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STRONG CONVERGENCE FOR ACCRETIVE OPERATORS IN BANACH SPACES

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Abstract. This paper introduces a composite iteration scheme for approximating a zero point of accretive operators in the framework of uniformly smooth Banach spaces and the reflexive Banach space which has a weak continuous duality map, respectively. Our results improve and extend results of Kim, Xu and some others.

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1. Introduction and Preliminaries

Let E be a real Banach space, C a nonempty closed convex subset of E, and $T : C \to C$ a mapping. Recall that T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$.

One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping [2], [9]. More precisely, take $t \in (0, 1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1 - t)Tx, \quad x \in C, \tag{1.1}$$

where $u \in C$ is a fixed point. Banach's Contraction Mapping Principle guarantees that T_t has a unique fixed point x_t in C. It is unclear, in general, what is the behavior of x_t as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder [2] proved that if E is a Hilbert space, then x_t converges strongly to a fixed point of T that is nearest to u. Reich [9] extended Broweder's result to the setting of Banach spaces and proved that if E is a uniformly smooth Banach space, then x_t converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from C onto $F(T)$.

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Recall that a (possibly multivalued) operator A with domain $D(A)$ and range $R(A)$ in E is accretive, if for each $x_i \in D(A)$ and $y_i \in Ax_i (i = 1, 2)$, there exists a $j(x_2 - x_1) \in J(x_2 - x_1)$ such that

$$\langle y_2 - y_1, j(x_2 - x_1)\rangle \geq 0,$$

where J is the duality map from E to the dual space E^* given by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x^2\| = \|x^*\|^2\},\ x \in E.$$

An accretive operator A is m-accretive if $R(I + rA) = E$ for each $r > 0$. Throughout this article we always assume that A is m-accretive and has a zero (i.e., the inclusion $0 \in A(z)$ is solvable). The set of zeros of A is denoted by F. Hence,

$$F = \{z \in D(A) : 0 \in A(z)\} = A^{-1}(0).$$

For each $r > 0$, we denote by J_r the resolvent of A, i.e., $J_r = (I + rA)^{-1}$. Note that if A is m-accretive, then $J_r : E \to E$ is nonexpansive and $F(J_r) = F$ for all $r > 0$. We also denote by A_r the Yosida approximation of A, i.e., $A_r = \frac{1}{r}(I - J_r)$. It is known that J_r is a nonexpansive mapping from X to $C := \overline{D(A)}$ which will be assumed convex.

Recently Kim and Xu [6] and Xu [12] studied the sequence generated by the algorithm

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)J_{r_n}x_n, \quad n \geq 0 \tag{1.2}$$

and proved strong convergence of the scheme (1.2) in the framework of uniformly smooth Banach spaces.

Inspired and motivated by the iterative sequences (1.2) given by Xu, this paper gives the following iterative sequences

$$\begin{cases} z_n = \gamma_n x_n + (1 - \gamma_n)J_{r_n}x_n, \\ y_n = \beta_n x_n + (1 - \beta_n)J_{r_n}z_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n, \end{cases} \tag{1.3}$$

where $u \in C$ is an arbitrary (but fixed) element in C, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $(0, 1)$. We prove, under certain appropriate assumptions on the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{r_n\}$, that $\{x_n\}$ defined by (1.3) converges to a fixed point of T.

If $\gamma_n = 1$ in (1.3) we have the iterative scheme as follows

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)J_{r_n}x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n. \end{cases} \tag{1.4}$$

If $\beta_n = 0$ in (1.3) we have the iterative sequence $\{x_n\}$ defined by (1.2).

It is our purpose in this paper to introduce this composite iteration scheme for approximating a zero point of accretive operators in the framework of uniformly smooth Banach spaces and the reflexive Banach space which has a weakly continuous duality map, respectively. We establish the strong convergence of the composite iteration scheme $\{x_n\}$ defined by (1.3). The results improve and extend results of Kim and Xu [6] and Xu [12] and others.

The norm of E is said to be Gâteaux differentiable (and E is said to be smooth) if

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \tag{1.5}$$

exists for each x, y in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be uniformly Fréchet differentiable (and E is said to be uniformly smooth) if the limit in (1.5) is attained uniformly for $(x, y) \in U \times U$.

