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HOMOGENEOUS DISTRIBUTIONS IN $\mathcal{D}'_{L^q}$ [1]

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Abstract. We study the homogeneity property on a scale of subspaces $\mathcal{D}'_{L^q}(\mathbb{R}^n)$, $1 \le q \le \infty$ of the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$. It is shown that a homogeneous distributions belong to $\mathcal{D}'_{L^q}(\mathbb{R}^n)$ if and only if its degree of homogeneity belongs to $(-\infty, -\frac{n}{q})$, $1 \le q \le \infty$ (if $q = \infty$, then $\frac{n}{q} = 0$).

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1. Introduction

The study of homogeneous distributions is usually motivated by their use in PDE, see for example [3, 5, 12] and the references given there. Another source of interest in homogeneous distributions is asymptotic analysis [6, 8, 10]. Namely, if f has a quasiasymptotic expansion (asymptotic separation of variable) given by $f(\lambda\,\cdot) = \sum_{j=1}^{N} \rho_j(\lambda) g_{\beta_j}(\cdot) + o(\rho_N(\lambda))$, where ρ_j are regularly varying function of order β_j, $j = 1, \ldots, N$, such that $\beta_1 \ge \beta_2 \ge \cdots \ge \beta_N$, then g_{β_j} are homogeneous distributions of degree β_j, $j = 1, \ldots, N$. Recall, distribution $g_\alpha \in \mathcal{D}'(\mathbb{R}^n)$ is homogeneous of degree $\alpha \in \mathbb{C}(Re\alpha > -1)$ if $g_\alpha(ax) = a^\alpha g_\alpha(x)$, $a > 0$. Note that a distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ is quasihomogeneous of degree $\alpha \in \mathbb{C}$ and type $p \in \mathbb{R}^n \setminus \{0\}$ if $f(a^p x) = a^\alpha f(x)$, $a > 0$. Quasihomogeneous distributions and their applications, e.g., quasihomogeneous fundamental solutions of PDE, can be found in [4].

In the construction of fundamental solutions of PDE it is often of interest to study different types of subspaces of the space of distributions $\mathcal{D}'(\mathbb{R}^n)$. The scale of spaces $\mathcal{D}'_{L^q}(\mathbb{R}^n)$, $1 \le q \le \infty$, is studied by many authors, see for example [1, 7, 9, 11, 12].

In this paper we study homogeneity property within that scale of subspaces and show the connection between integrability conditions on test functions and the degree of homogeneity of corresponding distributions. We show that a homogeneous distribution of degree α, denoted by g_α, belongs to $\mathcal{D}'_{L^q}(\mathbb{R}^n)$, $1 \le q \le \infty$ if and only if $\alpha \in (-\infty, n/q)$; if $q = \infty$, then $n/q = 0$.

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2. Notions and notation

We denote by $\mathbb{R}^n$ ($\mathbb{N}_0^n$) set of $n-$tuples $(x_1,\dots,x_n)$, $x_i \in \mathbb{R}$ ($x_i \in \mathbb{N}_0$), and $\mathbb{Z}_- = \{-1,-2,\dots\}$. By Ω we denote an open subset of $\mathbb{R}^n$. For a multi-index $\alpha \in \mathbb{N}_0^n$, we have $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ and $f^{(\alpha)}(x) = \partial^\alpha f(x) = \dfrac{\partial^{\alpha_1}}{\partial x^1}\cdots\dfrac{\partial^{\alpha_n}}{\partial x^n}f(x)$, $x = (x_1,\dots,x_n) \in \mathbb{R}^n$. Throughout the paper, the integrals are taken over $\mathbb{R}^n$ (or $\mathbb{R}$), unless otherwise indicated. The surface of the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = \sqrt{x_1^2 + \cdots + x_n^2} = 1\}$ is $|S^{n-1}| = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$, where Γ is the Gamma function. The letter C will denote a positive constant, not necessarily the same every time when it occurs. The definition and basic properties of spaces $L^p(\mathbb{R}^n)$ and $L^p_{loc}(\mathbb{R}^n)$, $1 \le p \le \infty$, can be found in [2].

The space of distributions $\mathcal{D}'(\Omega)$ is the dual of the space of compactly supported infinitely differentiable functions $C_0^\infty(\Omega) = \mathcal{D}(\Omega)$ and the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ is the dual of the space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$; $\mathcal{S}'_+(\mathbb{R}^n)$ denotes the space of tempered distributions supported by $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x_i \ge 0, \quad i = 1,\dots,n\}$.

