Novi Sad J. Math.
Vol. 36, No. 2, 2006, 137-152

ON A CLASS OF QUASI-DISTRIBUTION SEMIGROUPS

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Abstract. A class of $[r]$-semigroups, extending the class of smooth distribution semigroups of Balabane and Emami-Rad, is introduced. Relations with integrated semigroups of Arendt are given as well as a generalization of deLaubenfels and Jazar's result on relations between smooth semispectral distributions and integrated semigroups.

AMS Mathematics Subject Classification (2000): 47D60,

Key words and phrases: $[r]$-semigroups, quasi-distribution semigroups, integrated semigroups, well-posedness

0. Introduction

The aim of this paper is to give the structural characterizations of the class of $[r]$-semigroups, $r \geq 0$, analyze the relations of $[r]$-semigroups with other known classes of semigroups, and determine the functional calculus for the class of dense $[r]$-semigroups. If $r = 0$ and A is dense, A is the generator of an $[r]$-semigroup if and only if A is the generator of a smooth distribution semigroup of Balabane and Emami-Rad ([6], [7]). We also introduce a class of $\{r\}$-semigroups, $r \geq 0$, closely linked with the class of smooth distribution semigroups of exponential growth r (cf. [8]).

We give several characterizations of non-degenerate, polynomially bounded integrated semigroups and their relations with quasi-distribution semigroups and $[r]$-semigroups. In Section 4, quasi-distribution semigroups and $[r]$-semigroups are used in the analysis of smooth semispectral distributions and $\mathcal{A}_{n,k}$ functional calculi of deLaubenfels and Jazar ([11]). In the last section we present several examples of $[r]$-semigroups.

1. Preliminaries

Throughout this paper E denotes a complex Banach space and $L(E)$ denotes the space of all bounded linear operators from E into E. We will assume that $L(E)$ is equipped with the strong topology. For a linear operator A in E, its domain, range and null space are denoted by $D(A)$, $R(A)$ and $N(A)$, respectively. We will always assume that A is a closed operator.

Schwartz spaces of test functions on the real line $\mathbb{R}$ are denoted by $\mathcal{D} = C_0^\infty$ and $\mathcal{S}$. Their strong duals are $\mathcal{D}'$ and $\mathcal{S}'$, respectively; $\mathcal{D}_0$, resp. $\mathcal{D}_{(0,\infty)}$,

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denotes the subspace of $\mathcal{D}$ which consists of the elements supported by $[0,\infty)$, resp. $(0,\infty)$. Further on, $\mathcal{D}'(L(E)) = L(\mathcal{D}, L(E))$, is the space of continuous linear functions from $\mathcal{D}$ into $L(E)$, and we assume that it is equipped with the strong topology. $\mathcal{D}'_0(L(E))$ is the subspace of $\mathcal{D}'(L(E))$ containing the elements supported by $[0,\infty)$.

Definition 1.1. [19] A quasi-distribution semigroup G is an element $G \in \mathcal{D}'(L(E))$ satisfying

(QDSG1) $G(\varphi *_0 \psi) = G(\varphi)G(\psi),\ \varphi,\ \psi \in \mathcal{D},$ and

(QDSG2) $\mathcal{N}(G) := \bigcap_{\varphi\in\mathcal{D}_0} N(G(\varphi)) = \{0\},$

where $*_0$ is the convolution $f *_0 g(t) := \int_0^t f(t-s)g(s)ds,\ t \in \mathbb{R}$. If the set $\mathcal{R}(G) := \bigcup_{\varphi\in\mathcal{D}_0} R(G(\varphi))$ is dense in E, then G is called a dense (QDSG).

Conditions (QDSG1) and (QDSG2) imply that $G \in \mathcal{D}'_0(L(E))$, (cf. [16] and [19]) and Definition 1.1 is equivalent to the definition of the distribution semigroup G given on page 839 of [14]. The generator A of G is defined by $A := \{(x,y) \in E \times E : G(-\varphi')x = G(\varphi)y,\ \varphi \in \mathcal{D}_0\}$ and it is a closed linear operator in E.

If $\varphi \in \mathcal{D}$, let $\varphi_+(t) := \varphi(t)H(t),\ t \in \mathbb{R}$, where $H(\cdot)$ is the Heaviside function. Denote $\mathcal{D}_+ := \{\varphi_+ : \varphi \in \mathcal{D}\}$. Further, if G is a (QDSG) then it can be regarded as an element of $L(\mathcal{D}_+, L(E))$ (see [19]). If $\varphi_+ \in \mathcal{D}_+$, then $\frac{d^k}{dt^k}\varphi_+(t)$ means the kth right derivative.

If $f : [0,\infty) \mapsto E$ is a measurable function and if there exist $M > 0$ and $\omega \in \mathbb{R}$ such that $||f(t)|| \le Me^{\omega t}$, a.e. $t \ge 0$, then the Laplace transformation of f is defined by $\hat{f}(\lambda) := \mathcal{L}(f)\lambda = \int_0^\infty e^{-\lambda t} f(t)dt,\ Re\lambda > \omega$.

In the sequel, we shall also use φ, ψ, etc. to denote the elements in $\mathcal{D}_+$.

2. $[r]$-semigroups

We recall a result from [19]: Let $r > 0$. Then

$$(1)\qquad \int_0^\infty e^{rt}|\varphi^{(k)}(t)|dt \le \frac{1}{r}\int_0^\infty e^{rt}|\varphi^{(k+1)}(t)|dt,\ \varphi \in \mathcal{D}_+,\ k \in \mathbb{N}_0.$$

Lemma 2.1. *Let $r \ge 0$ and $k \in \mathbb{N}_0$. Define*

$$p_{rk}(\varphi) := \sum_{i=0}^{k} \left\| e^{rt} t^i \varphi^{(i)} \right\|_{L^1([0,\infty))};$$

$$q_{rk}(\varphi) := \sum_{i=0}^{k} \left\| t^i (e^{rt}\varphi)^{(i)} \right\|_{L^1([0,\infty))}, \ \varphi \in \mathcal{D}_+.$$

Then the inclusion mapping $id : (\mathcal{D}_+, p_{rk}) \to (\mathcal{D}_+, q_{rk})$ *is a continuous mapping between normed spaces. (We will use notation* $|| \cdot ||_1$ *for* $|| \cdot ||_{L^1([0,\infty))}$*.)*

