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CURVATURE THEORY OF GENERALIZED CONNECTION IN J_k^2M

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Abstract. The introduction of J_k^2M manifold and its geometrical presentation is given in [2]. Here, the generalized connection is defined, the torsion and curvature tensors are determined, and the Ricci equations are established.

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1. Manifold J_k^2M

Let M be a smooth manifold of dimension n and $J_{o,p}(\mathbf{R}^k, M)$ the set of germs of smooth mappings $f : \mathbf{R}^k \to M$ with $f(o) = p \in M$. We say that $f, g \in J_{o,p}(\mathbf{R}^k, M)$ are equivalent up to order q if there exists a chart (U, φ) around p such that

$$d_o^h(\varphi \circ f) = d_o^h(\varphi \circ g),\ 1 \leq h \leq q, \tag{1.1}$$

where d means Frechet differentiation. It can be seen that if (1.1) holds for a chart (U, φ), it holds for any other chart (V, ψ) around p.

We denote by $j_{o,p}^q f$ the equivalence class of f (the coset of f) and set $J_{o,p}^q = \{j_{o,p}^q f, f \in J_{o,p}(\mathbf{R}^k, M)\}$. Then we put $J_k^q M = \bigcup_{p\in M} J_{o,p}^q$ and define $\pi : J_k^q M \to M$ by $\pi(J_{o,p}^q) = p$.

One can see that $J_m^q M$ has a structure of smooth manifold.

We notice that for $k = 1$, this manifold is just the manifold $\mathrm{Osc}^q M$ studied by R. Miron [5], which reduces to the tangent manifold for $q = 1$. For $k = n$ and $q = 1$, we get the manifold of frames over M and for $k \in \{2, 3, \dots, n-1\}$ and $q = 1$ it can be identified to $TM \otimes \cdots \otimes TM$ (k times), which is the manifold supporting the k-Lagrange geometry, see R. [7].

For these reasons we confine ourselves to the cases $k = 2, 3, \dots, n-1$ and for the sake of simplicity we take $q = 2$. The case q greater than 2 can be similarly treated.

We also notice that J_k^2M is the manifold of 2-jets of the sections of the fibre bundle $\mathbf{R}^k \times M \to \mathbf{R}^k$ but the theory of jets from the book by D.J. Saunders [8] cannot be applied since the typical fibre M of this bundle is too general.

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Instead of that theory we follow the ideas and techniques from the k-Lagrange geometry and from the geometry of $\mathrm{Osc}^q M$ spaces as well, see [1], [3], [6].

Let us come back to (1.1) for $q = 2$. Letting $\varphi \circ g : \mathbf{R}^k \to \mathbf{R}^n$ as $f^i = f^i(t^1, \ldots, t^k)$, $g^i = g^i(t^1, \ldots, t^k)$ this condition becomes

$$(1.1)' \quad f^i(o) = g^i(o) = \varphi(p),\ \frac{\partial f^i}{\partial t^\alpha}(o) = \frac{\partial g^i}{\partial t^\alpha}(o),\ \frac{\partial^2 f^i}{\partial t^\alpha \partial t^\beta}(o) = \frac{\partial^2 g^i}{\partial t^\alpha \partial t^\beta}(o),$$

for $\alpha, \beta = 1, 2, \ldots, k$. Let us set $\partial_i : \frac{\partial}{\partial x^i}$, $\partial_\alpha := \frac{\partial}{\partial t^\alpha}$.

Now, for another local chart (V, ψ) around p such that $\psi \circ \varphi^{-1} : x^{i'} = x^{i'}(x^1, \ldots, x^n)$, rank $\left(\frac{\partial x'}{\partial x^k}\right) = n$, taking $\psi \circ f$ and $\psi \circ g$ as $f^{i'} = f^{i'}(t^1, \ldots, t^k)$ and $g^{i'} = g^{i'}(t^1, \ldots, t^k)$, respectively, we get $f^{i'} = x^{i'}(f^j(t^1, \ldots, t^k))$, $g^{i'} = x^{i'}(g^j(t^1, \ldots, t^k))$ as well as

$$(1.2) \quad \begin{aligned} &\tfrac{\partial f^{i'}}{\partial t^\alpha}(o) = \tfrac{\partial x^{i'}}{\partial x^j}(\varphi(p)) \tfrac{\partial f^j}{\partial t^\alpha}(o) \\ &\tfrac{\partial^2 f^{i'}}{\partial t^\alpha \partial t^\beta} = \tfrac{\partial^2 x^{i'}}{\partial x^j \partial x^k}(\varphi(p)) \tfrac{\partial f^j}{\partial t^\alpha}(o) \tfrac{\partial f^k}{\partial t^\beta}(o) + \tfrac{\partial x^{i'}}{\partial x^j} \tfrac{\partial^2 f^j}{\partial t^\alpha \partial t^\beta}. \end{aligned}$$

By (1.2) it follows the independence of (1.1) on the chosen local chart.

For $f : \mathbf{R}^k \to M$ with $f(o) = \varphi(p)(x^1, \ldots, x^n)$ we set $y^{\alpha i} = \frac{\partial f^i}{\partial t^\alpha}(o)$, $z^{\alpha\beta i} = \frac{\partial^2 f^i}{\partial t^\alpha \partial t^\beta}(o)$ and define a mapping $\phi : \pi^{-1}(U) \to \varphi(U) \times \mathbf{R}^{kn} \times \mathbf{R}^{\frac{k(k+1)}{2} n}$ by $\phi([f]_p) = (x^i, y^{\alpha i}, z^{\alpha\beta i})$.