We need the following definitions and lemmas for the proof of our main results.

Lemma 1. *A Banach space E is uniformly smooth if and only if the duality map J is the single-valued and norm-to-norm uniformly continuous on bounded sets of E.*

Lemma 2. *In a Banach space E, there holds the inequality*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad x, y \in E$$

where $j(x+y) \in J(x+y)$.

Lemma 3. *(Xu [11], [10]) Let $\sum_{n=0}^{\infty}\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \gamma_n\sigma_n, n \ge 0$$

where $\{\gamma_n\}_{n=0}^{\infty} \subset (0,1)$ and $\{\sigma_n\}_{n=0}^{\infty}$ such that
(i) $\lim_{n\to\infty}\gamma_n = 0$ and $\sum_{n=0}^{\infty}\gamma_n = \infty$,
(ii) either $\limsup_{n\to\infty}\sigma_n \le 0$ or $\sum_{n=0}^{\infty}|\gamma_n\sigma_n| < \infty$.
Then $\{\alpha_n\}_{n=0}^{\infty}$ converges to zero.

Lemma 4. *(The resolvent Identity [1]). For $\lambda > 0$ and $\mu > 0$ and $x \in E$,*

$$J_\lambda x = J_\mu\left(\frac{\mu}{\lambda}x + \left(1-\frac{\mu}{\lambda}\right)J_\lambda x\right).$$

Recall that if C and D are nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a map $Q : C \to D$ is sunny [4], [8] provided $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in C$ and $t \ge 0$ whenever $x + t(x - Q(x)) \in C$. A sunny nonexpansive retraction is a sunny retraction which is also nonexpansive. Sunny nonexpansive retractions play an important role in our argument. They are characterized as follows [4], [5], [8]: if E is a smooth Banach space, then $Q : C \to D$ is a sunny nonexpansive retraction if and only if there holds the inequality

$$\langle x - Qx, J(y - Qx)\rangle \le 0 \quad \text{for all} \quad x \in C \quad \text{and} \quad y \in D. \tag{1.6}$$

Reich [9] showed that if E is uniformly smooth and if D is the fixed point set of a nonexpansive mapping from C into itself, then there is a sunny nonexpansive retraction from C onto D and it can be constructed as follows.

Lemma 5. *(Reich [11]). Let E be a uniformly smooth Banach space and let $T : C \to C$ be a nonexpansive mapping with a fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1-t)tx$ converges strongly as $t \to 0$ to a fixed point of T. Define $Q : C \to F(T)$ by $Qu = s - \lim_{t\to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto $F(T)$; that is, Q satisfies the property:*

$$\langle u - Qu, J(z - Qu)\rangle \leq 0, \quad u \in C, z \in F(T).$$

Recall that a gauge is a continuous strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ *as* $t \to \infty$. Associated to a gauge φ is the duality map $J_\varphi : X \to X^*$ defined by

$$J_\varphi(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|\varphi(\|x\|). \ \|x^*\| = \varphi(\|x\|)\}, \ x \in X.$$

Following Browder [3], we say that a Banach space X has a weakly continuous duality map if there exists a gauge φ for which the duality map J_φ is single-valued and *weak*-to-*weak** sequentially continuous (i.e., if $\{x_n\}$ is a sequence in X weakly convergent to a point x, then the sequence $J_\varphi(x_n)$ converges weak*ly to J_φ). It is known that l^p has a weakly continuous duality map for all $1 < p < \infty$. Set

$$\Phi(t) = \int_0^t \varphi(\tau)d_\tau, \ t \geq 0.$$

Then

$$J_\varphi(x) = \partial\Phi(\|x\|), \ x \in X,$$

where ∂ denotes the subdifferential in the sense of convex analysis. The first part of the next lemma is an immediate consequence of the subdiferential inequality and the proof of the second part can be found in [7].

