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$(\oplus, \odot)$-LAPLACE TRANSFORM AS A BASIS FOR AGGREGATION TYPE OPERATORS

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Abstract. Pseudo-Laplace transform is an important notion from pseudo-analysis' framework that is often used in dealing with differential or integral equation. The $(\oplus, \odot)$-Laplace transform considered here is a generalization of the pseudo-Laplace transform based on a special class of generalized pseudo-operations that need not be commutative nor associative. This pseudo-Laplace type transform has been used for the construction of pseudo-aggregation operators.

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1. Introduction

The approach presented in this paper has been pursuaded in the pseudo-analysis' framework, where by pseudo-analysis is understood a mathematical theory that is a generalization of the classical analysis. Pseudo-analysis has appeared to be a useful tool for solving problems in different aspects of mathematics, as well as in various practical problems ([7, 10, 13, 14]). Using this apparatus over the years, some important notions that are analogous to their classical counterparts, i.e., notions such as $\oplus$-measure, pseudo-integral, pseudo-convolution, pseudo-Laplace transform, etc., have been introduced ([7, 10, 13, 17, 18, 19]). Generalized pseudo-convolution, based on $\oplus$-measure and pseudo-integral, has taken an important role in theory of fuzzy numbers (operations with fuzzy numbers), as well as in optimization, information theory, system theory, etc. ([19]). Also, pseudo-convolution and pseudo-Laplace transform have been successfully used for the determination of utility functions' extreme values ([5, 18]). Of special interest is the application of pseudo-analysis on non-linear partial differential equations. By using the pseudo-linear superposition principle ([4, 8, 10, 13, 14, 15, 16, 17]) some new solutions for considered nonlinear equation have been obtained. Additionally, the pseudo-analysis' approach has been successful in finding a weak solution of Hamilton-Jacobi equation with non-smooth Hamiltonian ([10, 16, 18]). A step further in this direction has been presented in [22, 23] where generalized pseudo-operations were introduced. A special class of these operations that need not be commutative nor associative

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has been used to extend the pseudo-linear superposition principle on generalized Burger's type nonlinear partial differential equations [23].

Another important problem addressed by pseudo-analysis is the construction of aggregation operators by means of different types of pseudo-integrals. It is a well known fact that a large class of idempotent aggregation operators can be constructed and represented by different types of integrals. Some of the integrals that have been used for this constructions are Lebesgue integral, Choquet and Sugeno integral, monotone set functions-based integrals, Choquet-like integrals, (S,U)-integral, etc. (see [1, 2, 3, 6, 11]). This paper presents a generalization of the pseudo-Laplace transform based on a special class of generalized pseudo-operations, i.e., on a pair of generated pseudo-operations with two parameters, and corresponding aggregation type operator. Also, the $\oplus$-integral as a core of pseudo-aggregation operator is considered.

Preliminary notions as generalized pseudo-operations, $\oplus$-integral and corresponding pseudo-convolutions are given in Section 2. The third section contains definition of the $(\oplus,\odot)$-Laplace transform, where $\oplus$ and $\odot$ are generated pseudo-operations with two parameters. Generalization of the exchange formula that transforms convolution into the product is given in Section 3. Aggregation type operator constructed by means of $(\oplus,\odot)$-Laplace transform is presented in the fourth section.

2. Preliminary notions

In this paper, as already mentioned, a special class of generalized pseudo-operations (see [22, 23]) will be considered. This class is given by the following definition.

Definition 1. *Let ε and γ be arbitrary but fixed positive real numbers and let g be a positive strictly monotone continuous function defined on $\mathbb{R}$ or* $[0,\infty)$. Generated pseudo-addition and pseudo-multiplication with two parameters, *denoted with $\oplus$ and $\odot$ respectively, are*

$$x \oplus y = g^{-1}(\varepsilon g(x) + g(y)) \quad \text{and} \quad x \odot y = g^{-1}(g(x)^{\gamma} g(y)). \tag{1}$$

Specially, for $\varepsilon = \gamma = 1$, operations from g-semiring are obtained ([9, 12, 13]).