Let $1 \le p \le \infty$. We denote by $\mathcal{D}_{L^p}(\mathbb{R}^n)$ the space of functions $\phi \in C^\infty(\mathbb{R}^n)$ such that $\phi^{(\alpha)} \in L^p(\mathbb{R}^n)$, $\alpha \in \mathbb{N}_0^n$. The topology in $\mathcal{D}_{L^p}$ is defined by the sequence of norms

$$\| \phi \|_{m,p} = \Big(\sum_{|\alpha| \le m} \| \phi^{(\alpha)} \|_p^p \Big)^{\frac{1}{p}},\ m \in \mathbb{N};$$

$\dot{\mathcal{D}}_{L^\infty}(\mathbb{R}^n)$ is the closure of $\mathcal{D}(\mathbb{R}^n)$ with respect to $\| \cdot \|_{m,\infty}$. Note that $\mathcal{D}_{L^p}(\mathbb{R}^n)$, $1 \le p < \infty$ and $\dot{\mathcal{D}}_{L^\infty}(\mathbb{R}^n)$ are Frechét spaces and normal spaces of distributions. We denote by $\mathcal{D}'_{L^q}(\mathbb{R}^n)$, $1 < q \le \infty$, the dual of $\mathcal{D}_{L^p}(\mathbb{R}^n)$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $\mathcal{D}'_{L^1}(\mathbb{R}^n)$ is the dual of $\dot{\mathcal{D}}_{L^\infty}(\mathbb{R}^n)$. Embedding $\mathcal{D}'_{L^q}(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\mathbb{R}^n)$ is strongly continuous and for $1 \le p_1 \le p_2 < \infty$ (and $q_1 = \frac{p_1}{p_1-1}$, $q_2 = \frac{p_2}{p_2-1}$) we have $C_0^\infty(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{D}_{L^{p_1}}(\mathbb{R}^n) \hookrightarrow \mathcal{D}_{L^{p_2}}(\mathbb{R}^n) \hookrightarrow \dot{\mathcal{D}}_{L^\infty}(\mathbb{R}^n) \hookrightarrow \mathcal{D}'_{L^1}(\mathbb{R}^n) \hookrightarrow \mathcal{D}'_{L^{q_2}}(\mathbb{R}^n) \hookrightarrow \mathcal{D}'_{L^{q_1}}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\mathbb{R}^n)$.

We will use the following structural theorem for $f \in \mathcal{D}'_{L^q}(\mathbb{R}^n), 1 \le q \le \infty$.

Let $f \in \mathcal{D}'(\mathbb{R}^n)$. Then $f \in \mathcal{D}'_{L^q}(\mathbb{R}^n), 1 \le q \le \infty$, if and only if there exists $m \in \mathbb{N}, h_\alpha \in \mathcal{D}_{L^\infty}(\mathbb{R}^n)$ and $F_\alpha \in L_q(\mathbb{R}^n)$ such that

$$f(x) = \sum_{|\alpha| \le m} h_\alpha(x)\, F_\alpha^{(\alpha)}(x),\ x \in \mathbb{R}^n. \tag{1}$$

If $f \in \mathcal{D}'_{L^q}(\mathbb{R}^n), 1 \le q \le \infty$ then a representation of f of the form (1) with $h_\alpha = 1$, $|\alpha| \le m$ is proved in [9].

Let us show that the right-hand side of (1) defines a distribution in $\mathcal{D}'_{L^q}(\mathbb{R}^n)$, $1 \le q \le \infty$. Note that, if $h \in \mathcal{D}_{L^\infty}(\mathbb{R}^n)$ and $\phi \in \mathcal{D}_{L^p}(\mathbb{R}^n)$ then $h\phi \in \mathcal{D}_{L^p}(\mathbb{R}^n)$, and the mapping $\phi \mapsto h\phi$, from $\mathcal{D}_{L^p}(\mathbb{R}^n)$ into itself, is continuous. By the Hölder inequality we have

$$|\langle f,\phi\rangle| \leq \sum_{|\alpha|\leq m}\int |F_\alpha(x)||(h_\alpha(x)\phi(x))^{(\alpha)}|\,dx \leq \sum_{|\alpha|\leq m} \parallel F_\alpha \parallel_{L^q}\parallel (h_\alpha\phi)^{(\alpha)} \parallel_{L^p}.$$

This completes the assertion that any f of the form (1) belongs to $\mathcal{D}'_{L^q}(\mathbb{R}^n)$.