Lemma 6. *Assume that X has a weakly continuous duality map J_φ with gauge φ.*
(i) For all $x, y \in X$, there holds the inequality

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, J_\Phi(x + y)\rangle.$$

(ii) Assume a sequence x_n in X is weakly convergent to a point x. Then there holds the identity

$$\limsup \lim_{n\to\infty} \Phi(\|x_n - y\|) = \limsup \lim_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|), \ x, y \in X.$$

Notation: " $\rightharpoonup$ " stands for weak convergence and " $\to$ " for strong convergence.

Lemma 7. *[7] Let X be a reflexive Banach space and have a weakly continuous duality map $J\varphi(x)$ with gauge φ. Let C be a closed convex subset of X and let $T : C \to C$ be a nonexpansive mapping. Fix $u \in C$ and $t \in (0, 1)$. Let $x_t \in C$ be the unique solution in C to Eq.(1.1). Then T has a fixed point if and only if x_t remains bounded as $t \to 0^+$, and in this case, x_t converges as $t \to 0^+$ strongly to a fixed point of T.*

Under the condition of Lemma 6, we define a map $Q : C \to F(T)$ by

$$Q(u) := \lim_{t\to 0} x_t, \quad u \in C.$$

From [11], Theorem 3.2, we know Q is the sunny nonexpansive retraction from C onto $F(T)$.

2. Main Results

Theorem 1. *Assume that E is a uniformly smooth Banach space and A is an m-accretive operator in E such that $A(0) \neq \emptyset$. Given a point $u \in C$ and given sequences $\{\alpha_n\}_{n=0}^{\infty}$ in (0,1) and $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ in [0,1] suppose $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=0}^{\infty}$ satisfy the conditions:*
(i) $\sum_{n=0}^{\infty} \alpha_n = \infty, \alpha_n \to 0$;
(ii) $r_n \geq \epsilon$ for all n and $\gamma_n - \beta_n(2 - \gamma_n) \in [0, a),\ for\ some\ a \in (0, 1)$;
(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty\ \sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ and $\sum_{n=0}^{\infty} |r_n - r_{n-1}| < \infty$.
Let $\{x_n\}_{n=1}^{\infty}$ be the composite process defined by

$$\begin{cases} z_n = \gamma_n x_n + (1 - \gamma_n) J_{r_n} x_n \\ y_n = \beta_n x_n + (1 - \beta_n) J_{r_n} z_n \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n \end{cases}$$

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a zero point of A.

Proof. First we observe that $\{x_n\}_{n=0}^{\infty}$ is bounded. Indeed, if we take a fixed point p of T, noting that

$$\|z_n - p\| \leq \gamma_n \|x_n - p\| + (1 - \gamma_n)\|J_{r_n} x_n - p\| \leq \|x_n - p\|,$$

and

$$\begin{aligned} \|y_n - p\| &\leq \beta_n \|x_n - p\| + (1 - \beta_n)\|J_{r_n} z_n - p\| \\ &\leq \beta_n \|x_n - p\| + (1 - \beta_n)\|z_n - p\| \\ &\leq \beta_n \|x_n - p\| + (1 - \beta_n)\|x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned}$$

Thus

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|u - p\| + (1 - \alpha_n)\|J_{r_n} y_n - p\| \\ &\leq \alpha_n \|u - p\| + (1 - \alpha_n)\|y_n - p\| \\ &\leq \alpha_n \|u - p\| + (1 - \alpha_n)\|x_n - p\| \\ &\leq \max\{\|u - p\|, \|x_n - p\|\}. \end{aligned}$$

Now, an induction gives that

$$\|x_n - p\| \le \max\{\|u - p\|, \|x_0 - p\|\}, \quad n \ge 0. \tag{2.1}$$

This implies that $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$. As a result, we obtain by condition (i),

$$\|x_{n+1} - J_{r_n} y_n\| = \alpha_n \|u - y_n\| \to 0, \quad as\ n \to \infty. \tag{2.2}$$

Next we prove

$$\|x_{n+1} - x_n\| \to 0. \tag{2.3}$$

In order to prove (2.3) we calculate $x_{n+1} - x_n$ first. From (1.3) we have

$$\begin{cases} z_n = \gamma_n x_n + (1 - \gamma_n) J_{r_n} x_n, \\ z_{n-1} = \gamma_{n-1} x_{n-1} + (1 - \gamma_{n-1}) J_{r_{n-1}} x_{n-1}, \end{cases}$$