Since the operations $\oplus$ and $\odot$ need not be commutative nor associative, it is necessary to define a pseudo-sum of n elements $\alpha_i \in [a,b]$, $i \in \{1,2,\ldots n\}$:

$$\bigoplus_{i=1}^{n} \alpha_i = (\ldots((\alpha_1 \oplus \alpha_2) \oplus \alpha_3) \oplus \ldots) \oplus \alpha_n.$$

Neutral elements from the left for $\oplus$ and $\odot$ are $\mathbf{0} = g^{-1}(0)$ and $\mathbf{1} = g^{-1}(1)$, respectively, i.e., $\mathbf{0} \oplus x = x$ and $\mathbf{1} \odot x = x$.

Remark 2. Operations of this type have been used in dealing with nonlinear PDE ([22, 23]). For example, if we consider the Burger's type of nonlinear partial differential equation $u_t - \alpha u_{xx} = \alpha\Phi(u)u_x^2$, where Φ is a given continuous function and $\alpha \in \mathbb{R}$, there exist generated pseudo-operations $\oplus$ and $\odot$ with two parameters given by a generating function

$$g(x) = \pm \int_0^x \exp(\int_0^t \Phi(s)\, ds)\, dt,$$

such that the pseudo-linear combination of solutions of considered equation is, again, the solution (see [23]).

Let $(a, b]$ be a subinterval of the real line and let, for some $n \in \mathbb{N}$, $P_n = \{(x_i, x_{i+1}]\}_{i=0}^{n-1}$ be its n-partition, where $a = x_0 < x_1 < \ldots < x_n = b$. Now, for ν being Lebesque measure, the $\oplus$-*measure* $\mu_{P_n} : P_n \to [0, \infty)$ is given by

$$\mu_{P_n}((x_i, x_{i+1}]) = g^{-1}\left(\frac{x_{i+1} - x_i}{\varepsilon^{n-i-1}}\right).$$

Some properties of this family of measures has been proved in [20]. Among them is the following:

$$\mu_{P_{n-r+j}}\left(\bigcup_{i=j}^{r} A_i\right) = \bigoplus_{i=j}^{r} \mu_{P_n}(A_i),$$

where $1 \leq j \leq r \leq n$, $P_n = \{A_i\}_{i=1}^n = \{(x_{i-1}, x_i]\}_{i=1}^n$ is an n-partition of interval $(a, b]$ and $P_{n-r+j} = \{B_s\}_{s=1}^{n-r+j}$ is a new $(n - r + j)$-partition, such that $B_s = A_s$ while $s = 1, 2, \cdots, j - 1$, $B_j = \cup_{i=j}^r A_i$ and $B_s = A_{s+r-j}$ for $s = j + 1, \cdots, n - r + j$.

Let $\varphi : [a, b] \to [0, \infty)$ be a step function that assumes finitely many values $\{u_1, u_2, \ldots, u_n\}$ in the following manner: φ assumes value u_i while $x \in (x_{i-1}, x_i]$, $i \in \{1, 2, \ldots, n\}$ and $a = x_0 < x_1 < \ldots < x_n = b$. Now, $a = x_0 < x_1 < \ldots < x_n = b$ is one n-partition of interval $(a, b]$. The $\oplus$-*integral of the function* φ with respect to the $\oplus$-measure μ_{P_n} is given by

$$\int_{[a,b]}^{(\oplus,\odot)} \varphi\, d\mu_{P_n} = \bigoplus_{i=1}^{n} u_i \odot \mu_{P_n}((x_{i-1}, x_i]). \tag{2}$$

Remark 3. Since the form of partition P_n from (2) follows directly from the form of step function φ, the integral in (2) will be denoted with $\int_{[a,b]}^{(\oplus,\odot)} \varphi$, and, by means of the generating function g, it can be written as

$$\int_{[a,b]}^{(\oplus,\odot)} \varphi = g^{-1}\left(\sum_{i=1}^{n} (g(u_i))^\gamma (x_i - x_{i-1})\right).$$