Proof. Clearly, p_{rk} and q_{rk} are norms on $\mathcal{D}_+$. Let us show that there exists $C > 0$ such that $q_{rk}(\varphi) \leq C p_{rk}(\varphi)$, $\varphi \in \mathcal{D}_+$. If $r = 0$ or $k = 0$, the proof is trivial. So let us assume $r > 0$ and $k \in \mathbb{N}$. Then

$$q_{rk}(\varphi) = \sum_{i=0}^{k} \left\| t^i (e^{rt}\varphi)^{(i)} \right\|_1 = \sum_{i=0}^{k} \left\| t^i \sum_{j=0}^{i} \binom{i}{j} r^{i-j} e^{rt} \varphi^{(j)} \right\|_1$$

$$\leq \sum_{i=0}^{k}\sum_{j=0}^{i} \left\| \binom{i}{j} r^{i-j} e^{rt} t^i \varphi^{(j)} \right\|_1 \leq C_1 \sum_{i=0}^{k}\sum_{j=0}^{i} \left\| e^{rt} t^i \varphi^{(j)} \right\|_1,$$

for a suitable constant $C_1 > 0$ which is independent of $\varphi \in \mathcal{D}_+$. Let

$$a_{i,j} := \left\| e^{rt} t^i \varphi^{(j)} \right\|_1, \ i, \ j = 0, 1, \ldots, k.$$

Then (1) implies that for all $i \in \{1, 2, \ldots, k\}$ and $j \in \{0, 1, \ldots, k-1\}$, one has

$$a_{i,j} \leq \frac{i}{r} a_{i-1,j} + \frac{1}{r} a_{i,j+1}.$$

Applying this inequality sufficiently many times, one obtains that if $i \in \{0, 1, \ldots, k\}$ and $j \in \{0, 1, \ldots, i\}$, then $\left\| e^{rt} t^i \varphi^{(j)} \right\|_1 \leq C_{ij} p_{rk}(\varphi)$, where C_{ij} is independent of $\varphi \in \mathcal{D}_+$. This ends the proof. □

Let T_{rk} and D_{rk} be the completions of $(\mathcal{D}_+, p_{rk})$ and $(\mathcal{D}_+, q_{rk})$, respectively. Denote $h_\lambda(t) = e^{-\lambda t} H(t)$, $t \in \mathbb{R}$. Then $h_\lambda(t)$ belongs to T_{rk} and D_{rk} for all $\lambda \in \mathbb{C}$ with $Re\lambda > r$. Similarly as in Proposition II. 4 in [6], we have that T_{rk} and D_{rk} are algebras for the convolution product $*_0$.

Definition 2.1. Let $r \geq 0$. A (QDSG) G is said to be an $[r, k]$-semigroup, respectively an $\{r, k\}$-semigroup, if G can be extended to a continuous linear mapping from T_{rk}, respectively D_{rk}, into $L(E)$. It is said that G is an $[r]$-semigroup, respectively an $\{r\}$-semigroup, if it is an $[r, k]$-semigroup, respectively an $\{r, k\}$-semigroup, for some $k \in \mathbb{N}_0$. If $\overline{\mathcal{R}(G)} = E$, then we say that G is a dense $[r]$-semigroup.

Lemma 2.1 implies that $\{r, k\}$-semigroups make a subclass of the class of $[r, k]$-semigroups, $r \geq 0$, $k \in \mathbb{N}_0$. We will show that, for every $r > 0$, there exists a densely defined operator A which generates an $[r, 1]$-semigroup but it is not the generator of an $\{r, k\}$-semigroup for any $k \in \mathbb{N}_0$.

If $r = 0$ and G is a dense $[0, k]$-semigroup for some $k \in \mathbb{N}$, then the extension of G onto T_{rk} is a smooth distribution semigroup of order k (see [6], [7] and [2]).

In this case, it is easily seen that the generator A of G in the sense of Definition 1.1 is just the generator of the corresponding smooth distribution semigroup of order k in the sense of Balabane and Emami-Rad ([6], [7]). Smooth distribution semigroups are related to integrated semigroups in [2]. The proof of Theorem 4.4 in [2] will be frequently used in this paper and we shall repeat it in our context in Proposition 3.1.

Let us recall (cf. [19]) that a (QDSG) G is of order (r,k), $r>0,\ k\in\mathbb{N}_0$, if there exists $C>0$ such that $\|G(\varphi)\|\leq C\sum_{i=0}^{k}\|e^{rt}\varphi^{(i)}\|_1,\ \varphi\in\mathcal{D}_+$. Notice:

1. Let $r>0$. Then it is clear that an element $G\in\mathcal{D}'(L(E))$ is a (QDSG) of order $(r,0)$ iff G is an $[r,0]$-semigroup iff G is an $\{r,0\}$-semigroup. By [19, Theorem 4.13], A generates an $[r,0]$-semigroup if and only if $(r,\infty)\subset\rho(A)$ and

$$\sup_{\lambda>r,\ n\in\mathbb{N}_0}\left\|\frac{(\lambda-r)^{n+1}}{n!}\frac{d^n}{d\lambda^n}[R(\lambda:A)]\right\|<\infty.$$

This remains true for $r=0$ because A generates a $[0,0]$-semigroup G if and only if $A+1$ generates a $[1,0]$-semigroup $e^{t\cdot}G$. Hence, every Hille-Yosida operator (see for instance [5], Definition 3.5.1, and [9]) is the generator of an $[r,0]$-semigroup for some $r\geq 0$.

2. Suppose $r>0$, $\omega\in(r,\ \infty)$, $k\in\mathbb{N}_0$ and G is an $[r,k]$-semigroup. By induction and (1) one can prove that for all $i\in\mathbb{N}_0$ there exists $C(i,r)$ such that

$$\sum_{j=0}^{i}\|e^{rt}t^j\varphi^{(j)}\|_1\leq C(i,r)\|e^{rt}(1+t^i)\varphi^{(i)}\|_1,\quad \varphi\in\mathcal{D}_+.$$

Consequently, $\|G(\varphi)\|\leq C\|e^{wt}\varphi^{(k)}\|_1$, $\varphi\in\mathcal{D}_+$ and G is of order (ω,k).

We refer to [11] for the definition of a smooth semispectral distribution of degree $n\in\mathbb{N}_0$. The next proposition makes clear the structural properties of a dense $\{r,k\}$-semigroup.