Simple calculations show that

$$\begin{aligned} z_n - z_{n-1} = (1 - \gamma_n)(J_{r_n} x_n - J_{r_{n-1}} x_{n-1}) + \gamma_n (x_n - x_{n-1}) \\ + (x_{n-1} - J_{r_{n-1}} x_{n-1})(\gamma_n - \gamma_{n-1}). \end{aligned} \tag{2.4}$$

It follows that

$$\begin{aligned} \|z_n - z_{n-1}\| \le (1 - \gamma_n) \|J_{r_n} x_n - J_{r_{n-1}} x_{n-1}\| + \gamma_n \|x_n - x_{n-1}\| \\ + \|x_{n-1} - J_{r_{n-1}} x_{n-1}\| |\gamma_n - \gamma_{n-1}|. \end{aligned} \tag{2.5}$$

Lemma 4 the resolvent identity implies that

$$J_{r_n} x_n = J_{r_{n-1}} (\frac{r_{n-1}}{r_n} x_n + (1 - \frac{r_{n-1}}{r_n}) J_{r_n} x_n).$$

If $r_{n-1} \le r_n$, which in turn implies that

(2.6)

$$\begin{aligned} \|J_{r_n} x_n - J_{r_{n-1}} x_{n-1}\| &\le \|\frac{r_{n-1}}{r_n} x_n + (1 - \frac{r_{n-1}}{r_n}) J_{r_n} x_n - x_{n-1}\| \\ &\le \|\frac{r_{n-1}}{r_n}(x_n - x_{n-1}) + (1 - \frac{r_{n-1}}{r_n})(J_{r_n} x_n - x_{n-1})\| \\ &\le \|x_n - x_{n-1}\| + (\frac{r_n - r_{n-1}}{r_n}) \|J_{r_n} x_n - x_{n-1}\| \\ &\le \|x_n - x_{n-1}\| + (\frac{r_n - r_{n-1}}{\epsilon}) \|J_{r_n} x_n - x_{n-1}\|. \end{aligned}$$

Substituting (2.6) into (2.5) we obtain

$$\begin{aligned} \|z_n - z_{n-1}\| &\le (1 - \gamma_n)(\|x_n - x_{n-1}\| + (\frac{r_n - r_{n-1}}{\epsilon}) \|J_{r_n} x_n - x_{n-1}\|) \\ &\quad + \gamma_n \|x_n - x_{n-1}\| + \|x_{n-1} - J_{r_{n-1}} x_{n-1}\| |\gamma_n - \gamma_{n-1}| \\ &\le \|x_n - x_{n-1}\| + M_1(|r_n - r_{n-1}| + |\gamma_n - \gamma_{n-1}|), \end{aligned} \tag{2.7}$$

where M_1 is a constant such that

$$M_1 > \max\{\frac{\|J_{r_n}x_n - x_{n-1}\|}{\epsilon},\ \|x_{n-1} - J_{r_{n-1}}x_{n-1}\|\}.$$

Similarly, we have

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)J_{r_n}z_n, \\ y_{n-1} = \beta_{n-1}x_{n-1} + (1-\beta_{n-1})J_{r_{n-1}}z_{n-1}, \end{cases}$$

Simple calculations show that

$$\begin{aligned} y_n - y_{n-1} = {} & (1-\beta_n)(J_{r_n}z_n - J_{r_{n-1}}z_{n-1}) + \beta_n(x_n - x_{n-1}) \\ & + (x_{n-1} - J_{r_{n-1}}z_{n-1})(\beta_n - \beta_{n-1}). \end{aligned} \tag{2.8}$$

It follows that

$$\begin{aligned} \|y_n - y_{n-1}\| \le {} & (1-\beta_n)\|J_{r_n}z_n - J_{r_{n-1}}z_{n-1}\| + \beta_n\|x_n - x_{n-1}\| \\ & + \|x_{n-1} - J_{r_{n-1}}z_{n-1}\||\beta_n - \beta_{n-1}|. \end{aligned} \tag{2.9}$$