The mapping ϕ is invertible, its inverse associating to $(x^i, y^{\alpha i}, z^{\alpha\beta i})$ the coset of the mapping $\varphi^{-1} \circ T$, where T is the Taylor polynomial of second order with respect to t.

2. Decomposition of $T(E)$. Integrability conditions

Let $E = J^2_k M$ be an $n + kn + 2^{-1}k(k+1)n$ dimensional C^∞ manifold. Some point $u \in J^2_k M$ in the local charts (U, φ) and (U', φ') has coordinates $(x^i, y^{\alpha i}, z^{\alpha\beta i})$ and $(x^{i'}, y^{\alpha i'}, z^{\alpha\beta i'})$ respectively. In $U \cap U'$, the allowable coordinate transformations are given by the equation

$$(2.1) \quad \begin{aligned} &x^{i'} = x^{i'}(x^1, x^2, \ldots, x^n),\ rank(\frac{\partial x^{i'}}{\partial x^i}) = n, \\ &y^{\alpha i'} = \frac{\partial x^{i'}}{\partial x^i} y^{\alpha i} = y^{\alpha i'}(x^i, y^{\alpha i}),\ \alpha \leq \beta, rank(y^{\alpha i}) = k, \\ &z^{\alpha\beta i'} = \frac{\partial^2 x^{i'}}{\partial x^j \partial x^h} y^{\alpha j} y^{\beta h} + \frac{\partial x^{i'}}{\partial x^i} z^{\alpha\beta i} = z^{\alpha\beta i'}(x^i, y^{\alpha i}, z^{\alpha\beta i}) \\ &i, j, h, k, l = 1, 2, \ldots, n, \quad \alpha, \beta, \gamma, \delta, \kappa, \varepsilon = 1, 2, \ldots, k. \end{aligned}$$

Proposition 2.1. *Transformations of type (2.1) form a pseudo group.*

Proof. The neutral element of the group is given by the transformation $x^{i'} = x^i$, $y^{\alpha i'} = y^{\alpha i}$, $z^{\alpha\beta i'} = z^{\alpha\beta i}$. From $rank(\frac{\partial x^{i'}}{\partial x^i}) = n$ it follows that (2.1) has inverse transformation of the same type. If the point u in some local chart (U'', φ'') has coordinates $(x^{i''}, y^{\alpha i''}, z^{\alpha\beta i''})$, then in $U' \cap U''$ the transformation law is given by equation obtained from (2.1) in such a way, that the latin indices obtain one more prime. After some calculation it follows that the coordinates of the point u in $U \cap U''$ satisfy the equation of type (2.1) if everywhere the index i' is substituted by i''.

Let us introduce the notations:

$$\partial_i = \frac{\partial}{\partial x^i},\ \partial_{\alpha i} = \frac{\partial}{\partial y^{\alpha i}},\ \partial_{\alpha\beta i} = \frac{\partial}{\partial z^{\alpha\beta i}}, (\alpha \leq \beta). \tag{2.2}$$

The natural basis $\bar{B}$ of $T(J_k^2M) = T(E)$ is $\bar{B} = \{\partial_i, \partial_{\alpha i}, \partial_{\alpha\beta i}\}$.

If a change of coordinates (2.1) is performed, the elements of $\bar{B}$ are transformed as follows:

$$\begin{aligned} \partial_i &= (\partial_i x^{i'})\partial_{i'} + (\partial_i\partial_j x^{i'})y^{\alpha j}\partial_{\alpha i'} + [(\partial_i\partial_j\partial_h x^{i'})y^{\beta j}y^{\gamma h} + (\partial_i\partial_j x^{i'})z^{\beta\gamma j}]\partial_{\beta\gamma i'} \\ \partial_{\alpha i} &= (\partial_i x^{i'})\partial_{\alpha i'} + 2(\partial_i\partial_j x^{i'})y^{\gamma j}\partial_{\alpha\gamma i'} \\ \partial_{\alpha\beta i} &= (\partial_i x^{i'})\partial_{\alpha\beta i'}. \end{aligned} \tag{2.3}$$

The adapted basis of $T(E)$ is $B = \{\delta_i, \delta_{\alpha i}, \delta_{\alpha\beta i}\}$, where

$$\begin{aligned} &\text{(a)} \quad \delta_i = \partial_i - N_i^{\alpha j}\partial_{\alpha j} - N_i^{\alpha\beta j}\partial_{\alpha\beta j}, (\alpha \leq \beta), \\ &\text{(b)} \quad \delta_{\alpha i} = \partial_{\alpha i} - N_{\alpha i}^{\beta\gamma j}\partial_{\beta\gamma j}, (\beta \leq \gamma), \\ &\text{(c)} \quad \delta_{\alpha\beta i} = \partial_{\alpha\beta i}. \end{aligned} \tag{2.4}$$

The summation is going over both types of indices.

From (2.4) it follows that $\partial_{\alpha\beta i}$ is transformed as d tensor field, i.e. $\partial_{\alpha\beta i} = (\partial_i x^{i'})\partial_{\alpha\beta i'}$.