Proposition 2.1. *Let $r\geq 0$, $k\in\mathbb{N}_0$ and let $D(A)$ be dense in E. The following assertions are equivalent.*

(a) A is the generator of an $\{r,k\}$-semigroup G.

(b) $A-r$ is the generator of an exponentially bounded k-times integrated semigroup $(S(t))_{t\geq 0}$ satisfying $\|S(t)\|=O(t^k)$.

(c) $(r,\infty)\subset\rho(A)$ and there exists a constant $M>0$ such that

$$\left\|\frac{d^j}{d\lambda^j}\left[\frac{R(\lambda+r:A)}{\lambda^k}\right]\right\|\leq M\frac{(k+j)!}{\lambda^{k+j+1}},\quad \lambda>0,\ j\in\mathbb{N}_0.$$

(d) $r-A$ poses a smooth semispectral distribution of degree k.

Proof. The proof in the case $k = 0$ is well known and it follows from the theory of C_0-semigroups. Suppose $k \in \mathbb{N}$.
(a) $\Rightarrow$ (b) One can easily conclude that $A - r$ is the generator of $e^{-rt}G$. Using the same arguments as in Theorem 4.4 of [2] we obtain this part.
(b) $\Rightarrow$ (a) $A - r$ generates a quasi-distribution semigroup G_1 given by $G_1(\varphi) := \int_0^\infty \varphi^{(k)}(t)S(t)dt$, $\varphi \in \mathcal{D}$, cf. [19]. Hence, A is the generator of a (QDSG) $G := e^{-rt}G_1$. Clearly, G is an $\{r, k\}$-semigroup.
(b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) This follows immediately from [11, Theorem 3.6]. *qed*

Example 2.1. Let $r \geq 0$, $1 < p < \infty$, $p \neq 2$, $k \in \mathbb{N}$, $k \geq n\left|\frac{1}{p} - \frac{1}{2}\right|$. Then $A := i\Delta + r$ with the maximal distributional domain is the generator of an $\{r, k+2\}$-semigroup on $L^p(\mathbb{R}^n)$.

Let us recall that if $\rho(A) \neq \emptyset$, then $n(A) = \inf\left\{k \in \mathbb{N}_0 : D(A^k) \subset \overline{D(A^{k+1})}\right\}$. Lemma 1.5 in [15] and the next theorem imply the estimate $n(A) \leq 1$, if A generates an $[r]$-semigroup.

Theorem 2.1. *Let A be the generator of an $[r, k]$-semigroup, $r \geq 0$, $k \in \mathbb{N}$. Then $\{\lambda \in \mathbb{C} : Re\lambda > r\} \subset \rho(A)$ and there exists $M > 0$ such that for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ with $Re\lambda > r$, the following holds*

$$\|R(\lambda : A)^n\| \leq \frac{Mn(n+1)\cdots(n+k-1)|\lambda|^k}{(Re\lambda - r)^{n+k}}.$$

Proof. We follow the proof of [6, Theorem III.8]. Clearly, if A is the generator of an $[r, k]$-semigroup G then $\{\lambda \in \mathbb{C} : Re\lambda > r\} \subset \rho(A)$ and $R(\lambda : A) = G(h_\lambda(t))$, $Re\lambda > r$. Let $k \in \mathbb{N}$ and $\lambda \in \mathbb{C}$, $Re\lambda > r$, be fixed. By induction, we have $\underbrace{h_\lambda(t) * \ldots * h_\lambda(t)}_{n} = \frac{t^{n-1}}{(n-1)!}h_\lambda(t)$, $t \in \mathbb{R}$. Hence,

$$||R(\lambda : A)^n|| = ||G(\frac{t^{n-1}}{(n-1)!}h_\lambda(t))|| \leq \frac{C}{(n-1)!}\sum_{i=0}^{k}\int_0^\infty e^{rt}t^i|(t^{n-1}e^{-\lambda t})^{(i)}(t)|dt$$

$$\leq \frac{C}{(n-1)!}\sum_{i=0}^{k}\int_0^\infty e^{rt}t^i|\sum_{j=0}^{i}\binom{i}{j}(n-1)\ldots(n-j)t^{n-j-1}|\lambda|^{i-j}e^{-Re\lambda t}|dt$$

$$\leq \frac{2^k C}{(n-1)!(Re\lambda - r)^n}\sum_{i=0}^{k}(n+k-1)!(i+1)\frac{|\lambda|^i}{(Re\lambda - r)^i}$$

$$\leq M_1 2^k (k+1)^2 \frac{n(n+1)\ldots(n+k-1)}{(Re\lambda - r)^n}\frac{|\lambda|^k}{(Re\lambda - r)^k}. \quad \square$$

Suppose that A generates an $[r]$-semigroup, $r \geq 0$ and $x \in E$. Then Theorem 2.1 implies that the sequence $(n||R(n : A)x||)_{n\in\mathbb{N},\ n>r}$ is bounded and that $\lim_{\lambda\to+\infty} R(\lambda : A)x = 0$. Now one can repeat literally the proof of [9, Proposition 1.1] to obtain the proof of the next proposition.

Proposition 2.2. *Assume that E is reflexive. If A is the generator of an $[r]$-semigroup, $r \geq 0$, then A is densely defined.*

3. Relations with integrated semigroups

Let $D_\infty(A) := \bigcap_{n\geq 0} D(A^n)$. In order to analyze relations of $[r]$-semigroups and integrated semigroups, we need the following lemma. See [2] for the proof.

Lemma 3.1. *Let $k,\ m \in \mathbb{N}$ and $m \geq k$. The set $\left\{\varphi^{(k)} : \varphi \in \mathcal{D}_{(0,\infty)}\right\}$ is dense in $L^1((0,\infty),\ (t^k + t^m)dt)$.*

Proposition 3.1. *Assume that $\overline{D(A)} = E$, $r \geq 0$ and $k \in \mathbb{N}_0$. If A is the generator of an $[r,k]$-semigroup G, then $A - r$ is the generator of a k-times integrated semigroup $(W(t))_{t\geq 0}$ satisfying $\|W(t)\| = O(t^k + t^{2k})$.*