A distribution $g_\alpha \in \mathcal{D}'(\mathbb{R}^n)$ is homogeneous of degree α if and only if

$$\langle g_\alpha(x), \phi(\frac{x}{a})\rangle = a^{\alpha+n}\,\langle g_\alpha(x),\phi(x)\rangle,\ a>0, \phi\in\mathcal{D}(\mathbb{R}^n).$$

For example, distributions $x_+^\alpha = x^\alpha\,H(x)$ and $x_-^\alpha = (-x)^\alpha\,H(-x), x\in\mathbb{R}$, are homogeneous distributions of degree α, for $\alpha \neq -1,-2,-3,\ldots$, where H is the Heaviside function. Also, if we put $x^\alpha = |x|^\alpha, x\in\mathbb{R}$, for $\alpha=-2,-4,\ldots$ and $x^\alpha = |x|^\alpha\ \mathrm{sgn}(x), x\in\mathbb{R}$, for $\alpha=-1,-3,\ldots$, then the distribution x^α is homogeneous of degree α, $\alpha=-1,-2,-3,\ldots$ The Dirac delta δ, is homogeneous of degree $-n$ in $\mathbb{R}^n$. Moreover, $\delta^{(\alpha)}$ is homogeneous of degree $-n-|\alpha|$, $\alpha\in\mathbb{N}_0^n$.

Recall [3], for $x_+^\alpha = x^\alpha H(x), x\in\mathbb{R}, \alpha > -(n+2)$, $\alpha\neq -1,-2,\ldots,-(n+1)$ and $n\in\mathbb{N}$,

$$\langle x_+^\alpha,\phi\rangle = \int_0^1 x^\alpha\Big(\phi(x)-\sum_{j=0}^n\frac{\phi^{(j)}(0)}{j!}x^j\Big)\,dx + \int_1^\infty x^\alpha\phi(x)\,dx$$

$$+\sum_{j=0}^n\frac{\phi^{(j)}(0)}{j!(\alpha+j+1)},\ \phi\in\mathcal{D}(\mathbb{R}).$$

For the later use we recall the following structural theorem for homogeneous distributions, [3, 5, 6].

Let $\phi\in C_0^\infty(\mathbb{R}^n)$ and $K\in\mathcal{S}'(S^{n-1})$. We define

$$S_\phi(r) := \frac{1}{|S^{n-1}|}\int_{S^{n-1}} K(\dot{x})\,\phi(r\dot{x})\,d\dot{x},\ r>0.$$

It is an element of $C_0^\infty(\mathbb{R}_+)$.

Theorem 1. *Let g_α be a homogeneous distribution of degree α in $\mathbb{R}^n$. If $\alpha\neq -n,-n-1,-n-2,\ldots$ then*

$$\langle g_\alpha,\,\phi\rangle = |S^{n-1}|\int_0^\infty r^{\alpha+n-1}S_\phi(r)\,dr. \tag{2}$$

If $\alpha=-n-m$, with $m=0,1,2,\ldots$ then

$$\langle g_\alpha,\,\phi\rangle = |S^{n-1}|\int_0^\infty r^{\alpha+n-1}S_\phi(r)\,dr + \langle\sum_{|k|=m}a_k\delta^{(k)}(x),\,\phi(x)\rangle, \tag{3}$$

for some constants $a_k, |k|=m$.

If $n = 1$ and $\alpha \neq -1, -2, -3, \dots$ then there exist constants C_1 and C_2 such that

$$\langle g_\alpha,\, \phi\rangle = \langle C_1\, x_+^\alpha + C_2\, x_-^\alpha,\, \phi(x)\rangle, \quad \phi \in \mathcal{D}(\mathbb{R}), \tag{4}$$

and if $\alpha = -n$, $n = 1, 2, 3, \dots$

$$\langle g_\alpha,\, \phi\rangle = \langle C_1 x^{-n} + C_2 \delta^{(n-1)}(x),\, \phi(x)\rangle, \quad \phi \in \mathcal{D}(\mathbb{R}). \tag{5}$$

3. Main results

Let g_α be a homogeneous distribution in $\mathcal{D}'(\mathbb{R}^n)$. We know that it is a tempered distribution and thus $|\langle g_\alpha, \phi\rangle| < \infty$ for any $\phi \in \mathcal{S}(\mathbb{R}^n)$. We start with the one-dimensional case.

Theorem 2. *Let $g_\alpha \neq 0$ be a homogeneous distribution on $\mathbb{R}$ of degree α, $\alpha \in \mathbb{R} \setminus (\mathbb{Z}_-)$.*

(i) Let $q \in [1, \infty)$. Then $g_\alpha \in \mathcal{D}'_{L^q}(\mathbb{R})$ if and only if $\alpha \in (-\infty, -\frac{1}{q})$.

(ii) $g_\alpha \in \mathcal{D}'_{L^\infty}(\mathbb{R})$ if and only if $\alpha \in (-\infty, 0)$.