With P_n' is denoted an $(n+1)$-partition of the interval $(a, b]$ obtained from n-partition P_n in the following manner: we keep all the points from the previous partition and add one more point and renumerate the points of the new partition in the increasing order. After s-repetition of this procedure an $(n+s)$-partition $P_n^{(s)}$ is obtained (see [20]). Now, if $f:[a,b]\to[0,\infty)$ is a continuous function, the $\oplus$-*integral* of the function f is

$$\int_{[a,b]}^{(\oplus,\odot)} f d\mu_{P_n} = \lim_{\substack{\mu_{P_n^{(s)}}\to\mathbf{0} \\ (s\to+\infty)}} \left(\bigoplus_{i=0}^{n+s-1} \Big(f(x_{i+1}) \odot \mu_{P_n^{(s)}}((x_i, x_{i+1}]) \Big) \right), \tag{3}$$

if the limit exists.

Remark 4. The limit in (3) is considered with respect to the metric based on generated pseudo-operations with two parameters.

Since it has been proved in [20] that the $\oplus$-integral does not depend on the partition of the interval $[a,b]$ and that it can be represented in the following manner

$$\int_{[a,b]}^{(\oplus,\odot)} f d\mu_{P_n} = g^{-1}\left(\int_a^b g^{\gamma} \circ f(x) dx \right),$$

further on the $\oplus$-integral will be denoted by $\int_{[a,b]}^{(\oplus,\odot)} f$.

Corresponding *pseudo-convolution* of the continuous functions $f, h : [0,\infty)\to[0,\infty)$ is

$$f \star h(x) = \int_{[0,x]}^{(\oplus,\odot)} \left([f]_g(x-t) \odot h(t)\right), \tag{4}$$

where $[\cdot]_g$ is a transform of the following form $[f]_g(x) = g^{-1}\left(g^{1/\gamma}(f(x))\right)$.

3. The $(\oplus,\odot)$-Laplace transform

Let $\oplus$ and $\odot$ be generated pseudo-operations with two parameters given by the generating function g.

Definition 5. T*he* $(\oplus,\odot)$*-Laplace transform of a continuous function* $f:[0,\infty)\to[0,\infty)$ *is*

$$\mathcal{L}_{\odot}^{\oplus}(f)(z) = \lim_{b\to\infty} \int_{[0,b]}^{(\oplus,\odot)} \left(\left[g^{-1}\right]_g (e^{-xz}) \odot f(x) \right), \tag{5}$$

if the limit exists.

Using the connection between the $\oplus$-integral and Riemann integral, the following form of $(\oplus,\odot)$-Laplace transform is obtained:

$$\mathcal{L}_{\odot}^{\oplus}(f)(z) = g^{-1}\left(\int_0^{\infty} e^{-xz\gamma} \left(g\left(f(x)\right)\right)^{\gamma} dx \right).$$

It can be proved that the pseudo-exchange formula for the $(\oplus,\odot)$-Laplace transform, i.e., the formula that transforms $\oplus$-convolution into pseudo-product, holds.

Theorem 6. *Let $\oplus$ and $\odot$ be generated pseudo-operations with two parameters given by the generating function g, $\mathcal{L}_{\odot}^{\oplus}$ corresponding transform given by (5), $\star$ pseudo-convolution given by (4) and $f_1, f_2 : [0,\infty) \to [0,\infty)$ continuous functions. Then, the following holds*

$$\mathcal{L}_{\odot}^{\oplus} \left[f_1 \star f_2\right]_g (z) = \left[\mathcal{L}_{\odot}^{\oplus} f_1\right]_g (z) \odot \mathcal{L}_{\odot}^{\oplus} f_2(z).$$

