,\xi) &= 0. \end{aligned}$$

From this it follows that

$$\begin{aligned} {}'P(U,V,W,Y) - \eta(Y)\eta(P(U,V)W) + \eta(U)\eta(P(Y,V)W)+ \\ +\eta(V)\eta(P(U,Y)W) + \eta(W)\eta(P(U,V)Y) - g(Y,U)\eta(P(\xi,V)W)- \\ -g(Y,V)\eta(P(U,\xi)W) - g(Y,W)\eta(P(U,V)\xi) = 0, \end{aligned} \tag{26}$$

where $g(P(U,V)W,Y) = {}'P(U,V,W,Y)$.

Putting $Y = U$ in (26), we get

$$\begin{aligned}&{}'P(U,V,W,U) - \eta(U)\eta(P(U,V,W) + \eta(U)\eta(P(U,V)W) + \\ &+\eta(V)\eta(P(U,U)W) + \eta(W)\eta(P(U,V)U) - g(U,U)\eta(P(\xi,V)W) - \\ &-g(U,V)\eta(P(U,\xi)W) - g(U,W)\eta(P(U,V)\xi) = 0.\end{aligned} \tag{27}$$

Let $\{e_i\}$, $i = 1,2,\dots,n$ be an orthonormal basis of the tangent space at any point. Then the sum for $1 \le i \le n$ of the relation (27) for $U = e_i$ gives

$$\begin{aligned}\eta(P(\xi,V)W) &= \tfrac{1}{n-1}[Ric(V,W)\tfrac{r}{n} - g(V,W) + \\ &+(\tfrac{k}{n(n-1)} - 1)(n-1)\eta(W)\eta(V)].\end{aligned} \tag{28}$$

Using (22) and (28), it follows from (26) that

$$\begin{aligned}&{}'p(U,V,W,Y) + \tfrac{k}{n(n-1)}g(Y,U)g(V,W) - \tfrac{k}{n(n-1)}g(U,W)g(Y,V) + \\ &+\tfrac{1}{n-1}[Ric(U,W)g(Y,V) - Ric(V,W)g(Y,U)] = 0.\end{aligned} \tag{29}$$

From (23) and (28), we get

$$\begin{aligned}(\tfrac{k}{n-1} - 1)[\eta(V)\eta(W) - g(W,V)] &= \tfrac{1}{n-1}[Ric(V,W) - \tfrac{r}{n}g(V,W) + \\ &+(\tfrac{k}{n(n-1)} - 1)(n-1)\eta(V)\eta(W)].\end{aligned}$$

For $r = nk$, we have

$$Ric(W,V) = (n-1)g(W,V). \tag{30}$$

Using (30) and taking $r = nk$, the relation (29) reduces to

$${}'P(U,V,W,Y) = (\frac{k}{n-1} - 1)[g(Y,V)g(U,W) - g(Y,U)g(V,W)]. \tag{31}$$

From (21) and (31), we get

$${}'K(U,V,W,Y) = [g(Y,U)g(V,W) - g(Y,V)g(U,W)], \tag{32}$$

where ${}'K(U,V,W,Y) = g(K(U,V)W,Y)$.

Thus we have the following:

Theorem 4. *If in an Einstein-Sasakian manifold, the relation $K(X,Y).P = 0$ holds, then it is locally isometric with a unit sphere $S^n(1)$.*

For a projectively symmetric Riemannian manifold we have $DP = 0$. Hence for such a manifold $K(X,Y).P = 0$ holds. Thus we have the following corollary of the above theorem:

Corollary 1. *A projectively symmetric Sasakian manifold M^n $(n \ge 2)$ is locally isometric with a unit sphere $S^n(1)$.*

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References

[1] Sasaki, S., Lecture notes on almost contact manifolds, part 1. Tohoku University, 1975.

[2] Blair, David E., Contact manifold in a Riemannian Geometry. Lecture notes in Mathematics, 509, Springer Verlag, 1976.

[3] Sekigawa, K., Almost Hermition manifold satisfying some curvature conditions. Kodai. Math. J. 2 (1979), 384-405.

[4] Szabo, Z. L., Structure theorem on Riemannian space satisfying K (X,Y). K=0, The local version. J. Diff. Geom., 17 (1982) 531-582.

[5] Singh, H., Khan, Q., On generalized recurrent Riemannian manifolds. Publ. Math. Debrceen, Hungary, 56/1-2 (2000), 87-95.

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