Proposition 2.2. *The elements of B are transformed as d-tensor field, i.e.*

$$\delta_i = (\partial_i x^{i'})\delta_{i'} \quad \delta_{\alpha i} = (\partial_i x^{i'})\delta_{\alpha i'}, \tag{2.5}$$

if the nonlinear connection coefficients obey the following transformation law:

$$\begin{aligned} &(a) \quad N_{\alpha i'}^{\beta\gamma j'}(\partial_i x^{i'}) = N_{\alpha i}^{\beta\gamma j}(\partial_j x^{j'}) - (\partial_{\alpha i} z^{\beta\gamma j'}), \\ &(b) \quad N_{i'}^{\alpha j'}(\partial_i x^{i'}) = N_i^{\alpha j}(\partial_j x^{j'}) - (\partial_i y^{\alpha j'}), \\ &(c) \quad N_{i'}^{\alpha\beta j'}(\partial_i x^{i'}) = N_i^{\alpha\beta j}(\partial_j x^{j'}) + N_i^{\gamma j}(\partial_{\gamma j} z^{\alpha\beta j'}) - \partial_i z^{\alpha\beta j'}. \end{aligned} \tag{2.6}$$

Proof. The proof follows from (2.2)–(2.5).

If we denote by T_H, T_{V_1} and T_{V_2} the subspaces of $T(E)$ (at the point u) spanned by $\{\delta_i\}$, $\{\delta_{\alpha i}\}$, $\{\delta_{\alpha\beta i}\}$, then we have

$$T(E) = T_H \oplus T_{V_1} \oplus T_{V_2},$$

where $dimT_H = n$, $dimT_{V_1} = nk$, $dimT_{V_2} = 2^{-1}k(k+1)n$.

The dual basis of $\bar{B}$ is $\bar{B}^* = \{dx^i, dy^{\alpha i}, dz^{\alpha\beta i}\}$. By a change of coordinates (2.1) the element of $\bar{B}^*$ are transformed as follows:

$$\begin{aligned} dx^{i'} &= (\partial_i x^{i'})dx^i, \\ dy^{\alpha i'} &= (\partial_i\partial_j x^{i'})y^{\alpha j}dx^i + (\partial_i x^{i'})dy^{\alpha i}, \\ dz^{\alpha\beta i'} &= [(\partial_i\partial_j\partial_h x^{i'})y^{\alpha j}y^{\beta h} + (\partial_i\partial_j x^{i'})z^{\alpha\beta j}]dx^i + \\ &\quad (\partial_j\partial_h x^{i'})(y^{\beta h}dy^{\alpha j} + y^{\alpha h}dy^{\beta j}) + (\partial_i x^{i'})dz^{\alpha\beta i}. \end{aligned} \tag{2.7}$$

The adapted basis of $T^*(E)$ is $B^* = \{dx^i, \delta y^{\alpha i}, \delta z^{\alpha\beta i}\}$, where

$$\begin{aligned} \delta y^{\alpha j} &= dy^{\alpha j} + M_i^{\alpha j}dx^i \\ \delta z^{\alpha\beta j} &= dz^{\alpha\beta j} + M_{\gamma i}^{\alpha\beta j}dy^{\gamma i} + M_i^{\alpha\beta j}dx^i. \end{aligned} \tag{2.8}$$

The functions M are, for the time being, undetermined.

Proposition 2.3. *The necessary and sufficient conditions that the bases B and B^* to be dual to each other (when $\bar{B}$ and $\bar{B}^*$ are dual) are the following equations:*

$$\begin{aligned} &(a)\quad M_i^{\alpha j} = N_i^{\alpha j} \quad (b)\ M_{\gamma i}^{\alpha\beta j} = N_{\gamma i}^{\alpha\beta j}, (\alpha \le \beta) \\ &(c)\quad M_i^{\alpha\beta j} = N_i^{\alpha\beta j} + N_i^{\gamma h}N_{\gamma h}^{\alpha\beta j}, (\alpha \le \beta). \end{aligned} \tag{2.9}$$

Proof. The proof follows from (2.4)–(2.9).

Remark. The bases B and B^* are more general than those used in [2], but they are not in accordance with the operator J_α, where

$$J_\alpha\delta_i = \delta_{\alpha i}, \quad J_\beta\delta_{\alpha i} = \delta_{\beta\alpha i}, \quad J_\gamma\delta_{\alpha\beta i} = 0. \tag{2.10}$$

If we take the basis $\tilde{B} = \{\delta_i, \delta_{\alpha i}, \delta_{\alpha\beta i}\}$, where δ_i and $\delta_{\alpha\beta i}$ are determined by (2.4a) and (2.4c) and

$$\delta_{\alpha i} = \partial_{\alpha i} - N_i^{\beta j}\partial_{\alpha\beta j}$$

then (2.10) are satisfied. The coresponding dual basis $\tilde{B}^*$ is determined by (2.8), but now (2.9) has different form (see (2.16) in [2]).