It follows from resolvent identity that
(2.10)

$$\begin{aligned} \|J_{r_n}z_n - J_{r_{n-1}}z_{n-1}\| & \le \|\frac{r_{n-1}}{r_n}z_n + (1-\frac{r_{n-1}}{r_n})J_{r_n}z_n - z_{n-1}\| \\ & \le \|\frac{r_{n-1}}{r_n}(z_n - z_{n-1}) + (1-\frac{r_{n-1}}{r_n})(J_{r_n}z_n - z_{n-1})\| \\ & \le \|z_n - z_{n-1}\| + (\frac{r_n - r_{n-1}}{r_n})\|J_{r_n}z_n - z_{n-1}\| \\ & \le \|z_n - z_{n-1}\| + (\frac{r_n - r_{n-1}}{\epsilon})\|J_{r_n}z_n - z_{n-1}\|. \end{aligned}$$

Substitute (2.7) into (2.10) yields that

$$\begin{aligned} \|J_{r_n}z_n - J_{r_{n-1}}z_{n-1}\| \le {} & \|x_n - x_{n-1}\| + M_1(|r_n - r_{n-1}| + |\gamma_n - \gamma_{n-1}|) \\ & + (\frac{r_n - r_{n-1}}{\epsilon})\|J_{r_n}z_n - z_{n-1}\|. \end{aligned} \tag{2.11}$$

Substitute (2.11) into (2.9) yields that
(2.12)

$$\begin{aligned} \|y_n - y_{n-1}\| & \le (1-\beta_n)[\|x_n - x_{n-1}\| + M_1(|r_n - r_{n-1}| + |\gamma_n - \gamma_{n-1}|) \\ & \quad + (\frac{r_n - r_{n-1}}{\epsilon})\|J_{r_n}z_n - z_{n-1}\|] + \beta_n\|x_n - x_{n-1}\| \\ & \quad + \|x_{n-1} - J_{r_{n-1}}z_{n-1}\||\beta_n - \beta_{n-1}| \\ & \le \|x_n - x_{n-1}\| + M_2(|\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| + 2|r_n - r_{n-1}|). \end{aligned}$$

where M_2 is a constant such that

$$M_2 > \max\{\frac{\|J_{r_n}z_n - z_{n-1}\|}{\epsilon},\ \|x_{n-1} - J_{r_{n-1}}z_{n-1}\|,\ M_1\}.$$

On the other hand, we have

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n)y_n, \\ x_n = \alpha_{n-1}u + (1-\alpha_{n-1})y_{n-1}, \end{cases}$$

Simple calculations show that

$$x_{n+1} - x_n = (1-\alpha_n)(J_{r_n}y_n - J_{r_n}y_{n-1}) + (\alpha_n - \alpha_{n-1})(u - y_{n-1}).$$

It follows that

$$\begin{aligned} \|x_{n+1} - x_n\| &\le (1-\alpha_n)\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|u - y_{n-1}\| \\ &\le (1-\alpha_n)\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|u - y_{n-1}\|. \end{aligned} \tag{2.13}$$

Substituting (2.12) into (2.13) we obtain

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ &\le (1-\alpha_n)(\|x_n - x_{n-1}\| + M_2(|\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| + 2|r_n - r_{n-1}|)) \\ &\quad + |\alpha_n - \alpha_{n-1}|\|u - y_{n-1}\| \\ &\le (1-\alpha_n)\|x_n - x_{n-1}\| \\ &\quad + M_3(2|r_n - r_{n-1}| + |\beta_n - \beta_{n-1}| + |\alpha_n - \alpha_{n-1}| + |\gamma_n - \gamma_{n-1}|), \end{aligned} \tag{2.14}$$

where M_3 is a constant such that

$$M_3 > \max\{\|u - y_{n-1}\|,\ M_2\}.$$

Similarly, we can prove (2.14) if $r_{n-1} \ge r_n$, by assumptions (i)-(iii), we have that

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

and

$$\sum_{n=1}^{\infty}(|\beta_n - \beta_{n-1}| + 2|r_n - r_{n-1}| + |\alpha_n - \alpha_{n-1}| + |\gamma_n - \gamma_{n-1}|) < \infty.$$