Proof. (*i*) **Step 1:** Let $\alpha < -1$, $\alpha \in \mathbb{R} \setminus (\mathbb{Z}_-)$, and let $g_\alpha \neq 0$ be a homogeneous distribution of degree α. We will show that it can be extended on $\mathcal{D}_{L^p}(\mathbb{R}), 1 < p \leq \infty$ (and that this extension is continuous). Let $\phi \in \mathcal{D}_{L^p}(\mathbb{R})$. By a suitable partition of unity we have

$$\phi = \phi_1 + \phi_2 + \phi_3,$$

where supp $\phi_1 \subset (-\infty, -1)$, supp $\phi_2 \subset (-2, 2)$, supp $\phi_3 \subset (1, \infty)$ and $\phi_2 \in C_0^\infty(\mathbb{R})$. By Theorem 1 we have

$$\langle g_\alpha, \phi_1 + \phi_2 + \phi_3\rangle = \langle g_\alpha, \phi_1\rangle + \langle g_\alpha, \phi_2\rangle + \langle g_\alpha, \phi_3\rangle =$$

$$\langle C_1 x_+^\alpha + C_2 x_-^\alpha, \phi_1\rangle + \langle g_\alpha, \phi_2\rangle + \langle C_1 x_+^\alpha + C_2 x_-^\alpha, \phi_3\rangle < \infty,$$

which implies that $g_\alpha \in \mathcal{D}'_{L^q}(\mathbb{R})$, $\alpha < -1, q \in [1, \infty)$.

Step 2: Let $-1 < \alpha < -\frac{1}{q}$, $q > 1$, and let g_α be a homogeneous distribution of degree α. For arbitrary $\phi \in \mathcal{D}_{L^p}(\mathbb{R})$, $1 < p < \infty$ and a_1, $a_2 > 0$, by the Hölder inequality, we have

$$\begin{aligned}|\langle g_\alpha, \phi\rangle| = |C_1 \int_{-\infty}^{-a_2} (-x)^\alpha\, \phi(x)\, dx + C_1 \int_{-a_2}^{0} (-x)^\alpha\, \phi(x)\, dx \\ + C_2 \int_0^{a_1} x^\alpha\, \phi(x)\, dx + C_2 \int_{a_1}^{\infty} x^\alpha\, \phi(x)\, dx| \\ \leq C\Big[\Big(\int_{-\infty}^{-a_2} (-x)^{\alpha\, q}\, dx\Big)^{\frac{1}{q}} \Big(\int_{-\infty}^{-a_2} |\phi(x)|^p\, dx\Big)^{\frac{1}{p}}\end{aligned}$$

$$+\sup_{x\in[-a_2,0]}\{|\phi(x)|\}\int_0^{a_2} x^\alpha\,dx+\sup_{x\in[0,a_1]}\{|\phi(x)|\}\int_0^{a_1} x^\alpha\,dx$$
$$+\Big(\int_{a_1}^{\infty} x^{\alpha\,q}\,dx\Big)^{\frac{1}{q}}\,\Big(\int_{a_1}^{\infty}|\phi(x)|^p\,dx\Big)^{\frac{1}{p}}\Big].$$

Since $\sup_{x\in[0,a_1]}|\phi(x)|\le C\parallel\phi\parallel_{m,p}$ and $\sup_{x\in[-a_2,0]}|\phi(x)|\le C\parallel\phi\parallel_{m,p}$, for arbitrary $m\in\mathbb{N}$, we have

$$|\langle g_\alpha,\phi\rangle|\le C\Big[\Big(\frac{x^{\alpha\,q+1}}{\alpha\,q+1}\Big|_{a_2}^{\infty}\Big)^{\frac{1}{q}}\parallel\phi\parallel_{L^p}+\parallel\phi\parallel_{m,p}\Big(\frac{x^{\alpha+1}}{\alpha+1}\Big|_0^{a_2}\Big)$$

$$+\parallel\phi\parallel_{m,p}\Big(\frac{x^{\alpha+1}}{\alpha+1}\Big|_0^{a_1}\Big)+\Big(\frac{x^{\alpha\,q+1}}{\alpha\,q+1}\Big|_{a_1}^{\infty}\Big)^{\frac{1}{q}}\parallel\phi\parallel_{L^p}\Big]\;\le\;C\parallel\phi\parallel_{m,p},$$

and therefore $g_\alpha\in\mathcal{D}'_{L^q}(\mathbb{R})$, $1<q<\infty$, for $\alpha\in(-1,-\frac{1}{q})$.

Step 3: Assume that $q\in(1,\infty)$, $\alpha\ge-\frac{1}{q}$ and let g_α be a homogeneous distribution of degree α.

