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ON THE PSEUDO-LEBESGUE-STIELTJES INTEGRAL

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Abstract. The aim of this paper is to create a theory of pseudo-Lebesgue-Stieltjes integral for pseudo-probability defined in [6]. We show the relations between the pseudo-Lebesgue integral and pseudo-Lebesgue-Stieltjes integral and some applications, too.

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1. Introduction

In many directions, use has been made of the pseudo-arithmetical operations based on the generator g (increasing bijection with $g(0) = 0$). Hence, it is interesting to study the integral with respect to the Pap g-calculus (see [2], [3], [5], [7], [8], [9], [13]).

In this paper we create a theory of pseudo-Lebesgue-Stieltjes integral for pseudo-probability defined in [6]. We show the relations between the pseudo-Lebesgue integral and pseudo-Lebesgue-Stieltjes integral and some applications, too.

The main motivation is the pseudo-probability theory. In [6] it was proved the weak law of large numbers, of course, only for continuous random variables, because the authors had no the general pseudo-Lebesgue-Stieltjes integral at their disponsal.

2. Basic notion

In this section we recall the basic notions like pseudo-operations, the pseudo-probability and pseudo-Lebesgue integral (see [2], [3], [5], [6], [7], [8], [9]), [13]).

Let $g : \mathbb{R} \to \mathbb{R}$ be an increasing bijection with $g(0) = 0$ and $g(1) = 1$. We define

pseudo-addition: $x \oplus y := g^{-1}(g(x) + g(y))$,

pseudo-substraction: $x \ominus y := g^{-1}(g(x) - g(y))$,

pseudo-multiplication: $x \odot y := g^{-1}(g(x) \cdot g(y))$,

pseudo-absolute value: $|x|^{\oplus} := g^{-1}(|g(x)|)$, for $x, y \in \mathbb{R}$.

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Let $(I, \oplus, \odot)$ be a semiring, I be a subinterval of $[-\infty, +\infty]$. Let Ω be a nonempty set and $\mathcal{S}$ be a σ-algebra of subsets of the set Ω. We define the **pseudo-probability** P as the function $P : \mathcal{S} \to I$ with the properties:

(i) $P(\emptyset) = 0$ and $P(\Omega) = 1$

(ii) $P(\bigcup_{i=1}^{\infty} A_i) = \bigoplus_{i=1}^{\infty} P(A_i)$ for pairwise disjoint sets $A_i \in \mathcal{S}$, $i \in \mathbb{N}$

The triple $(\Omega, \mathcal{S}, P)$ is the pseudo-probability space.

We can define the **induced probability** p as the function $p = g \circ P$. Then the triple $(\Omega, \mathcal{S}, p)$ is the probability space.

Let $(\Omega, \mathcal{S}, P)$ be a pseudo-probability space and p be the induced probability. Let $f : \Omega \to \mathbb{R}$ be a random variable. Then the **pseudo-Lebesgue integral** of the function f can be expressed by the formula

$$\int^{\oplus} f \, \mathrm{d}P = g^{-1}\Big(\int g \circ f \; dp \Big)$$

3. The pseudo-Lebesgue -Stieltjes integral

Let ξ be an integrable random variable, F be its distribution function and λ_F be the Lebesgue-Stieltjes measure corresponding to F.

In a general probability theory the mean value (expectation) of a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ (f is a random variable in a probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda_F)$) is called the Lebesgues-Stieltjes integral of the function f corresponding to F. It is denote by

$$\int f \; dF, \int_{-\infty}^{\infty} f(x) \; dF(x), \int f \; d\lambda_F.$$

We have $p_{\xi}(A) = p\,(\xi^{-1}(A)) = \lambda_F(A)$ for each $A \in \mathcal{B}(\mathbb{R})$.

In this section we create a theory of the pseudo-Lebesgue-Stieltjes integral. We define the notions like the pseudo-distribution function, the pseudo-Lebesgue-Stieltjes measure and the pseudo-Lebesgue-Stieltjes integral.