Hence, Lemma 3 is applicable to (2.14) and we obtain

$$\|x_{n+1} - x_n\| \to 0. \tag{2.15}$$

On the other hand, it follows from (1.3) that

$$
\begin{aligned}
\|J_{r_n}x_n - x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - J_{r_n}x_n\| \\
&\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - J_{r_n}z_n\| \\
&\quad + \|J_{r_n}z_n - J_{r_n}x_n\| \\
&\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \beta_n\|x_n - J_{r_n}z_n\| \\
&\quad + \|z_n - x_n\| \\
&\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \beta_n\|x_n - J_{r_n}x_n\| \\
&\quad + \beta_n\|J_{r_n}x_n - J_{r_n}z_n\| + \|z_n - x_n\| \\
&\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \beta_n\|x_n - J_{r_n}x_n\| \\
&\quad + (1+\beta_n)\|x_n - z_n\| \\
&\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \\
&\quad [\beta_n + (1+\beta_n)(1-\gamma_n)]\|x_n - J_{r_n}x_n\|.
\end{aligned} \tag{2.16}
$$

That is,

$$(\gamma_n - \beta_n(2-\gamma_n))\|J_{r_n}x_n - x_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|.$$

It follows from (2.2), (2.15) and condition (ii) that

$$\|J_{r_n}x_n - x_n\| \to 0,$$

Taking a fixed number r such that $\epsilon > r > 0$, from Lemma 4 we have

$$
\begin{aligned}
\|J_{r_n}x_n - J_r x_n\| &= \|J_r(\frac{r}{r_n}x_n + (1-\frac{r}{r_n})J_{r_n}x_n) - J_r x_n\| \\
&\le (1-\frac{r}{r_n})\|x_n - J_{r_n}x_n\| \\
&\le \|x_n - J_{r_n}x_n\|.
\end{aligned} \tag{2.17}
$$

Therefore, we obtain

$$
\begin{aligned}
\|x_n - J_r x_n\| &\le \|x_n - J_{r_n}x_n\| + \|J_{r_n}x_n - J_r x_n\| \\
&\le \|J_{r_n} - x_n\| + \|J_{r_n} - x_n\| \\
&\le 2\|J_{r_n} - x_n\|.
\end{aligned} \tag{2.18}
$$

Hence we have

$$\|x_n - J_r x_n\| \to 0.$$

Since in a uniformly smooth Banach space, the sunny nonexpansive retract Q from E onto the fixed point set $F(J_r)(= F = A^{-1}(0))$ of J_r is unique, it must be obtained from Reich's theorem (Lemma 5). Namely,

$$Qu = s - \lim_{t\to 0} z_t, \qquad u \in E,$$

where $t \in (0,1)$ and z_t solves the fixed point equation

$$z_t = tu + (1-t)J_r z_t.$$

Next, we claim that

$$\limsup_{n\to\infty}\langle u-Q(u),J(x_n-Q(u))\rangle\le 0. \tag{2.19}$$

Thus we have

$$\|z_t-x_n\|=\|(1-t)(J_rz_t-x_n)+t(u-x_n)\|.$$

It follows From Lemma 2 that

$$\begin{aligned}\|z_t-x_n\|^2&\le(1-t)^2\|J_rz_t-x_n\|^2+2t\langle u-x_n,J(z_t-x_n)\rangle\\&\le(1-2t+t^2)\|z_t-x_n\|^2+f_n(t)\\&\quad+2t\langle u-z_t,J(z_t-x_n)\rangle+2t\|z_t-x_n\|^2,\end{aligned} \tag{2.20}$$

where

$$f_n(t)=(2\|z_t-x_n\|+\|x_n-J_rx_n\|)\|x_n-J_rx_n\|\to 0,\ \ as\ n\to 0. \tag{2.21}$$

It follows that

$$\langle z_t-u,J(z_t-x_n)\rangle\le\frac{t}{2}\|z_t-x_n\|^2+\frac{1}{2t}f_n(t). \tag{2.22}$$