Proof. Follows from (5) and properties of the classical Laplace transform:

$$\begin{aligned}
\mathcal{L}_{\odot}^{\oplus} \left[f_1 \star f_2\right]_g (z) &= g^{-1}\left(\int_0^\infty e^{-xz\gamma} g^{\gamma}\left(\left[f_1 \star f_2\right]_g (x)\right) dx\right) \\
&= g^{-1}\left(\int_0^\infty e^{-xz\gamma} \left(h_1 \star_{cl} h_2(x)\right) dx\right) \\
&= g^{-1}\left(\mathcal{L}\left(h_1 \star_{cl} h_2\right)(z\gamma)\right) \\
&= g^{-1}\left(\mathcal{L}\left(h_1\right)(z\gamma) \cdot \mathcal{L}\left(h_2\right)(z\gamma)\right) \\
&= g^{-1}\left(\int_0^\infty e^{-xz\gamma} \left(h_1(x)\right) dx \cdot \int_0^\infty e^{-xz\gamma} \left(h_2(x)\right) dx\right) \\
&= \left[\mathcal{L}_{\odot}^{\oplus} f_1\right]_g (z) \odot \mathcal{L}_{\odot}^{\oplus} f_2(z),
\end{aligned}$$

where $h_i = g^{\gamma} \circ f_i$ are continuous functions, $\star_{cl}$ is the classical convolution and $\mathcal{L}$ is the classical Laplace transform. □

Example 7. Let $\oplus$ and $\odot$ be generated pseudo-operations with two parameters given by the generating function $g(x) = x^p$, $x \in [0,\infty)$, for some $p > 0$. Under this assumption, the corresponding $\mathcal{L}_{\odot}^{\oplus}$-transform of the function $f : [0,\infty) \to [0,\infty)$ is

$$\mathcal{L}_{\odot}^{\oplus}(f)(z) = \left(\int_0^\infty e^{-xz\gamma} \left(f(x)\right)^{p\gamma} dx\right)^{1/p}.$$

It can be easily shown that the exchange formula from the previous theorem holds.

Remark 8. Specially, for $\varepsilon = \gamma = 1$, the pseudo-Laplace transform from [18] can be obtained. In this case, the pseudo-exchange formula in cooperation with the inverse pseudo-Laplace transform has been used for the determination of utility functions' extreme values ([5, 18]).

Remark 9. Generalization of Laplace type transform of a measurable function $f : [0,\infty) \to [0,1]$, known as the (S,T)-Laplace transform, where $([0,1], S, T)$ is the conditionally distributive semiring, can be found in [5].

Remark 10. Another direction for generalization of the pseudo-Laplace type transform has been presented in [21]. This generalization is done on the domain of functions that pseudo-Laplace type transform has been applied to. In this case, $([a,b], \oplus, \odot)$ is a semiring from the first or second class (see [7, 10, 13, 16, 17, 18, 19]), $*$ is a binary operation on $[0,+\infty)$ which is non-decreasing in both coordinates, continuous on $[0,+\infty)^2$, commutative, associative, has 0 as identity, fulfills cancellation law and is given by the multiplicative generator $l : [0,\infty] \to [0,1]$ as $x * y = l^{-1}(l(x)l(y))$, and $\diamondsuit$ is another binary operation $[0,\infty)$ distributive with respect to $*$. For $\oplus = \max$ and $\odot$ being an Archimedean t-norm T given by the continuous and increasing generating function $\theta : [0,1] \to [0,1]$ (see [5]), generalized $(\max, T)$ -Laplace transform from [21] is mapping $\mathcal{L}^{\max}_{T,*}$ defined for all $F : [0,\infty) \to [0,1]$ as

$$\mathcal{L}^{\max}_{T,*}F(z) = \theta^{(-1)}\left(\sup_{x\geq 0} l(x\diamondsuit z)\theta\left(F(x)\right)\right), \quad z \geq 0.$$

If $\oplus$ and $\odot$ are strict pseudo-operations given by the generating function g (semiring of the second class, see [16, 17]), the generalized $(\oplus,\odot)$ -Laplace transform from [21] is mapping $\mathcal{L}^{\oplus}_{\odot,*}$ defined for $F : [0,\infty) \to [a,b]$ as

$$\mathcal{L}^{\oplus}_{\odot,*}F(z) = g^{-1}\left(\int_{[0,\infty)} l(x\diamondsuit z)g\left(F(x)\right)\, dx\right).$$

4. Pseudo-aggregation operators based on the $(\oplus,\odot)$-Laplace transforms

By an aggregation operator ([2]) is usually understood a function $A : \cup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$, such that

(i) $A(u_1,\dots,u_n) \leq A(v_1,\dots,v_2)$ when $u_i \leq v_i$ for all $i \in \{1,\dots,n\}$,

(ii) $A(u) = u$ for all $u \in [0,1]$,

(iii) $A(1,\dots,1) = 1$ and $A(0,\dots,0) = 0$.