Proposition 2.4. *The horizontal distribution T_H is integrable iff the following relations are satisfied*

$$K_{i\ j}^{\ \beta k} = \bar{K}_{i\ j}^{\ \beta k} = \delta_j N_i^{\beta k} - \delta_i N_j^{\beta k} = 0, \tag{2.11}$$

$$K_{i\ j}^{\ \gamma\delta k} = \bar{K}_{i\ j}^{\ \gamma\delta k} + K_{i\ j}^{\ \beta h} N_{\beta h}^{\gamma\delta k} = 0, \tag{2.12}$$

$$(\bar{K}_{i\ j}^{\ \gamma\delta k} = \delta_j N_i^{\gamma\delta k} - \delta_i N_j^{\gamma\delta k}). \tag{2.13}$$

Proof. A straightforward calculation gives

$$[\delta_i, \delta_j] = K_{i\ j}^{\ \kappa k}\,\delta_{\kappa k} + K_{i\ j}^{\ \kappa\rho k}\,\delta_{\kappa\rho k}, \tag{2.14}$$

and from (2.11)–(2.13) it follows the statement.

Proposition 2.5. *The vertical distribution T_{V_1} is integrable iff*

$$K_{\alpha i\ \beta j}^{\ \ \delta\kappa k} = \delta_{\beta j} N_{\alpha i}^{\ \delta\kappa k} - \delta_{\alpha i} N_{\beta j}^{\ \delta\kappa k} = 0.$$

The proof follows from

$$[\delta_{\alpha i}, \delta_{\beta j}] = K_{\alpha i\ \beta j}^{\ \ \delta\kappa k}\delta_{\delta\kappa k}. \tag{2.15}$$

T_{V_2} is integrable distribution because

$$[\delta_{\alpha\beta i}, \delta_{\gamma\delta j}] = 0. \tag{2.16}$$

3. The generalized connection on $T(E)$

Definition 3.1. *The generalized connection $D : T(E)\times T(E) \to T(E)$, $(X, Y) \to D_X Y$, $X, Y \in T(E)$ is the linear connection defined by:*

$$\begin{aligned}
D_{\delta_i}\delta_j &= \underline{F_{j\ i}^{\ k}\,\delta_k} + F_{j\ i}^{\ \kappa k}\,\delta_{\kappa k} + F_{j\ \ i}^{\ \kappa\rho k}\,\delta_{\kappa\rho k}\\
D_{\delta_i}\delta_{\gamma j} &= F_{\gamma j\ i}^{\ \ k}\delta_k + \underline{F_{\gamma j\ i}^{\ \ \kappa k}\,\delta_{\kappa k}} + F_{\gamma j\ \ i}^{\ \ \kappa\rho k}\,\delta_{\kappa\rho k}\\
D_{\delta_i}\delta_{\gamma\delta j} &= F_{\gamma\delta j\ i}^{\ \ \ k}\delta_k + F_{\gamma\delta j\ i}^{\ \ \ \kappa k}\delta_{\kappa k} + \underline{F_{\gamma\delta j\ \ i}^{\ \ \ \kappa\rho k}\delta_{\kappa\rho k}}\\
D_{\delta_{\alpha i}}\delta_j &= \underline{F_{j\ \alpha i}^{\ k}\delta_k} + F_{j\ \alpha i}^{\ \kappa k}\,\delta_{\kappa k} + F_{j\ \ \alpha i}^{\ \kappa\rho k}\delta_{\kappa\rho k}\\
D_{\delta_{\alpha i}}\delta_{\gamma j} &= F_{\gamma j\ \alpha i}^{\ \ k}\delta_k + \underline{F_{\gamma j\ \alpha i}^{\ \ \kappa k}\delta_{\kappa k}} + F_{\gamma j\ \ \alpha i}^{\ \ \kappa\rho k}\delta_{\kappa\rho k}\\
D_{\delta_{\alpha i}}\delta_{\gamma\delta j} &= F_{\gamma\delta j\ \alpha i}^{\ \ \ k}\delta_k + F_{\gamma\delta j\ \alpha i}^{\ \ \ \kappa k}\delta_{\kappa k} + \underline{F_{\gamma\delta j\ \ \alpha i}^{\ \ \ \kappa\rho k}\delta_{\kappa\rho k}}\\
D_{\delta_{\alpha\beta i}}\delta_j &= \underline{F_{j\ \alpha\beta i}^{\ k}\delta_k} + F_{j\ \alpha\beta i}^{\ \kappa k}\delta_{\kappa k} + F_{j\ \ \alpha\beta i}^{\ \kappa\rho k}\delta_{\kappa\rho k}\\
D_{\delta_{\alpha\beta i}}\delta_{\gamma j} &= F_{\gamma j\ \alpha\beta i}^{\ \ k}\delta_k + \underline{F_{\gamma j\ \alpha\beta i}^{\ \ \kappa k}\delta_{\kappa k}} + F_{\gamma j\ \ \alpha\beta i}^{\ \ \kappa\rho k}\delta_{\kappa\rho k}\\
D_{\delta_{\alpha\beta i}}\delta_{\gamma\delta j} &= F_{\gamma\delta j\ \alpha\beta i}^{\ \ \ k}\delta_k + F_{\gamma\delta j\ \alpha\beta i}^{\ \ \ \kappa k}\delta_{\kappa k} + \underline{F_{\gamma\delta j\ \ \alpha\beta i}^{\ \ \ \kappa\rho k}\delta_{\kappa\rho k}} \quad (\kappa \le \rho).
\end{aligned} \tag{3.1}$$

Different types of linear connection in higher order geometries are given in [4]–[7].