Let $s\in[\frac{1}{p},\infty)$, $p\in(1,\infty)$. We define a nonnegative test function $\phi\in C^\infty(\mathbb{R})$ such that

$$\phi(x)=\begin{cases}0 & ,\quad x<2\\ \frac{1}{x^s\,\ln x} & ,\quad x\ge3,\end{cases}\quad. \tag{6}$$

It is easy to check that $\phi\in\mathcal{D}_{L^p}(\mathbb{R})$, $p\in(1,\infty)$ for any given $s\ge\frac{1}{p}$. By Theorem 1 with ϕ given by (6) we have

$$\langle g_\alpha,\,\phi\rangle=\langle C_1x_+^\alpha+C_2x_-^\alpha,\,\phi(x)\rangle$$
$$=C_1\int_2^3\frac{1}{x^{-\alpha}}\,\phi(x)\,dx+C_1\int_3^\infty\frac{1}{x^{-\alpha}}\,\frac{1}{x^s\,\ln x}\,dx.$$

For given $\alpha\ge-\frac{1}{q}$ we choose $s=1+\alpha\in[\frac{1}{p},\infty)$, and obtain

$$\langle g_\alpha,\phi\rangle=C_1\int_2^3\frac{1}{x^{-\alpha}}\,\phi(x)\,dx+C_1\int_3^\infty\frac{1}{x\,\ln x}\,dx=\infty,$$

that is $g_\alpha\notin\mathcal{D}'_{L^q}(\mathbb{R})$.

The function ϕ given by (6) belongs to $\dot{\mathcal{D}}_{L^\infty}(\mathbb{R})$ when $s\in[0,\infty)$ and for $\alpha\ge-1$ we obtain that $g_\alpha\notin\mathcal{D}'_{L^1}(\mathbb{R})$.

(ii) Let g_α be a homogeneous distribution of degree α. If $\alpha>0$, one can prove that $g_\alpha\notin\mathcal{D}'_{L^\infty}(\mathbb{R})$ by the same way as it is done in Step 3 of (i). On the other hand, if $\alpha\le-1$, then, using the same idea as in Step 1 of (i) we obtain $g_\alpha\in\mathcal{D}'_{L^\infty}(\mathbb{R})$.

Let $\alpha\in(-1,0)$ and for some fixed $a>0$ we define the function

$$\theta(x) = \begin{cases} 1 & , \quad |x| \le a \\ 0 & , \quad |x| > a \end{cases} .$$

We have

$$x^\alpha = \theta(x)\, x^\alpha + (1-\theta(x))\, x^\alpha = \Big(\int_0^x \theta(t)\, t^\alpha \, dt \Big)' + (1-\theta(x))\, x^\alpha.$$

Since $\int_0^x \theta(t)\, t^\alpha \, dt$ and $(1-\theta(x))\, x^\alpha$ belong to $L^\infty(\mathbb{R})$, by (1) it follows that $x^\alpha \in \mathcal{D}'_{L^\infty}(\mathbb{R})$. Thus, for a given homogeneous distribution g_α, of degree $\alpha \in (-1,0)$, by Theorem 1 we have $g_\alpha \in \mathcal{D}'_{L^\infty}(\mathbb{R})$.

It remains to observe that for $\alpha = 0$

$$|\langle g_0, \phi \rangle| = |C_1 \int_{-\infty}^0 \phi(x)\, dx + C_2 \int_0^\infty \phi(x)\, dx| < \infty$$

for arbitrary $\phi \in \mathcal{D}_{L^1}(\mathbb{R})$. □

In order to analyze the multidimensional case, we need the following lemma.

Lemma 1. *Let* $1 \le p < \infty$. *If* $\phi \in \mathcal{D}_{L^p}(\mathbb{R}^n)$, *then* $r^{\frac{n-1}{p}} S_\phi(r) \in L^p(\mathbb{R}_+)$.

Proof. Let $p \in (1,\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. We have

$$\Big(\int_0^\infty |r^{\frac{n-1}{p}} S_\phi(r)|^p \, dr \Big)^{\frac{1}{p}} = \Big(\int_0^\infty \frac{r^{n-1}}{|S^{n-1}|^p} \Big| \int_{S^{n-1}} K(\dot{x}) \phi(r\dot{x})\, d\dot{x} \Big|^p dr \Big)^{\frac{1}{p}}$$

$$\le \int_{S^{n-1}} K(\dot{x}) \Big(\int_0^\infty \frac{r^{n-1}}{|S^{n-1}|^p} |\phi(r\dot{x})|^p \, dr \Big)^{\frac{1}{p}} d\dot{x}$$