A large class of aggregation operators have been constructed by different types of integrals ([1, 2, 6]). Now, a method similar to the construction of (S,U)-integral-based aggregation operators ([6]) can be applied to construct the following $\oplus$-integral-based aggregation type operator.

Pseudo-aggregation operator $\widetilde{A} : \cup_{n\in\mathbb{N}}[0,\infty)^n \to [0,\infty)$ based on the $\oplus$-integral is

$$\widetilde{A}(u_1,\dots,u_n) = \int^{(\oplus,\odot)}_{[0,1]} \varphi, \tag{6}$$

where $\varphi : (0,1] \to [0,\infty)$ is a function given by $\varphi(x) = u_i,\ \ x_{i-1} < x \leq x_i$, $i \in \{1,\dots,n\}$, for some n-partition $0 = x_0 < x_1 \ldots < x_n = 1$ (see [2, 6]). For operators given by (6) following hold:

(i) $\widetilde{A}(u_1, \ldots, u_n) \leq \widetilde{A}(v_1, \ldots, v_2)$ when $u_i \leq v_i$ for all $i \in \{1, \ldots, n\}$,

(ii) $\widetilde{A}(u) = u \odot \mathbf{1}$ for all input values u,

(iii) $\widetilde{A}(\mathbf{1}, \ldots, \mathbf{1}) = \mathbf{1}$ and $\widetilde{A}(\mathbf{0}, \ldots, \mathbf{0}) = \mathbf{0}$,

where $\mathbf{0}$ and $\mathbf{1}$ are neutral elements for the pseudo-addition $\oplus$ and pseudo-multiplication $\odot$, respectively.

Remark 11. Input values u_i and v_i from property (i) are associated with the same subinterval $(x_{i-1}, x_i]$, $i \in \{1, \ldots, n\}$. For each input value, the corresponding associated interval can be considered as an area of influence of the input value in question.

Proposition 12. *Let $\widetilde{A}$ be a pseudo-aggregation operator given by (6). For the input values $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ and parameters $\alpha, b \in [0, \infty)$ the following holds*

(i) $\widetilde{A}\left([u_1 \oplus b]_g, \ldots, [u_n \oplus b]_g\right) = \widetilde{A}\left([u_1]_g, \ldots, [u_n]_g\right) \oplus b$,

(ii) $\widetilde{A}\left([\alpha]_g \odot u_1, \ldots, [\alpha]_g \odot u_n\right) = \alpha \odot \widetilde{A}\left(u_1, \ldots, u_n\right)$,

(iii) $\widetilde{A}\left([u_1 \oplus v_1]_g, \ldots, [u_n \oplus v_n]_g\right) = \widetilde{A}\left([u_1]_g, \ldots, [u_n]_g\right) \oplus \widetilde{A}\left([v_1]_g, \ldots, [v_n]_g\right)$.

Proof. (i) Let us suppose that the subinterval $(x_{i-1}, x_i]$ is associated to the both input values $[u_1 \oplus b]_g$ and $[u_1]_g$. Now, this distorted shift invariant property follows from (6) and (2):

$$
\begin{aligned}
\widetilde{A}\left([u_1 \oplus b]_g, \ldots, [u_n \oplus b]_g\right) &= g^{-1}\left(\sum_{i=1}^{n} g^{\gamma}\left([u_i \oplus b]_g\right)(x_i - x_{i-1})\right) \\
&= g^{-1}\left(\varepsilon \sum_{i=1}^{n} g(u_i)(x_i - x_{i-1}) + g(b)\right) \\
&= g^{-1}\left(\varepsilon g\left(\widetilde{A}\left([u_1]_g, \ldots, [u_n]_g\right)\right) + g(b)\right) \\
&= \widetilde{A}\left([u_1]_g, \ldots, [u_n]_g\right) \oplus b.
\end{aligned}
$$