Definition 3.1. *Let $(\Omega, \mathcal{S}, P)$ be a pseudo-probability space and $\xi : \Omega \to \mathbb{R}$ be a random variable. Then the pseudo-distribution function $F_g : \mathbb{R} \to \mathbb{R}$ of a random variable ξ is defined by the formula*

$$F_g(x) = P(\xi < x) = P(\{\omega; \xi(\omega) < x\})$$

for each $x \in \mathbb{R}$.

Theorem 3.2. *Let $(\Omega, \mathcal{S}, P)$ be a pseudo-probability space, $F : \mathbb{R} \to \mathbb{R}$ be a distribution function of a random variable ξ. Then the pseudo-distribution function $F_g : \mathbb{R} \to \mathbb{R}$ of random variable ξ can be expressed by the formula*

$$F_g = g^{-1} \circ F$$

Proof. Evidently, $F_g(x) = P(\xi < x) = g^{-1}(p(\xi < x)) = g^{-1}(F(x))$, for each $x \in \mathbb{R}$.

Definition 3.3. *Let $F_g : \mathbb{R} \to \mathbb{R}$ be the pseudo-distribution function of a random variable ξ and $F : \mathbb{R} \to \mathbb{R}$ be its distribution function. Then the pseudo-Lebesgues-Stieltjes measure $\lambda_F^g : \mathcal{B}(\mathbb{R}) \to \mathbb{R}$ is defined by the formula*

$$\lambda_F^g = g^{-1} \circ \lambda_F$$

where λ_F is the Lebesgue-Stieltjes measure corresponding to F.

Theorem 3.4. *Let $F_g : \mathbb{R} \to \mathbb{R}$ be the pseudo-distribution function of a random variable ξ. Then there exists exactly one pseudo-measure $\lambda_F^g : \mathcal{B}(\mathbb{R}) \to \mathbb{R}$ such that*

$$\lambda_F^g([\,a, b)) = F_g(b) \ominus F_g(a)$$

for each $a, b \in \mathbb{R}$, $a < b$.

Proof.
1. Existence: Define λ_F^g by *Definition 3.3.* Evidently, for each $a, b \in \mathbb{R}$, $a < b$ we obtain

$$\begin{aligned} \lambda_F^g([\,a, b)) &= g^{-1}(\lambda_F([a, b))) = g^{-1}(F(b) - F(a)) = \\ &= g^{-1}\big(g(F_g(b)) - g(F_g(a))\big) = \\ &= F_g(b) \ominus F_g(a) \end{aligned}$$

2. Uniqueness: Let μ, ν be the pseudo-measures defined on $\mathcal{B}(\mathbb{R})$, such that $\mu = \nu$ on $J = \{[a, b)\ ;\ a < b,\ a, b \in \mathbb{R}\}$.

Since μ, ν are the pseudo-measures and $\mu = \nu$ on J, then $g \circ \mu$, $g \circ \nu$ are the measures and $g \circ \mu = g \circ \nu$ on J. If $g \circ \mu(A) = g \circ \nu(A)$ for each $A \in J$, then $g \circ \mu(A) = g \circ \nu(A)$ for each $A \in \mathcal{B}(\mathbb{R})$.

Hence for each $A \in \mathcal{B}(\mathbb{R})$ we obtain

$$\mu(A) = g^{-1}\big(g \circ \mu(A)\big) = g^{-1}\big(g \circ \nu(A)\big) = \nu(A)$$

Therefore, if μ, ν are the pseudo-measures defined on $\mathcal{B}(\mathbb{R})$, such that $\mu = \nu$ on J, then $\mu = \nu$ on $\mathcal{B}(\mathbb{R})$.