Let $n\to\infty$ in (2.22) and noting (2.21) yields

$$\limsup_{n\to\infty}\langle z_t-u,J(z_t-x_n)\rangle\le\frac{t}{2}M, \tag{2.23}$$

where $M>0$ is a constant such that $M\ge\|z_t-x_n\|^2$ for all $t\in(0,1)$ and $n\ge 1$. Letting $t\to 0$ from (2.23) we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle z_t-u,J(z_t-x_n)\rangle\le 0.$$

So, for any $\epsilon>0$, there exists a positive number δ_1 such that, for $t\in(0,\delta_1)$ we get

$$\limsup_{n\to\infty}\langle z_t-u,J(z_t-x_n)\rangle\le\frac{\epsilon}{2}. \tag{2.24}$$

On the other hand, since $z_t\to q$ as $t\to 0$, from Lemma 1, there exists $\delta_2>0$ such that, for $t\in(0,\delta_2)$ we have

$$\begin{aligned}&|\langle u-q,J(x_n-q)\rangle-\langle z_t-u,J(z_t-x_n)\rangle|\\\le\ &|\langle u-q,J(x_n-q)\rangle-\langle u-q,J(x_n-z_t)\rangle|\\&+|\langle u-q,J(x_n-z_t)\rangle-\langle z_t-u,J(z_t-x_n)\rangle|\\\le\ &|\langle u-q,J(x_n-q)-J(x_n-z_t)\rangle|+|\langle z_t-q,J(x_n-z_t)\rangle|\\\le\ &\|u-q\|\|J(x_n-q)-J(x_n-z_t)\|+\|z_t-q\|\|x_n-z_t\|<\tfrac{\epsilon}{2}.\end{aligned}$$

Choosing $\delta=\min\{\delta_1,\delta_2\},\forall t\in(0,\delta)$, we have

$$\langle u-Q(u),J(x_n-Q(u))\rangle\le\langle z_t-u,J(z_t-x_n)\rangle+\frac{\epsilon}{2}.$$

That is,

$$\limsup_{n\to\infty}\langle u-Q(u),J(x_n-Q(u))\rangle \le \limsup_{n\to\infty}\langle z_t-u,J(z_t-x_n)\rangle+\frac{\epsilon}{2}.$$

It follows from (2.24) that

$$\limsup_{n\to\infty}\langle u-Q(u),J(x_n-Q(u))\rangle \le \epsilon.$$

Since ϵ is chosen arbitrarily, we have

$$\limsup_{n\to\infty}\langle u-Q(u),J(x_n-Q(u))\rangle \le 0. \tag{2.25}$$

Finally, we show that $x_n \to Q(u)$ strongly and this concludes the proof. Indeed, using Lemma 2 again we obtain

$$\begin{aligned}\|x_{n+1}-Q(u)\|^2 &= \|(1-\alpha_n)(y_n-Q(u))+\alpha_n(u-Q(u))\|^2\\ &\le (1-\alpha_n)^2\|y_n-Q(u)\|^2+2\alpha_n\langle u-Q(u),J(x_{n+1}-Q(u))\rangle\\ &\le (1-\alpha_n)\|x_n-Q(u)\|^2+2\alpha_n\langle u-Q(u),J(x_{n+1}-Q(u))\rangle.\end{aligned}$$

Now we apply Lemma 3 and use (2.25) to see that $\|x_n-Q(u)\|\to 0$. □

As a corollary of Theorem 1, we have the following.

Corollary 1. *Assume that E is a uniformly smooth Banach space and A is an m-accretive operator in E such that $A(0)\neq\emptyset$. Given a point $u\in C$ and given sequences $\{\alpha_n\}_{n=0}^{\infty}$ in (0,1) $\{\beta_n\}_{n=0}^{\infty}$ in [0,1], suppose $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, and $\{r_n\}_{n=0}^{\infty}$ satisfy the conditions:*
(i) $\sum_{n=0}^{\infty}\alpha_n=\infty, \alpha_n\to 0$;
(ii) $r_n\ge\epsilon$ *for all* n *and* $(1-\beta_n)\in[0,a)$, *for some* $a\in(0,1)$;
(iii) $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty, \sum_{n=0}^{\infty}|\beta_{n+1}-\beta_n|<\infty$ *and* $\sum_{n=0}^{\infty}|r_n-r_{n-1}|<\infty$.
Let $\{x_n\}_{n=1}^{\infty}$ be the composite process defined by (1.4). Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to a zero point of A.