Definition 3.2. *If on the right-hand side of (3.1) all terms vanish except for the underlined then the generalized connection reduces to the distinguished d-connection.*

If on the right-hand side of (3.1) all terms vanish except for $F_{j\ i}^{\ k}$, $F_{\gamma j\ \alpha i}^{\ \ \kappa k}$, $F_{\gamma\delta j\ \ \alpha\beta i}^{\ \ \ \kappa\rho k}$, the generalized connection reduces to the strongly distinguished (s.d.) connection.

For the sake of brevity it is convenient to use new kind of indices, the latin capitals, which take values from 1 to $n+nk+2^{-1}k(k+1)n$. Using them, (3.1) can be written in the form

$$D_{\delta_I}\delta_J = F_{J\ I}^{\ K}\delta_K = F_{J\ I}^{\ k}\delta_k + F_{J\ I}^{\ \kappa k}\delta_{\kappa k} + F_{J\ I}^{\ \kappa\rho k}\delta_{\kappa\rho k}. \tag{3.2}$$

Let X and Y be vector fields determined on $T(E)$ by:

$$X = X^I\delta_I = X^i\delta_i + X^{\alpha i}\delta_{\alpha i} + X^{\alpha\beta i}\delta_{\alpha\beta i}, \tag{3.3}$$

$$Y = Y^J\delta_J = Y^j\delta_j + Y^{\gamma j}\delta_{\gamma j} + Y^{\gamma\delta j}\delta_{\gamma\delta j} \tag{3.4}$$

then (3.2) has the form

$$D_XY = D_{X^I\delta_I}Y^J\delta_J = X^I(\delta_IY^J + F_{HI}^JY^H)\delta_J = X^IY^J_{|I}\delta_I. \tag{3.5}$$

Using the explicit forms of (3.3) and (3.4) for (3.5) we get the following proposition.

Proposition 3.1. *The generalized connection D can be expressed by covariant derivatives in the form:*

$$\begin{aligned} D_XY \ = \ & (X^iY^j_{|i} + X^{\alpha i}Y^j_{|\alpha i} + X^{\alpha\beta i}Y^j_{|\alpha\beta i})\delta_j + \\ & (X^iY^{\gamma j}_{|i} + X^{\alpha i}Y^{\gamma j}_{|\alpha i} + X^{\alpha\beta i}Y^{\gamma j}_{|\alpha\beta i})\delta_{\gamma j} + \\ & (X^iY^{\gamma\delta j}_{|i} + X^{\alpha i}Y^{\gamma\delta j}_{|\alpha i} + X^{\alpha\beta i}Y^{\gamma\delta j}_{|\alpha\beta i})\delta_{\gamma\delta j}. \end{aligned} \tag{3.6}$$

In (3.1), all the connection coefficients F are arbitrary smooth function of x, y and z, but they should satisfy prescribed transformation law with respect to (2.1). Our intention is to find these laws of transformations. All covariant derivatives

$$Y^J_{|I} = \delta_IY^J + F_{H\ I}^{\ J}Y^H = \delta_IY^J + F_{h\ I}^{\ J}Y^h + F_{\nu h\ I}^{\ \ J}Y^{\nu h} + F_{\mu\nu h\ I}^{\ \ \ J}Y^{\mu\nu h} \tag{3.7}$$

which appear in (3.6) are d-tensor fields.

In (3.7) $I \in \{i, \alpha i, \alpha\beta i\}$, $J \in \{j, \gamma j, \gamma\delta j\}$.

For the d-connection (3.7) takes the form:

$$Y^j_{|I} = \delta_IY^j + F_{h\ I}^{\ j}Y^h$$

$$Y^{\gamma j}_{|I} = \delta_IY^{\gamma j} + F_{\delta d\ I}^{\ \ \gamma j}Y^{\delta d}$$

$$Y^{\alpha\beta j}_{|I} = \delta_IY^{\alpha\beta j} + F_{\mu\nu h\ I}^{\ \ \ \alpha\beta j}Y^{\mu\nu h}$$

$$I \in \{i, \kappa k, \gamma\kappa k\}.$$

The necessary and sufficient conditions that all covariant derivatives $Y^J_{|I}$ appeared in (3.6) be d-tensor fields are given in the following proposition.

Proposition 3.2. *All the connection coefficients $F_{H\ I}^{\ J}$ that appear in (3.7) transform as d-tensor fields except for the case when $I = i$. Then we have*

$$(F_{H'\ i'}^{\ J'})(\partial_i x^{i'})(\partial_h x^{h'}) = F_{H\ i}^{\ J}(\partial_j x^{j'}) - (\partial_h \partial_i x^{j'}). \tag{3.8}$$

Proof. If we suppose that $Y^J_{|i}$ ($I = i$ in (3.7)) is d-tensor field, then

$$Y^J_{|i}(\partial_j x^{j'}) = Y^{J'}_{|i'}(\partial_i x^{i'}),$$

i.e.

$$(\delta_i Y^J + F_{H\ i}^{\ J} Y^H)(\partial_j x^{j'}) = (\delta_{i'} Y^{J'} + F_{H'\ i'}^{\ J'} Y^{H'})(\partial_i x^{i'}). \tag{3.9}$$

As (see (2.4a) and (2.5))

$$\begin{aligned}&\delta_i Y^J(\partial_j x^{j'}) = \delta_i(Y^J \partial_j x^{j'}) - Y^J \delta_i(\partial_j x^{j'}) =\\&(\partial_i x^{i'})\delta_{i'} Y^{J'} - Y^J(\partial_i \partial_j x^{j'})\end{aligned}$$

the substitution of the above equation in (3.9) gives

$$F_{H\ i}^{\ J} Y^H \partial_j x^{j'} - Y^H(\partial_h \partial_i x^{j'}) = F_{H'\ i'}^{\ J'} Y^H(\partial_h x^{h'})(\partial_i x^{i'})$$

from which follows (3.8). The connection coefficients from (3.8) appeared in the first three lines of (3.1).