$$\le \Big(\int_{S^{n-1}} |K(\dot{x})|^q \, d\dot{x} \Big)^{\frac{1}{q}} \Big(\int_{S^{n-1}} \int_0^\infty \frac{r^{n-1}}{|S^{n-1}|^p} |\phi(r\dot{x})|^p \, dr \, d\dot{x} \Big)^{\frac{1}{p}}$$

$$\le C \frac{1}{|S^{n-1}|^{\frac{1}{q}}} \Big(\int_0^\infty r^{n-1} \frac{1}{|S^{n-1}|} \int_{S^{n-1}} |\phi(r\dot{x})|^p \, d\dot{x}\, dr \Big)^{\frac{1}{p}}$$

$$\le C \frac{1}{|S^{n-1}|^{\frac{1}{q}}} \Big(\int_{\mathbb{R}^n} |\phi(x)|^p \, dx \Big)^{\frac{1}{p}} < \infty,$$

where we used the generalized Minkowsky inequality and the Hölder inequality. The proof is similar if $p = 1$. □

Theorem 3. *Let* $g_\alpha,\ \alpha \in \mathbb{R},\ \alpha \neq -n, -n-1, \ldots$ *be a homogeneous distribution of degree* α, $g_\alpha \neq 0$. *Then*

(*i*) $g_\alpha \in \mathcal{D}'_{L^q}(\mathbb{R}^n)$ *if and only if* $\alpha \in (-\infty,\, -\frac{n}{q}),\ 1 \leq q < \infty$.

(*ii*) $g_\alpha \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ *if and only if* $\alpha \in (-\infty, 0)$.

Proof. (*i*) **Step 1:** Let $\alpha \leq -n$, and let $g_\alpha \in \mathcal{D}'_{L^q}(\mathbb{R}^n), 1 \leq q < \infty$, be a homogeneous distribution of degree α. By Theorem 1 we have

$$\langle g_\alpha, \phi\rangle = |S^{n-1}|\langle r_+^{\alpha+n-1}, S_\phi(r)\rangle$$

and from Lemma 1 and Step 1 (i) in Theorem 2 we obtain that $g_\alpha \in \mathcal{D}'_{L^q}(\mathbb{R}^n)$, $1 \leq q < \infty$.

Step 2: Let $-n < \alpha < -\frac{n}{q}$ and let $g_\alpha \in \mathcal{D}'_{L^q}(\mathbb{R}^n), 1 < q < \infty$ be a homogeneous distribution of degree α. Let $\phi \in \mathcal{D}_{L^p}(\mathbb{R}^n), p = \frac{q}{q-1}$, and let $a > 0$ be an arbitrary constant. We have

$$\begin{aligned}
|\langle g_\alpha,\, \phi\rangle| &\leq |S^{n-1}| \int_0^a r^{\alpha+n-1}|S_\phi(r)|\,dr + |S^{n-1}| \int_a^\infty r^{\alpha+n-1}|S_\phi(r)|\,dr \\
&\leq |S^{n-1}| \sup_{r\in(0,a)}\{|S_\phi(r)|\} \int_0^a r^{\alpha+n-1}\,dr \\
&+|S^{n-1}| \int_a^\infty r^{\alpha+n-1-\frac{n-1}{p}}\, r^{\frac{n-1}{p}}\, |S_\phi(r)|\,dr \\
&\leq |S^{n-1}| \sup_{r\in(0,a)}\{|S_\phi(r)|\} \int_0^a r^{\alpha+n-1}\,dr \\
&+|S^{n-1}|\Big(\int_a^\infty r^{\alpha q+n-1}\,dr\Big)^{\frac{1}{q}} \Big(\int_a^\infty |r^{\frac{n-1}{p}}\, S_\phi(r)|^p\,dr\Big)^{\frac{1}{p}}.
\end{aligned}$$

By Lemma 1 we know that $S_\phi(r) \in \mathcal{D}_{L^p}(\mathbb{R}_+)$ and $r^{\frac{n-1}{p}}\, S_\phi(r) \in L^p(\mathbb{R}_+)$. Also, for every $m \in \mathbb{N}_0$ there exists $C > 0$ such that $\sup_{r\in(0,a)}\{|S_\phi(r)|\} \leq C\|S_\phi\|_{m,p}$. Therefore

$$\begin{aligned}
|\langle g_\alpha,\, \phi\rangle| &\leq C|S^{n-1}|\, \|\, S_\phi\, \|_{m,p}\, \Big(\frac{r^{\alpha+n}}{\alpha+n}\Big|_0^a\Big) \\
&+|S^{n-1}|\, \|\, r^{\frac{n-1}{p}}\, S_\phi\, \|_p\, \Big(\frac{r^{\alpha q+n}}{\alpha q+n}\Big|_a^\infty\Big)^{\frac{1}{q}} \\
&\leq \tilde{C}\big(\, \|\, S_\phi\, \|_{m,p} + \|\, r^{\frac{n-1}{p}}\, S_\phi\, \|_p\, \big) < \infty,
\end{aligned}$$

that is $g_\alpha \in \mathcal{D}'_{L^q}(\mathbb{R}^n)$, $1 < q < \infty$, for $\alpha \in (-n, -\frac{n}{q})$.