Theorem 3.5. *Let $(\Omega, \mathcal{S}, P)$ be a pseudo-probability space. Then for each $A, B \in \mathcal{S}$, $B \subset A$ we have*

$$P(A \backslash B) = P(A) \ominus P(B)$$

Proof. Evidently, we obtain

$$\begin{aligned} P(A \backslash B) &= g^{-1}(p(A \backslash B)) = \\ &= g^{-1}(p(A) - p(B)) = \\ &= g^{-1}\Big(g(P(A)) - g(P(B))\Big) = \\ &= P(A) \ominus P(B) \end{aligned}$$

Theorem 3.6. *Let ξ be a random variable and F_g be its pseudo-distribution function. Then for each $A \in \mathcal{B}(\mathbb{R})$ we have*

$$P(\xi^{-1}(A)) = \lambda_F^g(A)$$

Proof. By *Theorem 3.5* we obtain

$$\begin{aligned} P\big(\xi^{-1}([a,b))\big) &= P\big(\xi^{-1}((-\infty,b) \backslash (-\infty,a))\big) = \\ &= P\big(\xi^{-1}((-\infty,b)) \backslash \xi^{-1}((-\infty,a))\big) = \\ &= P\big(\xi^{-1}((-\infty,b))\big) \ominus P\big(\xi^{-1}((-\infty,a))\big) = \\ &= F_g(b) \ominus F_g(a) = \\ &= \lambda_F^g([a,b)) \end{aligned}$$

hence *Theorem 3.4* is applicable.

Theorem 3.7. *Let ξ be an integrable random variable, F_g be its pseudo-distribution function and $F = g \circ F_g$ be its distribution function. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. Then the pseudo-Lebesgues-Stieltjes integral of the function f corresponding to F_g can be expressed by the formula*

$$\int^{\oplus} f \; dF_g = g^{-1}\bigg(\int g \circ f \; dF\bigg)$$

Proof.
Function f is a random variable in the pseudo-probability space $(\mathbb{R}, B(\mathbb{R}), \lambda_F^g)$.

1. Let f be a Borel measurable function defined by $f = \sum\limits_{i=1}^{n} \alpha_i \cdot \chi_{A_i}$, where A_i $(i = 1, \cdots, n)$ are disjoint sets from $\mathcal{S}$. Then, by *Theorem 3.6* we obtain

$$\begin{aligned}
\int^{\oplus} f\ dF_g &= \bigoplus_{i=1}^{n}(\alpha_i \otimes \lambda_F^g(A_i)) = \\
&= g^{-1}\Big(\sum_{i=1}^{n} g(\alpha_i \otimes \lambda_F^g(A_i))\Big) = \\
&= g^{-1}\Big(\sum_{i=1}^{n} g(g^{-1}(g(\alpha_i)\cdot g(\lambda_F^g(A_i))))\Big) = \\
&= g^{-1}\Big(\sum_{i=1}^{n} (g(\alpha_i)\cdot g(\lambda_F^g(A_i)))\Big) = \\
&= g^{-1}\Big(\sum_{i=1}^{n} (g(\alpha_i)\cdot g(P(\xi^{-1}(A_i))))\Big) = \\
&= g^{-1}\Big(\sum_{i=1}^{n} (g(\alpha_i)\cdot p(\xi^{-1}(A_i)))\Big) = \\
&= g^{-1}\Big(\sum_{i=1}^{n} (g(\alpha_i)\cdot \lambda_F(A_i))\Big) = \\
&= g^{-1}\Big(\int g\circ f\ dF\Big)
\end{aligned}$$

2. Let f be a nonnegative Borel measurable function and $(f_n)_{n=1}^{\infty}$ be a sequence of the simple Borel measurable functions, where $f_n \geq 0$, $f_n \nearrow f$.