If we put $I = \alpha i$ in (3.7), and suppose that $Y^J_{|\alpha i}$ is d-tensor field, then we get

$$Y^J_{|\alpha i}(\partial_j x^{j'}) = Y^{J'}_{|\alpha i'}(\partial_i x^{i'})$$

$$(\delta_{\alpha i} Y^J + F_{H\ \alpha i}^{\ J} Y^H)(\partial_j x^{j'}) = (\delta_{\alpha i'} Y^{J'} + F_{H'\ \alpha i'}^{\ J'} Y^{H'})(\partial_i x^{i'}). \tag{3.10}$$

As (see (2.4b) and (2.5))

$$\begin{aligned}&(\delta_{\alpha i} Y^J)(\partial_j x^{j'}) = \delta_{\alpha i}(Y^J \partial_j x^{j'}) - Y^J \delta_{\alpha i}(\partial_j x^{j'}) =\\&(\partial_i x^{i'})(\delta_{\alpha i'} Y^{J'}), \qquad ((\delta_{\alpha i} \partial_j x^{j'}) = 0)\end{aligned}$$

the substitution of the above equation in (3.10) results in

$$F_{H\ \alpha i}^{\ J}(\partial_j x^{j'}) = F_{H'\ \alpha i'}^{\ J'}(\partial_h x^{h'})(\partial_i x^{i'}). \tag{3.11}$$

From the above equation follows that all connection coefficients that appeared in the middle three lines of (3.1) are transformed as d-tensor fields.

In a similar way one can prove

$$F_{H\ \alpha\beta i}^{\ J}(\partial_j x^{j'}) = F_{H'\ \alpha\beta i'}^{\ J'}(\partial_h x^{h'})(\partial_i x^{i'}), \tag{3.12}$$

i.e. all connection coefficients that appeared in the last three lines of (3.1) are tensor fields.

The torsion tensor of the generalized connection D is determined by

$$T(X,Y) = D_X Y - D_Y X - [X,Y].$$

A straightforward calculation gives

$$T(X,Y) = \{(F_{J\,I}^{\,K} - F_{I\,J}^{\,K})\delta_K - [\delta_I, \delta_J]\} Y^J X^I.$$

If we introduce the notation

$$[\delta_I, \delta_J] = K_{I\,J}^{\,K} \delta_K, \tag{3.13}$$

we get

$$T(X,Y) = (F_{J\,I}^{\,K} - F_{I\,J}^{\,K} - K_{I\,J}^{\,K}) Y^J X^I \delta_K = T_{J\,I}^{\,K} Y^J X^I \delta_K. \tag{3.14}$$

Now the components of $K_{J\,I}^{\,K}$ should be determined. $[\delta_i, \delta_j]$, $[\delta_{\alpha i}, \delta_{\beta j}]$ and $[\delta_{\alpha\beta i}, \delta_{\gamma\delta j}]$ are determined by (2.14), (2.15) and (2.16). Further, we obtain

$$\begin{aligned}
&[\delta_i, \delta_{\gamma j}] = K_{i\;\gamma j}^{\,\kappa k} \delta_{\kappa k} + K_{i\;\;\gamma j}^{\,\kappa\rho k} \delta_{\kappa\rho k},\\
&K_{i\;\gamma j}^{\,\kappa k} = \delta_{\gamma j} N_i^{\kappa k}\\
&K_{i\;\;\gamma j}^{\,\kappa\rho k} = \bar{K}_{i\;\;\gamma j}^{\,\kappa\rho k} + \bar{K}_{i\;\gamma j}^{\,\delta k} M_{\delta h}^{\kappa\rho k}\\
&\bar{K}_{i\;\;\gamma j}^{\,\kappa\rho k} = \delta_{\gamma j} N_i^{\kappa\rho k} - \delta_i N_{\gamma j}^{\kappa\rho k}\\
&[\delta_i, \delta_{\gamma\delta j}] = K_{i\;\gamma\delta j}^{\,\kappa k} \partial_{\kappa k} + K_{i\;\;\gamma\delta k}^{\,\kappa\rho k} \partial_{\kappa\rho k}\\
&K_{i\;\gamma\delta j}^{\,\kappa k} = \partial_{\gamma\delta j} N_i^{\kappa k}\\
&K_{i\;\;\gamma\delta j}^{\,\kappa\rho k} = \partial_{\gamma\delta j} N_i^{\kappa\rho k}\\
&[\delta_{\alpha i}, \delta_{\gamma\delta j}] = K_{\alpha i\;\;\gamma\delta j}^{\;\;\kappa\rho k} \partial_{\kappa\rho k}\\
&K_{\alpha i\;\;\gamma\delta j}^{\;\;\kappa\rho k} = \partial_{\gamma\delta j} N_{\alpha i}^{\kappa\delta k}
\end{aligned} \tag{3.15}$$