Properties (ii) and (iii) are consequences of Theorem 3 from [20]. Input values u_1 and $[\alpha]_g \odot u_1$, i.e., input values $[u_1]_g$, $[v_1]_g$ and $[u_1 \oplus v_1]_g$, are associated with the same area of influence. □

Also, it can be easily shown that for the operator $\widetilde{A}$ idempotent property is distorted in the following manner:

$$
\widetilde{A}\left([u]_g, \ldots, [u]_g\right) = u.
$$

Example 13. Let us consider the generating function $g : [0,\infty) \to [0,\infty)$ such that $g(x) = x^p$ for some $p > 0$. Corresponding pseudo-operations are $x \oplus y = (\varepsilon x^p + y^p)^{1/p}$ and $x \odot y = x^\gamma y$. If the length of each interval $(x_{i-1}, x_i]$ associated to the input value u_i is denoted with $l_i,\ i \in \{1,\ldots,n\},$ the operator $\widetilde{A}$ of n inputs $u_1, u_2, \ldots, u_n$ has the following form

$$\widetilde{A}(u_1,\ldots,u_n) = \left(\sum_{i=1}^{n} l_i u_i^{\gamma p}\right)^{1/p},$$

where $0 = x_0 < x_1 < \ldots < x_n = 1,$ and $\sum_{i=1}^{n} l_i = 1$. If all intervals $(x_{i-1}, x_i]$ are of the equal length, the operator $\widetilde{A}$ is

$$\widetilde{A}(u_1,\ldots,u_n) = \left(\frac{1}{n}\sum_{i=1}^{n} u_i^{\gamma p}\right)^{1/p}.$$

The question is whether operators of aggregation type can be induced by means of the $(\oplus,\odot)$-Laplace transforms.

Let $u_1, u_2, \ldots, u_n$ be n input values from $[0,\infty)$. For each n input value and each n-partition where $0 = x_0 < x_1 < \ldots < x_n = 1$ of the interval $(0,1]$ is possible to form a step function $\varphi : (0,\infty) \to [0,\infty)$ as

$$\varphi(x) = \begin{cases} u_i, & \text{for } x \in (x_{i-1}, x_i], \\ g^{-1}(0), & \text{for } x > 1, \end{cases} \tag{7}$$

where g is a generating function for the pseudo-operations $\oplus$ and $\odot$ given by (1).

Definition 14. The pseudo-aggregation operator $\widetilde{A_L} : \cup_{n\in\mathbb{N}}[0,\infty)^n \to [0,\infty)$ based on the $(\oplus,\odot)$-Laplace transform *is*

$$\widetilde{A_L}(u_1,\ldots,u_n) = \mathcal{L}_{\odot}^{\oplus}(\varphi)(z), \tag{8}$$

where φ is a step function for input values $u_1, u_2, \ldots, u_n$ given by (7) and z is some real positive parameter.

Since $(\oplus,\odot)$-Laplace transform is based on non-associative and non-commutative pseudo-operations, the impact of some input value on the result can be determined by its index and by length of the associated subinterval of the unite interval.

It can be easily shown that the pseudo-aggregation operator $\widetilde{A_L}$ with parameter z has the following form

$$\widetilde{A_L}(u_1,\ldots,u_n) = \bigoplus_{i=1}^{n} u_i \odot \omega_{(x_{i-1},x_i],z}, \tag{9}$$

where $(x_{i-1}, x_i]$ is a subinterval of the unite interval associated to the input value u_i and

$$\omega_{(x_{i-1},x_i],z} = g^{-1}\left(\frac{e^{-z\gamma x_{i-1}} - e^{-z\gamma x_i}}{\varepsilon^{n-i}\gamma z}\right).$$

Basic properties of the pseudo-aggregation operator $\widetilde{A_L}$ with parameter z are given by next proposition.