Since f_n are the simple Borel measurable functions, then by the case 1 we have

$$\int^{\oplus} f_n\ dF_g = g^{-1}\Big(\int g\circ f_n\ dF\Big) \tag{1}$$

Hence, by limits theorems for pseudo-Lebesgue integral and by (1) can be obtained

$$\begin{aligned}
\int^{\oplus} f\ dF_g &= \lim_{n\to\infty}\int^{\oplus} f_n\ dF_g = \\
&= \lim_{n\to\infty} g^{-1}\Big(\int g\circ f_n\ dF\Big) = \\
&= g^{-1}\Big(\lim_{n\to\infty}\int g\circ f_n\ dF\Big) = \\
&= g^{-1}\Big(\int g\circ f\ dF\Big)
\end{aligned}$$

3. Let f be a Borel measurable function, $f = f^{+} - f^{-}$. Then, by property of

pseudo-Lebesgue integral and the case 2 we obtain

$$\begin{aligned}\int^{\oplus} f\ dF_g &= \int^{\oplus} f^+\ dF_g \ominus \int^{\oplus} f^-\ dF_g = \\ &= g^{-1}\Big(g\Big(\int^{\oplus} f^+\ dF_g\Big) - g\Big(\int^{\oplus} f^-\ dF_g\Big)\Big) = \\ &= g^{-1}\Bigg(g\Big(g^{-1}\Big(\int g\circ f^+\ dF\Big)\Big) - g\Big(g^{-1}\Big(\int g\circ f^-\ dF\Big)\Big)\Bigg) = \\ &= g^{-1}\Big(\int g\circ f^+\ dF - \int g\circ f^-\ dF\Big) = \\ &= g^{-1}\Big(\int g\circ f\ dF\Big)\end{aligned}$$

Proposition 3.1. *Let* $\bigoplus_{i=1}^{\infty} P_i < \infty$, $P_i \geq 0$, $P_i \in \mathbb{R}$ *and* $(x_i)_1^{\infty}$ *be a sequence of real numbers such that if* $i \neq j$ *then* $x_i \neq x_j$. *Denote*

$$F_g(x) = \bigoplus_{x_i<x} P_i$$

Then, F_g *is non-decreasing and left-continuous function.*

Proof. Since $\bigoplus_{i=1}^{\infty} P_i = g^{-1}\Big(\sum_{i=1}^{\infty} g\circ P_i\Big) < \infty$ and g is an increasing bijection, then $\sum_{i=1}^{\infty} g\circ P_i < \infty$. Hence by *Proposition 4.11.2* in [12] can be obtained that the function $F(x) = \sum_{x_i<x} g\circ P_i$ is non-decreasing and left-continuous.

Therefore

$$F_g(x) = \bigoplus_{x_i<x} P_i = g^{-1}\Big(\sum_{x_i<x} g\circ P_i\Big)$$

is a non-decreasing and left-continuous function.

Theorem 3.8. *If the function* F_g *is defined like in Proposition 3.1, then a measurable function* f *is integrable by pseudo-Lebesgue-Stieltjes measure* λ_F^g *if and only if* $\bigoplus_{i=1}^{\infty} (|f(x_i)|^{\oplus} \odot P_i) < \infty$. *It holds*

$$\int^{\oplus} f(x)\ \mathrm{d}F_g(x) = \bigoplus_{i=1}^{\infty} (f(x_i) \odot P_i).$$

Proof. "⇒" Let f be integrable by the pseudo-Lebesgue-Stieltjes measure λ_F^g. Then, there exists the pseudo-Lebesgue-Stieltjes integral $\int^{\oplus} f\ \mathrm{d}F_g = g^{-1}\big(\int g\circ f\ \mathrm{d}F\big)$. Hence, $g\circ f$ is integrable by the Lebesgue-Stieltjes measure λ_F.

Moreover, since F_g is defined like in *Proposition 3.1* and g is an increasing bijection, then the function F defined by

$$F(x) = \sum_{x_i < x} g \circ P_i$$

and the measurable function $g \circ f$ satisfy the assumptions of *Theorem 4.11.3* in [12]. Hence

$$\sum_{i=1}^{\infty} \big(|(g \circ f)(x_i)| \cdot (g \circ P_i) \big) < \infty$$

and

$$\int g \circ f \, \mathrm{d}F = \sum_{i=1}^{\infty} \big((g \circ f)(x_i) \cdot (g \circ P_i) \big).$$