Theorem 3.1. *The components of the torsion tensor*

$$T(X,Y) = T_{J\,I}^{\,K} Y^J X^I \delta_K$$

of the generalized connection D are determined by

$$T_{J\,I}^{\,K} = F_{J\,I}^{\,K} - F_{I\,J}^{\,K}$$

except for the case when $K_{I\,J}^{\,K} \neq 0$ (see (3.13), (3.14)), and then they have the *form (see (2.14), (2.15) and (3.15)):*

$$
\begin{aligned}
(3.16)\qquad &(a)\quad T_{j\ i}^{\ \kappa k} = F_{j\ i}^{\ \kappa k} - F_{i\ j}^{\ \kappa k} - K_{i\ j}^{\ \kappa k}\\
&(b)\quad T_{j\ \ i}^{\ \kappa\rho k} = F_{j\ \ i}^{\ \kappa\rho k} - F_{i\ \ j}^{\ \kappa\rho k} - K_{i\ \ j}^{\ \kappa\rho k}\\
&(c)\quad T_{\gamma j\ \ \alpha i}^{\ \ \kappa\rho k} = F_{\gamma j\ \ \alpha i}^{\ \ \kappa\rho k} - F_{\alpha i\ \ \gamma j}^{\ \ \kappa\rho k} - K_{\alpha i\ \ \gamma j}^{\ \ \kappa\rho k}\\
&(d)\quad T_{\gamma j\ i}^{\ \ \kappa k} = F_{\gamma j\ i}^{\ \ \kappa k} - F_{i\ \gamma j}^{\ \kappa k} - K_{i\ \gamma j}^{\ \kappa k}\\
&(e)\quad T_{\gamma j\ \ i}^{\ \ \kappa\rho k} = F_{\gamma j\ \ i}^{\ \ \kappa\rho k} - F_{i\ \ \gamma j}^{\ \kappa\rho k} - K_{i\ \ \gamma j}^{\ \kappa\rho k}\\
&(f)\quad T_{\gamma\delta j\ i}^{\ \ \ \kappa k} = F_{\gamma\delta j\ i}^{\ \ \ \kappa k} - F_{i\ \gamma\delta j}^{\ \kappa k} - K_{i\ \gamma\delta j}^{\ \kappa k}\\
&(g)\quad T_{\gamma\delta j\ \ i}^{\ \ \ \kappa\rho k} = F_{\gamma\delta j\ \ i}^{\ \ \ \kappa\rho k} - F_{i\ \ \gamma\delta k}^{\ \kappa\rho k} - K_{i\ \ \gamma\delta j}^{\ \kappa\rho k}\\
&(h)\quad T_{\gamma\delta j\ \ \alpha i}^{\ \ \ \kappa\rho k} = F_{\gamma\delta j\ \ \alpha i}^{\ \ \ \kappa\rho k} - F_{\alpha i\ \ \gamma\delta j}^{\ \ \kappa\rho k} - K_{\alpha i\ \ \gamma\delta j}^{\ \ \kappa\rho k}
\end{aligned}
$$

As $T_{J\ I}^{\ K}$ are components of the d-tensor field, using (3.16), (3.8), (3.11) and (3.12) we can obtain the transformation laws of $K_{J\ I}^{\ K}$.

Proposition 3.3. *$K_{i\ j}^{\ \kappa k}$, $K_{i\ \ j}^{\ \kappa\rho k}$, $K_{\alpha i\ \ \gamma j}^{\ \ \kappa\rho k}$, $K_{\alpha i\ \ \gamma\delta j}^{\ \ \kappa\rho k}$ are d-tensor fields,*

$$K_{i\ \gamma j}^{\ \kappa k},\ K_{i\ \ \gamma j}^{\ \kappa\rho k},\ K_{i\ \gamma\delta j}^{\ \kappa k},\ K_{i\ \ \gamma\delta j}^{\ \kappa\rho k}$$

are not d-tensor fields and they transform in the following way:

$$(3.17)\qquad K_{i\ \gamma j}^{\ \kappa k} = K_{i'\ \gamma j'}^{\ \kappa k'}(\partial_i x^{i'})(\partial_j x^{j'})(\partial_{k'} x^k) + (\partial_i \partial_j x^{k'})(\partial_{k'} x^k)$$

(similar for the next three K).

Proof. From (3.8) it follows, that $F_{j\ i}^{\ \kappa k} - F_{i\ j}^{\ \kappa k}$ is a d-tensor, and from (3.16a) follows that $K_{i\ j}^{\ \kappa k}$ is the difference of two d-tensors, so itself is a d-tensor.

As in (3.16d), $T_{\gamma j\ i}^{\ \ \kappa k}$ and $F_{i\ \gamma j}^{\ \kappa k}$ are d-tensors (see (3.11)), so $F_{\gamma j\ i}^{\ \ \kappa k} - K_{i\ \gamma j}^{\ \kappa k}$ is a d-tensor. Using this fact and (3.8) we obtain (3.17).

For the d-connection and s.d. connection in (3.15) all terms $K_{A\ B}^{\ C}$ remain, because they are not functions of different connection coefficients, they only depend on N and M, which are involved in adapted bases. In some components of $T_{J\ I}^{\ K}$ some $\Gamma_{J\ I}^{\ K}$ vanish (see definition 3.2).