Proof. We give here the proof in the case $k \in \mathbb{N}$. If $k = 0$, then it can be derived similarly. We follow the proof of [2, Theorem 4.4] with appropriate modifications. Since $\overline{D(A)} = E$ and $\rho(A) \neq \emptyset$, we have $\overline{D_\infty(A)} = E$ (cf. [3], [15]). Since $A - r$ is the generator of a (QDSG) $G_1 := e^{-rt}G$, an application of [19, Corollary 3.9] gives that for all $x \in D_\infty(A - r) = D_\infty(A)$ there exists $v_x \in C([0,\infty) : E)$ such that $v_x(0) = x$ and $G_1(\varphi)x = \int_0^\infty \varphi(t)v_x(t)dt, \quad \varphi \in \mathcal{D}_0$. Integration by parts gives that for every fixed $x \in D_\infty(A)$:

$$G_1(\varphi)x = (-1)^k \int_0^\infty \varphi^{(k)}(t)t^k H_x(t)dt, \quad \varphi \in \mathcal{D}_0,$$

where $H_x(t) := \frac{1}{t^k}\int_0^t \frac{(t-s)^{k-1}}{(k-1)!}v_x(s)ds,\ t > 0$. Let $t > 0$ be fixed. Since the function v_x is unique, the mapping $x \mapsto H_x(t)$ defines a linear operator from $D_\infty(A - r)$ to E. Let us show the continuity of this mapping. Let $x^* \in E^*$ be fixed. Then prescribed assumptions and Lemma 4.6 in [2] imply

$$\left|\int_0^\infty \varphi^{(k)}(t)t^k x^*(H_x(t))dt\right| = |x^*(G_1(\varphi)x)| \leq \|x\|\ \|x^*\|\ \|G_1(\varphi)\|$$
$$\leq C\,\|x\|\ \|x^*\| \sum_{i=0}^{k} \left\|e^{rt}t^i(e^{-rt}\varphi)^{(i)}\right\|_1$$
$$\leq C_1\,\|x\|\ \|x^*\| \sum_{i=0}^{k}\sum_{j=0}^{i} \left\|t^i\varphi^{(j)}\right\|_1 \leq C_2\,\|x\|\ \|x^*\| \sum_{i=0}^{k}\sum_{j=0}^{i} \left\|t^{k+i-j}\varphi^{(k)}\right\|_1$$
$$\leq C_3\,\|x\|\ \|x^*\|\ \left\|(t^k + t^{2k})\varphi^{(k)}\right\|_1 = C_3\,\|x\|\ \|x^*\|\ \left\|\varphi^{(k)}\right\|_{L^1((0,\infty):(t^k+t^{2k}))},$$

for some absolute constants C, C_1, C_2 and C_3. Hence, the functional

$$T : \varphi^{(k)} \mapsto (-1)^k \int\limits_0^\infty \varphi^{(k)}(t) x^*(H_x(t)) t^k dt,\ \varphi \in \mathcal{D}_{(0,\infty)},$$

can be extended to the whole space $L^1((0,\infty) : (t^k + t^{2k})dt)$ by virtue of Lemma 3.1. Moreover, $\|T\| \leq C_3 \|x\|\,\|x^*\|$ and $|x^*(H_x(t))| \leq C_3 ||x||\,||x^*||(1+t^k),\ t > 0$. Let $t > 0$ be fixed again. Choose $x^* \in E^*$ with $(H_x(t),\ x^*) \in F$, where $F = \{(x,\ x^*) \in E \times E^* : x^*(x) = ||x||^2 = ||x^*||^2\}$. Then one obtains $||H_x(t)|| \leq C_3 ||x||(1 + t^k),\ t > 0$. Thus, for fixed $t > 0$, $x \mapsto H_x(t)$ defines the bounded linear operator from $D_\infty(A - r)$ into E, with the norm $\leq C_3(1 + t^k)$. If $t > 0$, then we define $W(t)$ as a bounded extension of $x \mapsto t^k H_x(t)$ from $D_\infty(A-r)$ to E. Define $W(0) := 0$. Then $(W(t))_{t\geq 0}$ is a strongly continuous operator family with $\|W(t)\| = O(t^k + t^{2k})$. Since $G_1 = e^{-rt}G$, we have

$G(e^{-rt}\varphi) = (-1)^k \int\limits_0^\infty \varphi^{(k)}(t) W(t) dt,\quad \varphi \in \mathcal{D}_0$. Let $\lambda \in \mathbb{C}$ with $Re\lambda > 0$ be fixed. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}_0$ such that $\lim\limits_{n\to\infty} \varphi_n^{(k)} = h_\lambda^{(k)}$, in the sense of convergence in $L^1((0,\infty) :\ (t^k + t^{2k})dt)$. Since $||W(t)|| = O(t^k + t^{2k})$, one obtains

$$(-1)^k \int\limits_0^\infty \varphi_n^{(k)}(t) W(t) dt \to \lambda^k \int\limits_0^\infty e^{-\lambda t} W(t) dt,\ n \to \infty.$$

By the previous arguments, one has (for appropriate $M > 0$)

$$p_{rk}(e^{-rt}(\varphi_n - h_\lambda)) \leq M ||\varphi_n^{(k)} - h_\lambda^{(k)}||_{L^1((0,\infty):\ (t^k+t^{2k}))} \to 0,\ n \to \infty.$$

As a consequence, one obtains $\lim\limits_{n\to\infty} e^{-rt}\varphi_n = h_{\lambda+r}$ in T_{rk}. This implies

$$R(\lambda : A - r)x = \lambda^k \int\limits_0^\infty e^{-\lambda t} W(t) x dt,\ x \in E,\ Re\lambda > 0. \ \square$$

Proposition 3.2. *Suppose m, $m - k \in \mathbb{N}_0$ and $r > 0$. Then:*

(a) If A is the generator of a k-times integrated semigroup $(S(t))_{t\geq 0}$ satisfying $\|S(t)\| = O(e^{rt}(t^k + t^m))$, then A is the generator of an $[r, m]$-semigroup.

(b) If $A - r$ is the generator of a k-times integrated semigroup $(W(t))_{t\geq 0}$ satisfying $\|W(t)\| = O(t^k + t^m)$, then A is the generator of an $[r, m]$-semigroup.