Therefore

$$\begin{aligned}
\bigoplus_{i=1}^{\infty} (|f(x_i)|^{\oplus} \odot P_i) &= g^{-1}\Bigg(\sum_{i=1}^{\infty} g\Big(|f(x_i)|^{\oplus} \odot P_i \Big) \Bigg) = \\
&= g^{-1}\Bigg(\sum_{i=1}^{\infty} g \circ g^{-1}\Big(g(|f(x_i)|^{\oplus}) \cdot g(P_i) \Big) \Bigg) = \\
&= g^{-1}\Bigg(\sum_{i=1}^{\infty} \big(g(|f(x_i)|^{\oplus}) \cdot g(P_i) \big) \Bigg) = \\
&= g^{-1}\Bigg(\sum_{i=1}^{\infty} \big(g \circ g^{-1}(|(g \circ f)(x_i)|) \cdot g(P_i) \big) \Bigg) = \\
&= g^{-1}\Bigg(\sum_{i=1}^{\infty} \big(|(g \circ f)(x_i)| \cdot (g \circ P_i) \big) \Bigg) < g^{-1}(\infty) = \infty
\end{aligned}$$

and

$$\begin{aligned}
\int^{\oplus} f(x) \, \mathrm{d}F_g(x) &= g^{-1}\Bigg(\int (g \circ f)(x) \, \mathrm{d}F(x) \Bigg) = \\
&= g^{-1}\Bigg(\sum_{i=1}^{\infty} \big((g \circ f)(x_i) \cdot (g \circ P_i) \big) \Bigg) = \\
&= \bigoplus_{i=1}^{\infty} (f(x_i) \odot P_i).
\end{aligned}$$

"$\Leftarrow$" It is analogy to the proof "$\Rightarrow$".

Remark 3.9. *The pseudo-Lebesgue-Stieltjes integral* $\int^{\oplus} f(x) \, \mathrm{d}F_g(x)$ *defined in Theorem 3.8 can be expressed in the form*

$$\int^{\oplus} f(x) \, \mathrm{d}F_g(x) = g^{-1}\Bigg(\sum_{i=1}^{\infty} \big((g \circ f)(x_i) \cdot (g \circ P_i) \big) \Bigg)$$

Proposition 3.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative function such that there exists the pseudo-Lebesgue integral $\int^{\oplus} f \; \mathrm{d}T < \infty$. Then the function F_g defined by*

$$F_g(x) = \int_{(-\infty,x)}^{\oplus} f \; \mathrm{d}T$$

is non-decreasing and left-continuous.

Proof. Since there exists the pseudo-Lebesgue integral $\int^{\oplus} f \; \mathrm{d}T < \infty$ and g is an increasing bijection, then there exists the Lebesgue integral $\int g \circ f \; \mathrm{d}g \circ T < \infty$. Hence by *Proposition 4.11.4* in [12] can be obtained that the function $F(x) = \int_{(-\infty,x)} g \circ f \; \mathrm{d}g \circ T$ is non-decreasing and left-continuous.

Therefore

$$F_g(x) = \int_{(-\infty,x)}^{\oplus} f \; \mathrm{d}T = g^{-1}\Big(\int_{(-\infty,x)} g \circ f \; \mathrm{d}g \circ T \Big)$$

is non-decreasing and left-continuous function.

Theorem 3.10. *Let F_g be a function defined in Proposition 3.2. Then a Borel measurable function $h : \mathbb{R} \to \mathbb{R}$ is integrable by λ_F^g if and only if there exists the pseudo-Lebesgue integral $\int_R^{\oplus} h(x) \odot f(x) \; \mathrm{d}x < \infty$. It holds*

$$\int_R^{\oplus} h(x) \; \mathrm{d}F_g(x) = \int_R^{\oplus} h(x) \odot f(x) \; \mathrm{d}x$$

Proof. "$\Rightarrow$" Let h be a Borel measurable function such that it is integrable by the pseudo-Lebesgue-Stieltjes measure λ_F^g. Then, there exists the pseudo-Lebesgue-Stieltjes integral $\int_R^{\oplus} h(x) \; \mathrm{d}F_g(x) = g^{-1}\big(\int_R (g \circ h)(x) \; \mathrm{d}F(x) \big)$. Hence, $g \circ h$ is integrable by the Lebesgue-Stieltjes measure λ_F.