4. The curvature theory of generalized connection

The curvature tensor

$$(4.1)\qquad R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z$$

can be calculated in the usual way. For $X = X^A \delta_A$, $Y = Y^B \delta_B$, $Z = Z^C \delta_C$ we get

$$(4.2)\qquad D_Y Z = D_{Y^B \delta_B} Z^C \delta_C = Y^B(\delta_B Z^C)\delta_C + Y^B Z^C F_{C\ B}^{\ D}\delta_D,$$

$$D_X D_Y Z = D_{X^A\delta_A}[Y^B(\delta_B Z^C)\delta_C + Y^B Z^C F_{C\ B}^{\ D}\delta_D] = \tag{4.3}$$
$$X^A(\delta_A Y^B)(\delta_B Z^C)\delta_C + X^A Y^B \delta_A(\delta_B Z^C)\delta_C +$$
$$X^A Y^B(\delta_B Z^C)F_{C\ A}^{\ D}\delta_D + X^A(\delta_A Y^B)Z^C F_{C\ B}^{\ D}\delta_D +$$
$$X^A Y^B(\delta_A Z^C)F_{C\ B}^{\ D}\delta_D + X^A Y^B Z^C(\delta_A F_{C\ B}^{\ D})\delta_D +$$
$$X^A Y^B Z^C F_{C\ B}^{\ E} F_{E\ A}^{\ D}\delta_D.$$

Further, using the notation $[\delta_A, \delta_B] = K_{A\ B}^{\ D}\delta_D$, we get

$$D_{[X,Y]}Z = D_{[X^A\delta_A, Y^B\delta_B]}Z^C\delta_C = \tag{4.4}$$
$$X^A(\delta_A Y^B)[(\delta_B Z^C)\delta_C + Z^C\Gamma_{C\ B}^{\ D}\delta_D] -$$
$$Y^B(\delta_B X^A)[(\delta_A Z^C)\delta_C + Z^C\Gamma_{C\ A}^{\ D}\delta_D] +$$
$$X^A Y^B\{[\delta_A, \delta_B]Z^C\}\delta_C + X^A Y^B Z^C K_{A\ B}^{\ E} F_{C\ E}^{\ D}\delta_D.$$

Finally we obtain

Theorem 4.1. *The curvature tensor of the generalized connection D on $T(E)$ is given by*

$$R(X,Y)Z = R_{C\ BA}^{\ D} X^A Y^B Z^C \delta_D, \tag{4.5}$$

where

$$R_{C\ BA}^{\ D} = K_{C\ BA}^{\ D} + F_{C\ E}^{\ D} K_{B\ A}^{\ E}, \tag{4.6}$$

$$K_{C\ BA}^{\ D} = \delta_A F_{C\ B}^{\ D} + F_{C\ B}^{\ E} F_{E\ A}^{\ D} - \delta_B F_{C\ A}^{\ D} - F_{C\ A}^{\ E} F_{E\ B}^{\ D}. \tag{4.7}$$

The values of $K_{A\ B}^{\ E}$ which are different from zero are determined by (3.16). Since the latin capitals as indices are connected with $T_H(i,j,k,h,\ldots)$, $T_{V_1}(\alpha i, \beta j, \gamma k, \ldots)$ or $T_{V_2}(\alpha\beta i, \gamma\delta j, \kappa\rho k, \ldots)$ there are 3^4 types of curvature tensors.

From (4.2) it follows

$$D_Y Z = Y^B Z^C_{\ |B}\delta_C,$$

$$D_X(D_Y Z) = X^A(Y^B Z^C_{\ |B})_{|A}\delta_C = \tag{4.8}$$
$$X^A(Y^B_{\ |A} Z^C_{\ |B} + Y^B Z^C_{\ |B|A})\delta_C.$$

hand,side from (4.2) we get

$$D_{[X,Y]}Z = Z^C_{\ |D}[X,Y]^D\delta_C = A + B, \tag{4.9}$$

where

$$A = Z^C_{\ |B}(X^A\delta_A Y^B - Y^A\delta_A X^B)\delta_C = \tag{4.10}$$
$$[X^A Y^B_A Z^C_{\ |B} - Y^B X^A_{\ |B} Z^C_{\ |A} -$$
$$(F_{B\ A}^{\ D} - F_{A\ B}^{\ D}) X^A Y^B Z^C_{\ |D}]\delta_C,$$

$$B = K_{A\ B}^{\ D}X^AY^BZ^C_{|D}\delta_C. \tag{4.11}$$

Substituting (4.10) and (4.11) into (4.9), then (4.9) and (4.4) into (4.1), we obtain

$$R(X,Y)Z = (Z^C_{|B|A} - Z^C_{|A|B} + T_{B\ A}^{\ D}Z^C_{|D})X^AY^B\delta_C. \tag{4.12}$$

From (4.12) and (4.5) it follows

Theorem 4.2. *The Ricci equations for the generalized connection D have the form:*

$$Z^C_{|B|A} - Z^C_{|A|B} + T_{B\ A}^{\ D}Z^C_{|D} = R_{D\ BA}^{\ C}Z^D,$$

where

$$A \in \{i, \alpha i, \alpha\beta i\}, \quad B \in \{j, \gamma j, \gamma\delta j\},$$

$$C \in \{k, \varepsilon k, \varepsilon\rho k\}, \quad D \in \{h, \nu h, \nu\mu h\}.$$

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