Step 3: Let $\alpha \geq -\frac{n}{q}$ and let $g_\alpha \in \mathcal{D}'_{L^q}(\mathbb{R}^n)$, $1 < q < \infty$, be a homogeneous distribution of degree α. We define the test function $\phi(x) \in \mathcal{D}(\mathbb{R}^n)$ as follows:

$$(7) \qquad \phi(x) = \begin{cases} 0 & , \quad |x| < 2 \\ \frac{1}{|x|^s \, \ln|x|} & , \quad |x| \geq 3, \end{cases} ,$$

where $s \in [\frac{n}{p}, \infty)$. Since

$$\int_{\mathbf{R}^n} |\phi(x)|^p \, dx = \int_{2 \leq |x| \leq 3} |\phi(x)|^p \, dx + \int_3^\infty r^{n-1} \Big| \frac{1}{r^s \ln r} \Big|^p dr \leq C + \int_3^\infty \frac{1}{r \ln^p r} \, dr$$

we have that $\phi \in L^p(\mathbb{R}^n),\ 1 < p < \infty$. Moreover, in a similar way we prove that $\phi \in \mathcal{D}_{L^p}(\mathbb{R}^n)$. On the other hand, for the given $\alpha \in [-\frac{n}{q}, \infty)$ take $s \in [\frac{n}{p}, \infty)$ such that

$$|\langle g_\alpha, \phi \rangle| = C + \int_{S^{n-1}} K(\dot{x}) \int_3^\infty r^{\alpha+n-1} \frac{1}{r^s \ln r} \, dr \, d\dot{x} = \infty,$$

and therefore $g_\alpha \notin \mathcal{D}'_{L^q}(\mathbb{R}^n),\ \alpha \in [-\frac{n}{q}, \infty),\ 1 < q < \infty$.

When $s \in [0, \infty)$, the test function ϕ, given by (7) belongs to $\dot{\mathcal{D}}_{L^\infty}(\mathbb{R}^n)$ and we conclude that $g_\alpha \notin \mathcal{D}'_{L^1}(\mathbb{R}^n)$, $\alpha \in (-n, \infty)$.

(*ii*) Let g_α be a homogeneous distribution of degree $\alpha \in \mathbb{R}$. By a slight modification of the proof of (*i*) we conclude that $g_\alpha \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$, when $\alpha \in (-\infty, -n)$. The test function

$$\phi(x) = \begin{cases} 0 & , \quad |x| < 2 \\ \frac{1}{|x|^s} & , \quad |x| \geq 3, \end{cases}$$

$\phi(x) \in C^\infty(\mathbb{R}^n)$, belongs to $\mathcal{D}_{L^1}(\mathbb{R}^n)$ when $s \in (n, \infty)$. Furthermore, $\langle g_\alpha, \phi \rangle = \infty$, when $\alpha \in (0, \infty)$, so that $g_\alpha \notin \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$.
Let $\alpha \in (-n, 0)$. We have

$$\langle g_\alpha, \phi \rangle = \langle g_\alpha \, \theta, \phi \rangle + \langle g_\alpha \, (1 - \theta), \phi \rangle,$$

where $\theta(x) \in \mathcal{D}(\mathbb{R}^n)$ is given by

$$\theta(x) = \begin{cases} 1 & , \quad |x| \leq 2 \\ 0 & , \quad |x| > 3, \end{cases}$$

We first observe $\langle g_\alpha \, \theta, \phi \rangle$. Put

$$F_\alpha(x) = F_\alpha(r, \dot{x}) := \int_0^r (g_\alpha \, \theta)(t, \dot{x}) \, dt, r \in [0, \infty), \dot{x} \in S^{n-1}, x = r\dot{x}.$$