Proof. (a) Define $G(\varphi) := (-1)^k \int\limits_0^\infty \varphi^{(k)}(t) S(t) dt,\ \varphi \in \mathcal{D}$. Then G is a (QDSG) with the generator A. Moreover, the definition of $G(\varphi),\ \varphi \in \mathcal{D}_+$ is clear, and one obtains the estimate

$$\|G(\varphi)\| \leq C(\|e^{rt}t^k\varphi^{(k)}\|_1 + \|e^{rt}t^m\varphi^{(k)}\|_1), \quad \varphi \in \mathcal{D}_+,$$

where C is independent of $\varphi \in \mathcal{D}_+$. Applying the same arguments as in the proof of Lemma 2.1, we have that G is an $[r,m]$-semigroup.
(b) Similarly as in the first part, we have that A is the generator of a (QDSG) G given by $G(\varphi) := (-1)^k \int\limits_0^\infty (e^{rt}\varphi)^{(k)}W(t)dt$, $\varphi \in \mathcal{D}$. Additionally,

$$\|G(\varphi)\| \leq C(\|t^k(e^{rt}\varphi)^{(k)}\|_1 + \|t^m(e^{rt}\varphi)^{(k)}\|_1), \quad \varphi \in \mathcal{D}_+,$$

where C is independent of $\varphi \in \mathcal{D}_+$. Using Leibniz's rule and the proof of Lemma 2.1 we obtain that G is an $[r,m]$-semigroup. This completes the proof. □

Now we state the assertion which corresponds to Proposition 3.1 in the case when A is not densely defined.

Theorem 3.1. *Suppose that A generates an $[r,k]$-semigroup G, $r \geq 0$, $k \in \mathbb{N}_0$. Then the part of $A - r$ in $\overline{D(A)}$ generates a k-times integrated semigroup $(S(t))_{t\geq 0}$ in $\overline{D(A)}$ which satisfies $||S(t)|| = O(t^k + t^{2k})$.*

Proof. Define $H(\varphi)x := G(\varphi)x$, $\varphi \in \mathcal{D}$, $x \in \overline{D(A)}$. It can be easily seen that H is an $[r,k]$-semigroup in $\overline{D(A)}$ and that its generator is the part of A in $\overline{D(A)}$. Since G is an $[r,k]$-semigroup, it follows $n(A) \leq 1$ and $D(A) \subset \overline{D(A^2)}$. Thus, the generator of H is densely defined in $\overline{D(A)}$ because its domain $\{x \in D(A) : Ax \in \overline{D(A)}\}$ contains $D(A^2)$. Now the claim follows by an application of Proposition 3.1. □

Let us state now the assertion which naturally corresponds to Theorem 2.1. It will be proved here with the help of integrated semigroups. We also refer to the proof of Theorem III. 9 in [6] where more complicated arguments are used.

Theorem 3.2. *Let A be a closed linear operator with $\{\lambda \in \mathbb{C} : Re\lambda > r\} \subset \rho(A)$, for some $r \geq 0$. If*

$$\|R(\lambda : A)\| \leq M\frac{|\lambda|^k}{(Re\lambda - r)^{k+1}}, \quad Re\lambda > r,$$

for some $k \in \mathbb{N}_0$ and $M > 0$, then A is the generator of a $(k+2)$-times integrated semigroup $(S(t))_{t\geq 0}$ with the growth rate $O(e^{rt}t^{k+2})$ as well as A is the generator of an $[r, k+2]$-semigroup. Moreover, if $r > 0$, then $||S(t)|| = O(e^{rt}t^{k+1})$.

Proof. Let $a > r$ be an arbitrary real number. Define

$$S(t) := \frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty} e^{\lambda t}\lambda^{-k-2}R(\lambda : A)d\lambda, \ t \geq 0.$$

Then $(S(t))_{t\geq 0}$ is a strongly continuous operator family with

$$\|S(t)\| \leq \frac{Me^{at}}{2a(a-r)^{k+1}},\ t\geq 0. \tag{2}$$

Using the same arguments as in [20, Theorem 1.12], we have that $(S(t))_{t\geq 0}$ is a $(k+2)$-times integrated semigroup generated by A. Let $t>0$ be fixed. Cauchy formula implies

$$S(t) = \frac{1}{2\pi i}\int\limits_{r+\frac{1}{t}-i\infty}^{r+\frac{1}{t}+i\infty} e^{\lambda t}\lambda^{-k-2}R(\lambda : A)d\lambda.$$

Putting $a = r+\frac{1}{t}$ in (2), we obtain $\|S(t)\| = O(\frac{e^{rt}t^{k+2}}{rt+1})$. Then, with

$$G(\varphi)x := (-1)^k\int\limits_0^\infty \varphi^{(k+2)}(t)S(t)xdt,\ x\in E,\ \varphi\in\mathcal{D},$$

is defined an $[r,k+2]-$semigroup G generated by A. □

Let $r\geq 0$. Then Theorem 2.1 and Theorem 3.2 imply that A is the generator of an $[r]$-semigroup iff there exist $k\in\mathbb{N}$ and $M>0$ such that
$\{\lambda\in\mathbb{C} : Re\lambda > r\}\subset\rho(A)$ and $\|R(\lambda : A)\|\leq M\frac{|\lambda|^k}{(Re\lambda-r)^{k+1}},\ Re\lambda>r$. Note that $\{r\}$-semigroups can be described in the similar manner. In this sense, we also refer to Proposition 1 and Proposition 2 of [8].

4. Relations with functional calculi

Throughout this section, we investigate relations between $[r]$-semigroups and functional calculi of deLaubenfels and Jazar. We need the following definition.

Definition 4.1. [11] Denote by $\mathcal{A}$ the space of all Laplace transforms of functions in the Schwartz space $\mathcal{S}$, supplied with the following family of seminorms $\|g\|_{j,k} := \|t^j\varphi^{(k)}(t)\|_{L^1([0,\infty))},\ j,\ k\in\mathbb{N}_0,\ g=\mathcal{L}(\varphi)\in\mathcal{A}$. A smooth semispectral distribution for A is a continuous algebra homomorphism
$f:\mathcal{A}\to L(E)$, such that

(i) $\{\lambda\in\mathbb{C} : Re\lambda<0\}\subset\rho(A)$, with $f\left(\frac{1}{\lambda-\cdot}\right) = R(\lambda : A)$ whenever $Re\lambda<0$;

(ii) $f\left(g\left(\frac{\cdot}{n}\right)\right)x\to x,\ n\to\infty$; for all $x\in E$ and $g\in\mathcal{A}$ such that $g(0)=1$.

Let $D(A)$ be dense in E and let A be the generator of a global k-times integrated semigroup with the growth order $O(t^k(1+t^n))$, for some $n,\ k\in\mathbb{N}_0$. Then $-A$ admits a smooth semispectral distribution, see [11, Theorem 3.2].