Proof. By taking $\gamma_n=1$ in Theorem 1, we can obtain the desired conclusion. □

Theorem 2. *Suppose that X is reflexive and has a weakly continuous duality map J_φ with gauge φ. Suppose that A is an m-accretive operator in X such that $C=\overline{D(A)}$ is convex and $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{r_n\}$ are as in Theorem 1. Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to a zero point of A.*

Proof. We only include the differences. From the assumptions we obtain

$$\begin{aligned}\|x_{n+1}-J_{r_n}x_n\| &= \|x_{n+1}-y_n\|+\|y_n-J_{r_n}x_n\|\\ &\le \alpha_n\|u-J_{r_n}y_n\|+\|y_n-J_{r_n}z_n\|+\|J_{r_n}z_n-J_{r_n}x_n\|\\ &\le \alpha_n\|u-J_{r_n}y_n\|+(\gamma_n-\beta_n(2-\gamma_n))\|J_{r_n}x_n-x_n\|.\end{aligned}$$

That is,

$$\|x_{n+1} - J_{r_n} x_n\| \to 0. \tag{2.26}$$

We next prove that

$$\limsup_{n\to\infty} \langle u - Q(u), J_\varphi(x_n - Q(u)\rangle \le 0. \tag{2.27}$$

By Lemma 6, we have the sunny nonexpansive retraction $Q : C \to Fix(T)$. Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle u - Q(u), J_\varphi(x_n - Q(u)\rangle = \lim_{k\to\infty} \langle u - Q(u), J_\varphi(x_{n_k} - Q(u)\rangle. \tag{2.28}$$

Since X is reflexive, we may further assume that $x_{n_k} \rightharpoonup \widetilde{x}$. Moreover, since

$$\|x_{n+1} - J_{r_n}\| \to 0,$$

we obtain

$$J_{r_{n_k-1}} x_{n_k-1} \rightharpoonup \widetilde{x}.$$

Taking the limit as $k \to \infty$ in the relation

$$[J_{r_{n_k-1}} x_{n_k-1}, A_{r_{n_k-1}} x_{n_k-1}] \in A,$$

we get $[\widetilde{x}, 0] \in A$. That is, $\widetilde{x} \in F$. Hence by (2.28) and (1.5) we have

$$\limsup_{n\to\infty} \langle u - Q(u), J_\varphi(x_n - Q(u))\rangle = \langle u - Q(u), J_\varphi(\widetilde{x} - Q(u))\rangle \le 0.$$

That is (2.27) holds. Finally to prove that $x_n \to p$.

$$\begin{aligned}\Phi(\|y_n - p\|) &= \Phi(\|\beta_n(x_n - p) + (1-\beta_n)(J_{r_n}x_n - p)\|)\\ &\le \Phi(\|\beta_n\|x_n - p\| + (1-\beta_n)\|J_{r_n}x_n - p\|)\\ &\le \Phi(\|x_n - p\|),\end{aligned}$$

that is,

$$\Phi(\|y_n - p\|) \le \Phi(\|x_n - p\|).$$

Therefore, we obtain

$$\begin{aligned}\Phi(\|x_{n+1} - p\|) &= \Phi(\|\alpha_n(u - p) + (1-\alpha_n)(y_n - p)\|)\\ &\le \Phi((1-\alpha_n)\|y_n - p\|) + \alpha_n\langle u - p, J_\varphi(x_{n+1} - p)\rangle\\ &\le (1-\alpha_n)\Phi(\|y_n - p\|) + \alpha_n\langle u - p, J_\varphi(x_{n+1} - p)\rangle\\ &\le (1-\alpha_n)\Phi(\|x_n - p\|) + \alpha_n\langle u - p, J_\varphi(x_{n+1} - p)\rangle.\end{aligned}$$

An application of Lemma 3 yields that $\Phi(\|x_n - p\|) \to 0$; that is $\|x_n - p\| \to 0$. This completes the proof. □

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