Moreover, since F_g is defined as in *Proposition 3.2* and g is an increasing bijection, then the function F, defined by

$$F(x) = \int_{(-\infty,x)} g \circ f \; \mathrm{d}g \circ T$$

and the Borel measurable function $g \circ h$ satisfy the assumptions of *Theorem 4.11.5* in [12]. Hence there exists the Lebesgue integral

$$\int_R (g \circ h)(x) \cdot (g \circ f)(x) \; \mathrm{d}g \circ x$$

and

$$\int_R (g \circ h)(x)\ \mathrm{d}F(x) = \int_R (g \circ h)(x) \cdot (g \circ f)(x)\ \mathrm{d}g \circ x.$$

Therefore, there exists the pseudo-Lebesgue integral

$$\begin{aligned}\int_R^{\oplus} h(x) \odot f(x)\ \mathrm{d}x &= g^{-1}\Big(\int_R g(h(x) \odot f(x))\ \mathrm{d}g \circ x\Big) = \\ &= g^{-1}\Big(\int_R g \circ g^{-1}\big(g(h(x)) \cdot g(f(x))\big)\ \mathrm{d}g \circ x\Big) = \\ &= g^{-1}\Big(\int_R (g \circ h)(x) \cdot (g \circ f)(x)\ \mathrm{d}g \circ x\Big)\end{aligned}$$

and

$$\begin{aligned}\int_R^{\oplus} h(x)\ \mathrm{d}F_g(x) &= g^{-1}\Big(\int_R (g \circ h)(x)\ \mathrm{d}F(x)\Big) = \\ &= g^{-1}\Big(\int_R (g \circ h)(x) \cdot (g \circ f)(x)\ \mathrm{d}g \circ x\Big) = \\ &= \int_R^{\oplus} h(x) \odot f(x)\ \mathrm{d}x.\end{aligned}$$

"$\Leftarrow$" It is analogy to the proof "$\Rightarrow$".

4. Relation between the pseudo-Lebesgue and pseudo-Lebesgue-Stieltjes integral

In the general probability theory there exists a relation between the Lebesgue integral and the Lebesgue-Stieltjes integral. If ξ is a random variable with a distribution function F and $f : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function, then the function $f \circ \xi$ is a random variable and

$$\int f \circ \xi\ dp = \int f\ dF$$

if one of these integrals exists.

In this section we show that in the pseudo-probability there exists a relation between the pseudo-Lebesgue integral and the pseudo-Lebesgue-Stieltjes integral, too.

Theorem 4.1. *Let $(\Omega, \mathcal{S}, P)$ be a pseudo-probability space and p be the induced probability. Let ξ be a random variable with the pseudo-distribution function F_g, $F = g \circ F_g$ be its distribution function and $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. Then the function $f \circ \xi$ is a random variable and*

$$\int^{\oplus} f \circ \xi\ dP = \int^{\oplus} f\ dF_g$$

if one of these integrals exists.

Proof. For each $A \in \mathcal{B}(\mathbb{R})$ we have that

$$\big(f \circ \xi\big)^{-1}(A) = \xi^{-1}\big(f^{-1}(A)\big) \in \mathcal{S}$$

hence $f \circ \xi$ is a random variable.