We give the result which generalizes [11, Theorem 3.2] so that a global k-times integrated semigroup that is $O(t^k+t^m)$, for some $k,\ m\in\mathbb{N}$ with $m\geq k$, is described in terms of a quasi-distribution semigroup and a smooth semispectral distribution.

Theorem 4.1. *Suppose that $D(A)$ is dense in E and $m,\ k \in \mathbb{N}$, $m \geq k$. Then the following assertions are equivalent.*

(a) A is the generator of a (QDSG) G satisfying, for some $C > 0$,

$$\|G(\varphi)\| \leq C\|(t^k + t^m)\varphi^{(k)}\|_1,\ \varphi \in \mathcal{D}.$$

(b) A is the generator of a k-times integrated semigroup $(S(t))_{t\geq 0}$ with $\|S(t)\| = O(t^k + t^m)$.

(c) $-A$ admits a smooth semispectral distribution f such that for some $C > 0$:

$$\|f(\hat{\varphi})\| \leq C\|(t^k + t^m)\varphi^{(k)}\|_1,\ \varphi \in \mathcal{D}. \text{ (Recall, } \hat{\varphi} = \mathcal{L}(\varphi).)$$

Proof. (a) $\Leftrightarrow$ (b) This follows similarly as in [2, Theorem 4.4]. Note only that one must use Lemma 3.1 to prove the denseness argument which appears in the proofs of Theorem 4.4 in [2] and Proposition 3.1. With this observation, one can repeat literally the proof of Theorem 4.4 in [2].

(b) $\Rightarrow$ (c) It follows from the proof of Theorem 3.2 in [11].

(c) $\Rightarrow$ (a) Define $G(\varphi) := f(\hat{\varphi})$, $\varphi \in \mathcal{D}$. Since f is a continuous algebra homomorphism, we have $G \in \mathcal{D}_0'(L(E))$. Moreover,

$$G(\varphi *_0 \psi) = f(\widehat{\varphi *_0 \psi}) = f(\hat{\varphi}\hat{\psi}) = f(\hat{\varphi})f(\hat{\psi}) = G(\varphi)G(\psi),\ \varphi,\ \psi \in \mathcal{D},$$

and (QDSG1) holds. In order to prove (QDSG2), let $x \in E$ and $f(\hat{\varphi})x = 0$, $\varphi \in \mathcal{D}_0$. Let $(\varphi_n)_{n\geq 0}$ be a $\mathcal{D}_0$-sequence such that $\lim\limits_{n\to\infty} \widehat{\varphi_n} = \frac{1}{\lambda - \cdot}$ in $\mathcal{A}$, for some fixed $\lambda \in \mathbb{C}$ with $Re\lambda < 0$. Consequently, $\lim\limits_{n\to\infty} f(\widehat{\varphi}_n)x = (\lambda + A)^{-1}x = 0$ and $x = 0$. Thus, (QDSG2) holds and G is a (QDSG) with

$$\|G(\varphi)\| \leq C\|(t^k + t^m)\varphi^{(k)}\|_1,\ \varphi \in \mathcal{D}.$$

Let us show that A is the generator of G. Suppose $(x, y) \in B$, where B is the generator of G. Then one has $G(-\varphi')x = G(\varphi)y$, $\varphi \in \mathcal{D}_0$. Let $(\psi_n)_{n\geq 0}$ be a $\mathcal{D}_0$-sequence with $\lim\limits_{n\to\infty} \widehat{\psi_n} = \frac{1}{(-1-\cdot)^2}$ in $\mathcal{A}$. This implies $\lim\limits_{n\to\infty} z\widehat{\psi_n} = \frac{z}{(-1-z)^2}$ in $\mathcal{A}$. Now we obtain $\lim\limits_{n\to\infty} f(\widehat{\psi_n})y = (-1 + A)^{-2}y$ and, by the definition of G,

$$\begin{aligned}(-1 + A)^{-2}y &= \lim_{n\to\infty} f(-\widehat{\psi_n'})x = \lim_{n\to\infty} f(-z\widehat{\psi_n})x = f\left(-\tfrac{z}{(-1-z)^2}\right)x \\ &= f\left(\tfrac{1}{-1-z} + \tfrac{1}{(-1-z)^2}\right)x = (-1 + A)^{-1}x + (-1 + A)^{-2}x.\end{aligned}$$

This implies $(x, y) \in A$ and $B \subset A$. Assume now $(x, y) \in A$. By the partial integration, we have

$$(\mathcal{L}\varphi)(z) = \frac{1}{(1 + z)^n}\mathcal{L}\left(\left(1 + \frac{d}{dt}\right)^n \varphi\right)(z),\ n \in \mathbb{N}_0,\ \varphi \in \mathcal{D}_0. \tag{3}$$

Let $x = R(1 : A)v$, for some $v \in E$. Applying (3), we have

$$f(\hat{\varphi})v = f\left(\frac{1}{1+z}(\hat{\varphi} + \widehat{\varphi'})\right) = -f(\hat{\varphi} + \widehat{\varphi'})(-1 + A)^{-1}v = f(\hat{\varphi} + \widehat{\varphi'})x, \text{ and}$$

$f(\hat{\varphi})v = f(\hat{\varphi})x + f(\widehat{\varphi'})x,\ \varphi \in \mathcal{D}_0$. Then $x - Ax = v$ implies $f(-\widehat{\varphi'})x = f(\hat{\varphi})Ax$, $\varphi \in \mathcal{D}_0$. Hence, $(x, y) \in B$. The proof is now complete. □

Remark 4.1. Suppose that $-A$ admits a smooth semispectral distribution f. Then A is densely defined. To prove this, let G be defined as above. By the proof of (c) ⇒ (a), we obtain that G is a (QDSG) generated by A. Let $\varphi \in \mathcal{D}_0$ satisfy $\int_0^\infty \varphi(x)dx = 1$. Define $\varphi_n := n\varphi(n\cdot)$, for all $n \in \mathbb{N}$. Then $\hat{\varphi}(0) = 1$ and (ii) of Definition 4.1 implies $\lim_{n\to\infty} G(\varphi_n)x = x$, for all $x \in E$. Since $\mathcal{R}(G) \subset D(A)$, it implies $\overline{D(A)} = E$.