It is easy to see that $F_\alpha \in L^\infty(\mathbb{R}^n)$. Furthermore we have

$$g_\alpha \, \theta(x) = \frac{\partial}{\partial r} F_\alpha(r, \dot{x}) = \frac{\partial F_\alpha}{\partial x_1} \frac{\partial x_1}{\partial r} + \frac{\partial F_\alpha}{\partial x_2} \frac{\partial x_2}{\partial r} + \cdots + \frac{\partial F_\alpha}{\partial x_n} \frac{\partial x_n}{\partial r}$$

$$= \frac{\partial F_\alpha}{\partial x_1}(r,\dot{x})\varphi_1(\dot{x}) + \frac{\partial F_\alpha}{\partial x_2}(r,\dot{x})\varphi_2(\dot{x}) + \cdots + \frac{\partial F_\alpha}{\partial x_n}(r,\dot{x})\varphi_n(\dot{x}),$$

where the terms $\dfrac{\partial x_i}{\partial r} = \varphi_i(\dot{x})$, $i = 1,2,\ldots,n$ do not depend on r. Moreover, $\varphi_i(\dot{x}) \in \mathcal{D}_{L^\infty}(\mathbb{R}^n)$, $i = 1,2,\ldots,n$.
Now, by the representation (1) we have that $g_\alpha\theta \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$.
It remains to show that $\langle g_\alpha\,(1-\theta), \phi\rangle < \infty$. Let $\phi \in \mathcal{D}_{L^1}(\mathbb{R}^n)$. We have

$$\langle g_\alpha\,(1-\theta), \phi\rangle = \langle g_\alpha, \phi\,(1-\theta)\rangle = |S^{n-1}| \int_0^\infty r^{\alpha+n-1} S_{\phi(1-\theta)}(r)\,dr$$

$$= |S^{n-1}| \int_0^\infty r^{\alpha+n-1} \frac{1}{|S^{n-1}|}\Big(\int_{S^{n-1}} K(\dot{x})\phi(r\dot{x})(1-\theta)(r)\,d\dot{x}\Big)\,dr$$

$$= C + \int_3^\infty r^{\alpha+n-1}\,S_\phi(r)\,dr = C + \int_3^\infty r^\alpha\,r^{n-1}\,S_\phi(r)\,dr.$$

By Lemma 1 we have $r^{n-1}\,S_\phi(r) \in L^1(\mathbb{R}_+)$, and therefore, since $\alpha \in (-n,0)$, the last integral is convergent. We conclude that $g_\alpha \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ for $\alpha \in (-n,0)$.
□

Corollary 1. *(i) Let g_α be a homogeneous distribution of degree $\alpha = -n$, $n \in \mathbb{N}$. Then $g_\alpha \in \mathcal{D}'_{L^q}(\mathbb{R})$, $1 \le q \le \infty$.*

(ii) Let g_α be a homogeneous distribution of degree $\alpha = -n-m$, $m \in \mathbb{N}$. Then $g_\alpha \in \mathcal{D}'_{L^q}(\mathbb{R}^n)$, $1 \le q \le \infty$.

Proof. (i) Let $n \in \mathbb{N}$. A homogeneous distribution g_α of degree $\alpha = -n$ has the representation

$$\langle g_\alpha,\,\phi\rangle = \langle C_1 x^{-n} + C_2\delta^{(n-1)}(x),\,\phi(x)\rangle, \quad \phi \in \mathcal{D}(\mathbb{R}).$$

By the suitable partition of unity we have

$$\phi = \phi_1 + \phi_2 + \phi_3,$$

where *supp* $\phi_1 \subset (-\infty,-1)$, *supp* $\phi_2 \subset (-2,2)$, *supp* $\phi_3 \subset (1,\infty)$ and $\phi_2 \in C_0^\infty(\mathbb{R})$. Now,

$$\langle g_\alpha, \phi_1+\phi_2+\phi_3\rangle = \langle g_\alpha,\phi_1\rangle + \langle g_\alpha,\phi_2\rangle + \langle g_\alpha,\phi_3\rangle$$

$$= \langle C_1x^{-n} + C_2\delta^{(n-1)}(x),\,\phi_1(x)\rangle + \langle g_\alpha,\phi_2\rangle + \langle C_1x^{-n} + C_2\delta^{(n-1)}(x),\,\phi_3(x)\rangle$$

$$= \langle C_1x^{-n},\,\phi_1\rangle + \langle g_\alpha,\phi_2\rangle + \langle C_1x^{-n},\,\phi_3\rangle.$$

Since $-n < -\frac{n}{q}$ we have that $|\langle x^{-n}, \phi_1\rangle| < \infty$ and $|\langle x^{-n}, \phi_3\rangle| < \infty$, and since ϕ_2 is compactly supported $|\langle g_\alpha, \phi_2\rangle| < \infty$.

The proof of $g_\alpha \in \mathcal{D}'_{L^q}(\mathbb{R}^n)$, $\alpha = -n-m$, $m \in \mathbb{N}$ is now a simple modification of the one-dimensional case. □

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