Since $g : \mathbb{R} \to \mathbb{R}$ is an increasing bijection with $g(0) = 0$, $g(1) = 1$ and $f : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function, then $g \circ f : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function. Hence

$$\int (g \circ f) \circ \xi \; dp = \int g \circ f \; dF \tag{2}$$

By property (2) and *Theorem 3.7* we obtain

$$\begin{aligned} \int^{\oplus} f \circ \xi \; dP &= g^{-1}\Big(\int g \circ (f \circ \xi) \; dp\Big) = g^{-1}\Big(\int (g \circ f) \circ \xi \; dp\Big) = \\ &= g^{-1}\Big(\int g \circ f \; dF\Big) = \int^{\oplus} f \; dF_g \end{aligned}$$

5. Applications

It can see that *Theorem 3.8*, *Theorem 3.10* and *Theorem 4.1* can be used to calculate the pseudo-dispersion $\sigma^{2_\oplus}$ in the same sense like their analogues in the general probability theory.

Now we recall the notions like pseudo-mean value (expectation) $E^{\oplus}$, pseudo-dispersion $\sigma^{2_\oplus}$ and $L_2^{\oplus}$ space, introduced in [4].

Let $(\Omega, \mathcal{S}, P)$ be a pseudo-probability space. If a random variable $\xi : \Omega \to \mathbb{R}$ is integrable, then the number

$$E^{\oplus}(\xi) = \int_{\Omega}^{\oplus} \xi \; dP < \infty$$

is called the pseudo-mean value (expectation) of the random variable ξ.

Let $\xi : \Omega \to \mathbb{R}$ be an integrable random variable in $(\Omega, \mathcal{S}, P)$. If the random variable $\big(\xi \ominus E^{\oplus}(\xi)\big)^{2\oplus}$ is integrable, then we say that ξ has a pseudo-dispersion $\sigma^{2_\oplus}$ defined by the formula

$$\sigma^{2_\oplus}(\xi) = E^{\oplus}\Big(\big(\xi \ominus E^{\oplus}(\xi)\big)^{2\oplus}\Big)$$

Let $(\Omega, \mathcal{S}, P)$ be a g-measure space (i.e. P is defined by the formula $P = g^{-1} \circ p$, where p is the measure). We denote $L_2^{\oplus} = L_2^{\oplus}(\Omega, \mathcal{S}, P)$ the class of all measurable real functions f for which $f \odot f$ is integrable. We write

$$L_2^{\oplus} = \Big\{ f : \Omega \to \mathbb{R}; \int^{\oplus} f \odot f \; dP < \infty \Big\}$$

The following theorems are the key to the calculation of the pseudo dispersion $\sigma^{2_\oplus}$ of random variables.

Proposition 5.1. *For each $\xi \in L_2^{\oplus}$ hold*

$$\sigma^{2\oplus}(\xi) = \int_R^{\oplus} \left(x \ominus E^{\oplus}(\xi)\right)^{2\oplus} \, dF_g(x),$$

Proof. If we denote $f(x) = \left(x \ominus E^{\oplus}(\xi)\right)^{2\oplus}$ in *Theorem 4.1*, then, by the definition of pseudo-dispersion and pseudo-mean value we obtain

$$\sigma^{2\oplus}(\xi) = \int_R^{\oplus} \left(x \ominus E^{\oplus}(\xi)\right)^{2\oplus} \, dF_g(x).$$

Proposition 5.2. *If ξ is a discrete random variable from the $L_2^{\oplus}$ space with the values $x_1, x_2, \dots$ and their pseudo-probabilities $P_1, P_2, \dots$, then*

$$\sigma^{2\oplus}(\xi) = \bigoplus_{i=1}^{\infty} \left(\left(x_i \ominus E^{\oplus}(\xi)\right)^{2\oplus} \odot P_i\right).$$

Proof. We obtain this result from *Proposition 5.1* and *Theorem 3.8.*

Proposition 5.3. *If ξ is a continuous random variable with a density f (i.e. $F_g(x) = \int\limits_{(-\infty,x)}^{\oplus} f \; \mathrm{d}T$), then*

$$\sigma^{2\oplus}(\xi) = \int_R^{\oplus} \left(\left(x \ominus E^{\oplus}(\xi)\right)^{2\oplus} \odot f(x)\right) \, \mathrm{d}x.$$

Proof. We obtain this result from *Proposition 5.1* and *Theorem 3.10.*

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