As an immediate consequence we have:

Theorem 4.2. *Let A be a closed, densely defined operator in E and let $r > 0$. Then the following statements are equivalent.*

(a) *A is the generator of an $[r]$-semigroup.*

(b) *$r - A$ admits a smooth semispectral distribution f such that*

$$\|f(\hat{\varphi})\| \le C\|(t^k + t^m)\varphi^{(k)}\|_1,\ \varphi \in \mathcal{D},$$

for some $k,\ m \in \mathbb{N}$ with $m \ge k$, and a suitable $C > 0$.

(c) *A is the generator of a k-times integrated semigroup $(S(t))_{t\ge 0}$ satisfying $\|S(t)\| = O(e^{rt}(t^k + t^m))$, for some $k,\ m \in \mathbb{N}$ with $m \ge k$.*

(d) *$A - r$ is the generator of a k-times integrated semigroup $(W(t))_{t\ge 0}$ satisfying $\|W(t)\| = O(t^k + t^m)$, for some $k,\ m \in \mathbb{N}$ with $m \ge k$.*

Proof. Proposition 3.1 implies that (d) is a consequence of (a). The implication (d) ⇒ (c) follows from the rescaling result for integrated semigroups, see [5, Proposition 3.2.6]. The implication (c) ⇒ (a) follows by an application of Proposition 3.2. The equivalence of (b) and (d) follows from Theorem 4.1. □

Recall [11], if $n,\ k \in \mathbb{N}$, then

$$W^{1,n}([0,\infty)) := \{F \in C^{n-1}([0,\infty)) : F^{(j)} \in L^1([0,\infty)) \text{ for } j = 0, 1, \dots, n\}, \text{ and}$$

$$\mathcal{A}_{n,k} = \{g = \mathcal{L}(F) : (1+t)^k F(t) \in W^{1,n}([0,\infty))\}.$$

It is topologized by the norm

$$\|f\|_{\mathcal{A}_{n,k}} = \sum_{j=0}^{n} \frac{1}{j!}\|(1+t)^k F^{(j)}(t)\|_{L^1([0,\infty))},\ f = \mathcal{L}(F) \in \mathcal{A}_{n,k}.$$

In the next proposition, $\mathcal{A}_{n,k}$ functional calculus is taken in the sense of [11, Definition 1.1].

Proposition 4.1. *Let A be the generator of an $[r,k]$-semigroup, $r \geq 0,\ k \in \mathbb{N}$. Then the following holds:*

(a) A is the generator of an $R(r+1:A)^{k+2}$-regularized semigroup $(C(t))_{t\geq 0}$ satisfying $\|C(t)\| = O(e^{rt}(1+t^{k+1}))$.

(b) $r-A$ admits an $\mathcal{A}_{k+2,n}$ functional calculus for all $n \in \mathbb{N}$ with $n \geq k+1$.

Proof. (a) By Theorem 2.1, we have $\{z \in \mathbb{C} : Rez > 0\} \subset \rho(A-r)$ and

$$\|(z-(A-r))^{-1}\| = \|R(z+r:A)\| \leq C\frac{|z+r|^k}{(Rez)^{k+1}} \leq C_1\frac{(1+|z|)^k}{(Rez)^{k+1}},$$

for some constants $C,\ C_1 > 0$ independent of z with $Rez > 0$. Applying [11, Theorem 2.7] we have that $A-r$ is the generator of an $R(r+1:A)^{k+2}$-regularized semigroup $(T(t))_{t\geq 0}$ with the growth order $O(1+t^{k+1})$. Put $C(t) = e^{rt}T(t),\ t \geq 0$. Then $(C(t))_{t\geq 0}$ is an $R(r+1:A)^{k+2}$-regularized semigroup generated by A and $||C(t)|| = O(e^{rt}(1+t^{k+1}))$.
(b) Similar estimates for the resolvent of $A-r$ and [11, Theorem 2.7] imply this part. □

Finally, we give several examples of $[r]$-semigroups.

Example 4.1. [4] Let $1 < p < \infty$. Denote by J_p the Riemann-Liouville semigroup on $L^p((0,1))$;

$$(J_p(z)f)(x) := \frac{1}{\Gamma(z)}\int_0^x (x-y)^{z-1}f(y)dy,\ f \in L^p((0,1)),\ x \in (0,1),\ Rez > 0.$$

Denote by A_p the generator of J_p. It is proved in [4] that the operator iA_p generates a C_0-group $(T_p(t))_{t\in\mathbb{R}}$ on $L^p((0,1))$ which satisfies $\|T_p(t)\| = O((1+t^2)e^{|t|\frac{\pi}{2}}),\ t \in \mathbb{R}$. Proposition 3.2 implies that with

$$G_p(\varphi) := \int_0^\infty \varphi(t)T_p(t)dt,\ \varphi \in \mathcal{D},$$

is defined a dense $[\frac{\pi}{2}, 2]$-semigroup G_p on $L^p((0,1))$ generated by iA_p. Evidently, $-iA_p$ also generates a dense $[\frac{\pi}{2}, 2]$-semigroup on $L^p((0,1))$.

Example 4.2. [12] Let $1 \leq p < \infty$ and $m : \mathbb{R} \to (0,\infty)$ be a measurable function such that

$$(\sup_{s\in\mathbb{R}}\frac{m(s-t)}{m(s)})^{\frac{1}{p}} \leq M(1+t^k),\ t \geq 0, \tag{4}$$

for some $k \in \mathbb{N}$ and $M > 0$. Let $r > 0$ be fixed. A simple observation (as in [12]) gives that

$$(T_p(t)f)(x) := e^{rt}f(x+t),\ x \in \mathbb{R},\ t \geq 0,\ f \in L^p(\mathbb{R},\ m(x)dx),$$

defines a C_0-semigroup $(T_p(t))_{t\geq 0}$ on $L^p(\mathbb{R},\ m(x)dx)$ satisfying

$$\|T_p(t)\| = e^{rt}(\sup_{s\in\mathbb{R}} \tfrac{m(s-t)}{m(s)})^{\frac{1}{p}} = O(e^{rt}(1+t^k)).$$

Thus, with $G_p(\varphi) := \int\limits_0^\infty \varphi(t)T_p(t)dt$, $\varphi \in \mathcal{D}$, is defined a dense $[r,k]$-semigroup G_p on $L^p(\mathbb{R},\ m(x)dx)$. If m is a positive polynomial, then (4) is satisfied for some $k \in \mathbb{N}$ and $M > 0$.