Proposition 15. *Let $\widetilde{A_L}$ be a pseudo-aggregation operator given by (8). Then*

(i) $\widetilde{A_L}(u_1, \ldots, u_n) \leq \widetilde{A_L}(v_1, \ldots, v_2)$ *when* $u_i \leq v_i$ *and* u_i *and* v_i *are associated to the same subinterval* $(x_{i-1}, x_i]$, $i \in \{1, \ldots, n\}$,

(ii) $\widetilde{A_L}(u) = u \odot \omega_{(0,1],z}$ *for all input values* u,

(iii) $\widetilde{A_L}(\mathbf{1}, \ldots, \mathbf{1}) = \mathbf{1} \odot \omega_{(0,1],z}$ *and* $\widetilde{A_L}(\mathbf{0}, \ldots, \mathbf{0}) = \mathbf{0} \odot \omega_{(0,1],z}$,

where $\omega_{(0,1],z} = g^{-1}\left((1 - e^{-z\gamma})/\gamma z\right)$.

Proof. Follows directly from the definition of pseudo-aggregation operators, properties of the generating function g and (9). □

Example 16. Let $\oplus$ and $\odot$ be generated pseudo-operations with two parameters given by the generating function $g(x) = x^p$, $x \in [0, \infty)$ for some $p > 0$. Now, the corresponding pseudo-aggregation operator $\widetilde{A_L}$ with parameter z for input values $u_1, \ldots, u_n$ is

$$\widetilde{A_L}\left(u_1, \ldots, u_n\right) = \left(\frac{1}{z\gamma}\sum_{i=1}^{n} u_i^{p\gamma}\left(e^{-z\gamma x_{i-1}} - e^{-z\gamma x_i}\right)\right)^{\frac{1}{p}}.$$

Some other properties of the pseudo-aggregation operators $\widetilde{A_L}$ are given by next theorem.

Theorem 17. *Let $\widetilde{A_L}$ be a pseudo-aggregation operator given by (8). For input the values $u, u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ and real parameters $\alpha, b \in [0, \infty)$, the following holds*

(i) $\widetilde{A_L}(u, \ldots, u) = u \odot \omega_{(0,1],z}$,

(ii) $\widetilde{A_L}\left([u_1 \oplus b]_g, \ldots, [u_n \oplus b]_g\right) = \widetilde{A_L}\left([u_1]_g, \ldots, [u_n]_g\right) \oplus \left([b]_g \odot \omega_{(0,1],z}\right)$,

(iii) $\widetilde{A_L}\left([\alpha]_g \odot u_1, \ldots, [\alpha]_g \odot u_n\right) = \alpha \odot \widetilde{A_L}\left(u_1, \ldots, u_n\right)$,

(iv) $\widetilde{A_L}\left([u_1 \oplus v_1]_g, \ldots, [u_n \oplus v_n]_g\right)$

$$= \widetilde{A_L}\left([u_1]_g, \ldots, [u_n]_g\right) \oplus \widetilde{A_L}\left([v_1]_g, \ldots, [v_n]_g\right).$$

Proof. Distorted idempotent property given by (i) follows from (9):

$$\begin{aligned}
\widetilde{A_L}(u,\ldots,u) &= \bigoplus_{i=1}^{n} u \odot \omega_{(x_{i-1},x_i],z} \\
&= g^{-1}\left(\sum_{i=1}^{n} \varepsilon^{n-i} g^{\gamma}(u) g(\omega_{(x_{i-1},x_i],z})\right) \\
&= g^{-1}\left(g^{\gamma}(u) g(\omega_{(0,1],z})\right) \\
&= u \odot \omega_{(0,1],z}.
\end{aligned}$$

Properties (ii), (iii) and (iv) can be easily proven in a similar manner. □

5. Conclusion

The main aim of this paper has been to present further possible steps in the generalization, based on the pseudo-analysis' apparatus, of well known notions as Laplace transform and aggregation operators that could broaden the area of applications. Some further research of this problem should concern properties of the $(\oplus, \odot)$-Laplace transform and pseudo-aggregation operators, and possible applications.

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