Let us show now that the class of $[r]$-semigroups does not coincide with the class of $\{r\}$-semigroups, if $r > 0$.

Example 4.3. [5] Let $r > 0$ be fixed and

$$E := \{f \in C([0,\infty)) : \lim_{x\to\infty} \tfrac{f(x)}{x+1} = 0\},$$
$$\|f\| := \sup_{x\geq 0} \tfrac{|f(x)|}{x+1},\ \ f \in E,$$
$$(T(t)f)(x) := f(x+t),\ f \in E,\ t \geq 0,\ x \geq 0.$$

Then $(T(t))_{t\geq 0}$ is a C_0-semigroup on E, and $\|T(t)\| = t+1,\ t \geq 0$, see Example 5.4.5 of [5]. Its generator A is just the operator $\frac{d}{dx}$ with the maximal domain. Accordingly, with

$$G(\varphi) := \int\limits_0^\infty \varphi(t)e^{rt}T(t)dt,\ \ \varphi \in \mathcal{D},$$

is defined a dense $[r,1]$-semigroup G on E generated by $A+r$. Suppose that G is an $\{r,k\}$-semigroup for some $k \in \mathbb{N}$. Then the use of Proposition 2.1 gives that A generates a k-times integrated semigroup $(S(t))_{t\geq 0}$ on E satisfying $\|S(t)\| \leq Mt^k,\ t \geq 0$, for some $M > 0$. Since $S(t) = \int\limits_0^t \frac{(t-s)^{k-1}}{(k-1)!}T(s)ds,\ t \geq 0$, it follows

$$\sup_{x\geq 0} \frac{\left|\int\limits_0^t \frac{(t-s)^{k-1}}{(k-1)!} f(x+s)ds\right|}{x+1} \leq Mt^k \sup_{x\geq 0} \tfrac{|f(x)|}{x+1},\ \ f \in E,\ t \geq 0.$$

Choose $f(\cdot) = \sqrt{\cdot}$ to obtain

$$\int\limits_0^t \frac{(t-s)^{k-1}}{(k-1)!}\sqrt{s}ds \leq \sup_{x\geq 0} \frac{\left|\int\limits_0^t \frac{(t-s)^{k-1}}{(k-1)!}\sqrt{x+s}ds\right|}{x+1} \leq Mt^k/2,\ \ t \geq 0.$$

This is a contradiction. Moreover, for every $k \in \mathbb{N}_0$ the operator A generates a k-times integrated semigroup $(S(t))_{t\geq 0}$ on E such that $||S(t)|| = O(t^k + t^{k+1})$. Note that there does not exist $\alpha \in [0, k+1)$ such that $\|S(t)\| = O(t^k + t^\alpha + 1)$.

Example 4.4. Suppose that A generates a k-times integrated semigroup $(S(t))_{t\geq 0}$ on E. If there exists $a > 0$ such that $\|Ax\| \leq a\|x\|$, $x \in D(A)$, then A generates an $[a, k+1]$-semigroup. Since

$$A\int_0^t S(s)xds = S(t)x - \tfrac{t^k}{k!}x, \;\; t \geq 0, \, x \in E,$$

we obtain

$$\|S(t)x\| \leq \tfrac{t^k}{k!}\|x\| + a\int_0^t \|S(s)x\|\, ds, \;\; t \geq 0, \, x \in E.$$

Gronwall's inequality implies

$$\|S(t)x\| \leq \tfrac{t^k}{k!}\|x\| + ae^{at}\int_0^t e^{-as}\tfrac{s^k}{k!}\|x\|\, ds, \;\; t \geq 0, \, x \in E.$$

This gives $\|S(t)\| = O(e^{at}(t^k + t^{k+1}))$ and now one may apply Proposition 3.2 to obtain that A generates an $[a, k+1]$-semigroup.

Next we show that for every $r > 0$ and $k \in \mathbb{N}$ there exists a dense $[r, k]$-semigroup that is not an $[r, k-1]$-semigroup.

Example 4.5. **Example 4.5.** Let $r > 0$ and $T \in L(E)$ satisfy $T^{k+1} = 0$, for some $k \in \mathbb{N}$. Define

$$T(t) := e^{rt}\sum_{i=0}^{k}\frac{T^i t^i}{i!}, \;\; t \geq 0.$$

Then $(T(t))_{t\geq 0}$ is a C_0-semigroup generated by $T + r$. Moreover, $||T(t)|| = O(e^{rt}(1 + t^k))$, and $T + r$ generates a dense $[r, k]$-semigroup.

Choose now $E := \mathbb{R}^{k+1}$ with the sup-norm, and

$$T(x_1, x_2, \ldots, x_{k+1}) := (x_2, \ldots, x_{k+1}, 0), \;\; x_i \in \mathbb{R}, \;\; i = 1, 2, \ldots, k+1.$$

Then $T^{k+1} = 0$ and $T + r$ generates a dense $[r, k]$-semigroup G. Suppose that G is an $[r, k-1]$-semigroup. Then Proposition 3.1 implies that T generates a $(k-1)$-times integrated semigroup $(S(t))_{t\geq 0}$ satisfying $||S(t)|| = O(t^{k-1} + t^{2k-2})$. If $k = 1$, it means that T generates a bounded C_0-semigroup. Then the contradiction is obvious since $||e^{-rt}T(t)|| = 1 + t + \ldots + \frac{t^k}{k!}$, $t \geq 0$. If $k > 1$, then

$$S(t)(x_1, x_2, \ldots, x_{k+1}) = \int_0^t \frac{(t-s)^{k-2}}{(k-2)!}e^{-rs}T(s)(x_1, x_2, \ldots, x_{k+1})ds.$$

Direct computation shows that $||S(t)|| = \frac{t^{k-1}}{(k-1)!} + \ldots + \frac{t^{2k-1}}{(2k-1)!}$, $t \geq 0$. This is in contradiction with $||S(t)|| = O(t^{k-1} + t^{2k-2})$.

At the end, we note that many other examples of dense $[r]$-semigroups, $r \geq 0$, can be derived through the analysis of Petrovsky correct parabolic systems of differential equations given in [21]. In this sense, Theorem 2.2 (a), Corollary 2.3 and Example 2.4 of [21], can be used for the construction of $[r]$-semigroups.

Acknowledgement

The author is very grateful to Professor Stevan Pilipović for his valuable suggestions.

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Received by the editors October 9, 2006